SINGULAR HAHN–HAMILTONIAN SYSTEMS

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Abstract. In this work, we study a Hahn–Hamiltonian system in the singular case. For this system, the Titchmarsh–Weyl theory is established. In this context, the first part provides a summary of the relevant literature and some necessary fundamental concepts of the Hahn calculus. To pass from the Hahn difference expression to operators, we define the Hilbert space $L^2_{\omega,q,W}((\omega_0,\infty);\mathbb{C}^{2n})$ in the second part of the work. The corresponding maximal operator $L_{\max}$ are introduced. For the Hahn–Hamiltonian system, we proved Green formula. Then we introduce a regular self-adjoint Hahn–Hamiltonian system. In the third part of the work, we study Titchmarsh-Weyl functions $M(\lambda)$ and circles $\mathcal{C}(a,\lambda)$ for this system. These circles proved to be embedded one to another. The number of square-integrable solutions of the Hahn–Hamilton system is studied. In the fourth part of the work, we obtain boundary conditions in the singular case. Finally, we define a self-adjoint operator in the fifth part of the work.

Keywords: Hahn–Hamiltonian system, singular point, Titchmarsh–Weyl theory.

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1. Introduction

In this paper, we consider singular Hahn–Hamiltonian systems defined as

$$J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x) = \lambda W(x)\mathcal{Z}(x),\ x \in [\omega_0,\infty), \tag{1.1}$$

where the matrices

$$B(x) = \begin{pmatrix} B_1(x) & B_2^*(x) \\ B_2(x) & B_3(x) \end{pmatrix}$$

and $W(\cdot)$ are $2n\times 2n$ complex Hermitian matrix-valued functions defined on $[\omega_0,\infty)$ and are continuous at ω_0; $\mathcal{Z}(x)$ is $2n\times 1$ vector-valued function;

$$\mathcal{Z}^{[h]}(x) = \begin{pmatrix} D_{\omega,q}\mathcal{Z}_1(x) \\ \frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{Z}_2(x) \end{pmatrix} = \begin{pmatrix} D_{\omega,q}\mathcal{Z}_1(x) \\ \frac{1}{q}D_{\omega,q}\mathcal{Z}_2\left(h^{-1}(x)\right) \end{pmatrix},$$

and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

where I_n is the $n\times n$ identity matrix. The theory of Hamiltonian systems is well developed, see [5], [6], [9]–[12], [14]–[16] and it plays important role in modeling various physical systems, for example, in the study of electromechanical, electrical, and complex network systems with negligible dissipation, see [18]. However, to the best knowledge of the authors of this paper, there is no study on the Hahn–Hamiltonian system, though there are some results about the Hahn–Dirac systems in the literature, see [1], [2], [13]. In this paper, our main aim is to develop the Titchmarsh–Weyl theory for singular Hahn–Hamiltonian systems. In our analysis we mostly follow the development of the theory in [14], [15], [17].

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For the reader's convenience, we recall main concepts. For further details, we refer the reader to [1]–[4], [7], [8], [13]. Throughout the paper, we let $\omega > 0$, $h(x) := \omega + qx$ and $q \in (0,1)$. Let I be a real interval containing ω_0, where $\omega_0 := \frac{\omega}{1-q}$.

Definition 1.1 ([7],[8]). *Let $u : I \to \mathbb{R}$ be a function. If u is differentiable at ω_0, then the Hahn operator $D_{\omega,q}$ is given by the formula*

$$D_{\omega,q}u(x) = \begin{cases} (\omega + (q-1)x)^{-1}(u(\omega + qx) - u(x)), & x \neq \omega_0, \\ u'(\omega_0), & x = \omega_0. \end{cases}$$

We have the following theorem.

Theorem 1.1 ([3]). *Let $u, v : I \to \mathbb{R}$ be Hahn-differentiable at $x \in I$. Then*

i) $D_{\omega,q}(uv)(x) = (D_{\omega,q}u(x))v(x) + u(\omega + xq) D_{\omega,q}v(x)$,

ii) $D_{\omega,q}(au + bv)(x) = aD_{\omega,q}u(x) + bD_{\omega,q}v(x), \qquad a, b \in I$,

iii) $D_{\omega,q}(u/v)(x) = (v(x)v(\omega + xq))^{-1}(D_{\omega,q}(u(x))v(x) - u(x)D_{\omega,q}v(x))$,

iv) $D_{\omega,q}u(h^{-1}(x)) = D_{-\omega q^{-1}, q^{-1}}u(x)$,

where $h^{-1}(x) = q^{-1}(x - \omega)$, and $x \in I$.

Definition 1.2 ([3]). *Let $u : I \to \mathbb{R}$ be a function and $a, b, \omega_0 \in I$. The ω, q-integral of the function u is given by*

$$\int_a^b u(x) d_{\omega,q}x := \int_{\omega_0}^b u(x) d_{\omega,q}x - \int_{\omega_0}^a u(x) d_{\omega,q}x,$$

where

$$\int_{\omega_0}^x u(x) d_{\omega,q}x := ((1-q)x - \omega) \sum_{n=0}^{\infty} q^n u\left(\omega \frac{1-q^n}{1-q} + xq^n\right), \qquad x \in I,$$

provided the series converges.

2. Singular Hahn–Hamiltonian system

We consider the following system:

$$\Gamma(\mathcal{Z}) := J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x) = \lambda W(x)\mathcal{Z}(x), \qquad x \in [\omega_0, \infty), \tag{2.1}$$

assuming that λ is a complex spectral parameter, $I + ((q-1)x + \omega) B_2(x)$ is invertible, and $W(\cdot)$ is nonnegative definite.

By $L^2_{\omega,q,W}((\omega_0, \infty); \mathbb{C}^{2n})$ we denote the Hilbert space of all $2n$-dimensional vector-valued functions $\mathcal{Z}$ defined on $[\omega_0, \infty)$ satisfying the condition

$$\int_{\omega_0}^{\infty} (W\mathcal{Z}, \mathcal{Z})_{\mathbb{C}^{2n}} d_{\omega,q}x < \infty$$

with the scalar product

$$(\mathcal{Z}, \mathcal{Y}) := \int_{\omega_0}^{\infty} (W\mathcal{Z}, \mathcal{Y})_{\mathbb{C}^{2n}} d_{\omega,q}x$$

$$= \int_{\omega_0}^{\infty} \mathcal{Y}^*(x)W(x)\mathcal{Z}(x)d_{\omega,q}x.$$

We assume that if $\Gamma(\mathcal{Z}) = WF$ and $W\mathcal{Z} = 0$, then $\mathcal{Z} = 0$. Furthermore, throughout this work, we assume that the following definiteness condition holds: for every nontrivial solution $\mathcal{Z}$ of (2.1), we have

$$\int_{\omega_0}^{\infty} \mathcal{Z}^*(x)W(x)\mathcal{Z}(x)d_{\omega,q}x > 0.$$

We define a maximal operator $L_{\max}$ by the formula $L_{\max}\mathcal{Z} = F$ for all $\mathcal{Z} \in \mathcal{D}_{\max}$, where

$$\mathcal{D}_{\max} := \left\{ \begin{array}{l} \mathcal{Z} \in L^2_{\omega,q,W}((\omega_0,\infty);\mathbb{C}^{2n}) : \mathcal{Z} \text{ is a continuous at } \omega_0, \\ J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x) = W(x)F(x) \text{ is well-defined in } (\omega_0,\infty), \\ F \in L^2_{\omega,q,W}((\omega_0,\infty);\mathbb{C}^{2n}) \end{array} \right\}.$$

The next theorem introduces a Green formula.

Theorem 2.1. *For all functions $\mathcal{U},\mathcal{V} \in D_{\max}$ we have the following relation:*

$$(L_{\max}\mathcal{U},\mathcal{V}) - (\mathcal{U}, L_{\max}\mathcal{V}) = \widehat{\mathcal{V}}^*(t)J\widehat{\mathcal{U}}(t) - \widehat{\mathcal{V}}^*(\omega_0)J\widehat{\mathcal{U}}(\omega_0), \tag{2.2}$$

where $t \in [\omega_0,\infty)$.

Proof. For $\mathcal{U},\mathcal{V} \in D_{\max}$, there exist $F, G \in \mathcal{H}$ such that $L_{\max}\mathcal{U} = F$ and $L_{\max}\mathcal{V} = G$. Then we get

$$\begin{aligned}
(L_{\max}\mathcal{U},\mathcal{V}) - (\mathcal{U}, L_{\max}\mathcal{V}) &= (F,\mathcal{V}) - (\mathcal{U},G) \\
&= \int_{\omega_0}^{t} \mathcal{V}^*(x)W(x)F(x)d_{\omega,q}x - \int_{\omega_0}^{t} G^*(x)W(x)\mathcal{U}(x)d_{\omega,q}x \\
&= \int_{\omega_0}^{t} \mathcal{V}^*(x)\Gamma(\mathcal{U})\,d_{\omega,q}x - \int_{\omega_0}^{t} (\Gamma(\mathcal{V}))^*\,\mathcal{U}(x)d_{\omega,q}x \\
&= \int_{\omega_0}^{t} \mathcal{V}^*(x)\left(J\mathcal{U}^{[h]}(x) + (\lambda W(x) + B(x))\mathcal{U}(x)\right)d_{\omega,q}x \\
&\quad - \int_{\omega_0}^{t} \left(J\mathcal{V}^{[h]}(x) + (\lambda W(x) + B(x))\,\mathcal{V}(x)\right)^*\mathcal{U}(x)d_{\omega,q}x \\
&= \int_{\omega_0}^{t} \mathcal{V}^*(x)J\mathcal{U}^{[h]}(x)d_{\omega,q}x - \int_{\omega_0}^{t} \left(J\mathcal{V}^{[h]}(x)\right)^*\mathcal{U}(x)d_{\omega,q}x \\
&= \int_{\omega_0}^{t} \left(-\frac{1}{q}\mathcal{V}_1^*(x)D_{-\omega q^{-1},q^{-1}}\mathcal{U}_2(x) + \mathcal{V}_2^*(x)D_{\omega,q}\mathcal{U}_1(x)\right)d_{\omega,q}x \\
&\quad - \int_{\omega_0}^{t} \left(\left(-\frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{V}_2^*(x)\right)\mathcal{U}_1(x) + D_{\omega,q}\mathcal{V}_1^*(x)\mathcal{U}_2(x)\right)d_{\omega,q}x
\end{aligned}$$

$$=\int_{\omega_0}^{t}\left(\mathcal{V}_1^*(x)\left(-\frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{U}_2(x)\right)-D_{\omega,q}\mathcal{V}_1^*(x)\mathcal{U}_2(x)\right)d_{\omega,q}x$$

$$+\int_{\omega_0}^{t}\left(\mathcal{V}_2^*(x)D_{\omega,q}\mathcal{U}_1(x)-\left(-\frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{V}_2^*(x)\right)\mathcal{U}_1(x)\right)d_{\omega,q}x.$$

On the other hand,

$$\begin{aligned}D_{\omega,q}\left(\mathcal{V}_1^*(x)\mathcal{U}_2\left(h^{-1}(x)\right)\right)&=\mathcal{V}_1^*(x)D_{\omega,q}\mathcal{U}_2\left(h^{-1}(x)\right)D_{\omega,q}h^{-1}(x)+D_{\omega,q}\mathcal{V}_1^*(x)\mathcal{U}_2(x)\\&=\mathcal{V}_1^*(x)\frac{1}{q}\left(D_{-\omega q^{-1},q^{-1}}\mathcal{U}_2(x)\right)+\left(D_{\omega,q}\mathcal{V}_1(x)\right)^*\mathcal{U}_2(x)\end{aligned}$$

and

$$\begin{aligned}D_{\omega,q}\left(\mathcal{V}_2^*\left(h^{-1}(x)\right)\mathcal{U}_1(x)\right)&=D_{\omega,q}\mathcal{V}_2^*\left(h^{-1}(x)\right)D_{\omega,q}\left(h^{-1}(x)\right)\mathcal{U}_1(x)+\mathcal{V}_2^*(x)D_{\omega,q}\mathcal{U}_1(x)\\&=\frac{1}{q}\left(D_{-\omega q^{-1},q^{-1}}\mathcal{V}_2^*(x)\right)\mathcal{U}_1(x)+\mathcal{V}_2^*(x)D_{\omega,q}\mathcal{U}_1(x).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\omega_0}^{t}\mathcal{V}^*(x)\left(\Gamma\left(\mathcal{U}\right)\right)d_{\omega,q}x-\int_{\omega_0}^{t}\left(\Gamma\left(\mathcal{V}\right)\right)^*\mathcal{U}(x)d_{\omega,q}x&=\int_{\omega_0}^{t}D_{\omega,q}\begin{pmatrix}-\mathcal{V}_1^*(x)\mathcal{U}_2\left(h^{-1}(x)\right)\\+\mathcal{V}_2^*\left(h^{-1}(x)\right)\mathcal{U}_1(x)\end{pmatrix}d_{\omega,q}x\\&=\widehat{\mathcal{V}}^*(t)J\widehat{\mathcal{U}}(t)-\widehat{\mathcal{V}}^*(\omega_0)J\widehat{y}(\omega_0).\end{aligned}$$

The proof is complete. □

Let $\zeta_1,\zeta_2,\gamma_1,\gamma_2$ be matrices satisfying

$$\zeta_1\zeta_1^*+\zeta_2\zeta_2^*=I_n,\qquad \zeta_1\zeta_2^*-\zeta_2\zeta_1^*=0, \tag{2.3}$$

$$\gamma_1\gamma_1^*+\gamma_2\gamma_2^*=I_n,\qquad \gamma_1\gamma_2^*-\gamma_2\gamma_1^*=0, \tag{2.4}$$

and

$$\operatorname{rank}\begin{pmatrix}\zeta_1 & \zeta_2\end{pmatrix}=\operatorname{rank}\begin{pmatrix}\gamma_1 & \gamma_2\end{pmatrix}=n.$$

We impose the following boundary conditions:

$$\Sigma\widehat{\mathcal{Z}}\left(\omega_0\right)=0, \tag{2.5}$$

$$\Xi\widehat{\mathcal{Z}}(a)=0, \tag{2.6}$$

where

$$\Sigma=\begin{pmatrix}\zeta_1 & \zeta_2\\ 0 & 0\end{pmatrix},\qquad \Xi=\begin{pmatrix}0 & 0\\ \gamma_1 & \gamma_2\end{pmatrix},$$

and

$$\widehat{\mathcal{Z}}(x)=\begin{pmatrix}\mathcal{Z}_1(x)\\ \mathcal{Z}_2\left(h^{-1}(x)\right)\end{pmatrix}.$$

It follows from (2.5) that $\Sigma J\Sigma^*=0$ and $\Xi J\Xi^*=0$. It is obvious that (2.1) with conditions (2.5), (2.6) defines a regular self-adjoint problem.

We denote by

$$Z=\begin{pmatrix}\varphi & \psi\end{pmatrix}=\begin{pmatrix}\varphi_1 & \psi_1\\ \varphi_2 & \psi_2\end{pmatrix} \tag{2.7}$$

the fundamental matrix for $\Gamma\left(\mathcal{Z}\right)=\lambda W\mathcal{Z}$ satisfying

$$\widehat{Z}\left(\omega_0\right)=E:=\begin{pmatrix}\zeta_1^* & -\zeta_2^*\\ \zeta_2^* & \zeta_1^*\end{pmatrix}.$$

Thus, $\begin{pmatrix}\zeta_1 & \zeta_2\end{pmatrix}\widehat{\varphi}(\omega_0) = I_n$, and $\begin{pmatrix}\zeta_1 & \zeta_2\end{pmatrix}\widehat{\psi}(\omega_0) = 0$.

Lemma 2.1. *The following relation holds*

$$\widehat{Z}^*(x,\lambda)\,J\widehat{Z}(x,\lambda) = J. \tag{2.8}$$

Proof. From Theorem 2.1, we see that

$$\begin{aligned} 0 &= \int_{\omega_0}^{x} Z^*(t,\lambda)\,\Gamma(Z(t,\lambda))d_{\omega,q}t - \int_{\omega_0}^{x} \Gamma(Z^*(t,\lambda)Z(t,\lambda)d_{\omega,q}t \\ &= \widehat{Z}^*(x,\lambda)\,J\widehat{Z}(x,\lambda) - \widehat{Z}^*(\omega_0,\lambda)\,J\widehat{Z}(\omega_0,\lambda). \end{aligned}$$

Thus,

$$\widehat{Z}^*(x,\lambda)\,J\widehat{Z}(x,\lambda) = \widehat{Z}^*(\omega_0,\lambda)\,J\widehat{Z}(\omega_0,\lambda).$$

Since $\widehat{Z}(\omega_0,\lambda) = E$, we obtain

$$\widehat{Z}^*(x,\lambda)\,J\widehat{Z}(x,\lambda) = J.$$

The proof is complete. □

3. The Titchmarsh–Weyl function

In this section, we construct the Titchmarsh–Weyl function $M(\lambda)$ for system (2.1), (2.5).

Definition 3.1. *Let*

$$\widehat{Y}_a(x,\lambda) = \widehat{Z}(x,\lambda)\begin{pmatrix} I_n \\ M(a,\lambda)\end{pmatrix},$$

where $\operatorname{Im}\lambda \neq 0$ *and* $M(a,\lambda)$ *is a* $n\times n$ *matrix-valued function. Then* $M(a,\lambda)$ *is called the Titchmarsh–Weyl function for boundary value problem (2.1), (2.5), (2.6).*

The following theorem holds true.

Theorem 3.1. *Let*

$$\begin{pmatrix}\gamma_1 & \gamma_2\end{pmatrix}\widehat{Y}_a(a,\lambda) = 0. \tag{3.1}$$

Then

$$M(a,\lambda) = -\left(\gamma_1\psi_1(a) + \gamma_2\psi_2\left(h^{-1}(a)\right)\right)^{-1}\left(\gamma_1\varphi_1(a) + \gamma_2\varphi_2\left(h^{-1}(a)\right)\right),$$

and

$$\widehat{Y}_a^*(a,\lambda)\,J\widehat{Y}_a(a,\lambda) = 0,$$

where γ_1 *and* γ_2 *are defined in (2.4). And vice versa, if* $\widehat{Y}_a$ *satisfies*

$$\widehat{Y}_a^*(a,\lambda)\,J\widehat{Y}_a(a,\lambda) = 0,$$

then there exists γ_1,γ_2 *satisfying (2.4) such that*

$$\begin{pmatrix}\gamma_1 & \gamma_2\end{pmatrix}\widehat{Y}_a(a,\lambda) = 0,$$

and

$$M(a,\lambda) = -\left(\gamma_1\psi_1(a) + \gamma_2\psi_2\left(h^{-1}(a)\right)\right)^{-1}\left(\gamma_1\varphi_1(a) + \gamma_2\varphi_2\left(h^{-1}(a)\right)\right).$$

Proof. Let $\begin{pmatrix}\gamma_1 & \gamma_2\end{pmatrix}\widehat{Y}_a(a,\lambda) = 0$. Then we get

$$\left[\gamma_1\psi_1(a) + \gamma_2\psi_2\left(h^{-1}(a)\right)\right]M(a,\lambda) = -\left(\gamma_1\varphi_1(a) + \gamma_2\varphi_2\left(h^{-1}(a)\right)\right),$$

and

$$M(a,\lambda) = -\left(\gamma_1\psi_1(a) + \gamma_2\psi_2\left(h^{-1}(a)\right)\right)^{-1}\left(\gamma_1\varphi_1(a) + \gamma_2\varphi_2\left(h^{-1}(a)\right)\right).$$

Since λ is not an eigenvalue of the self-adjoint problem on $[\omega_0, a]$, the inverse of the matrix $\gamma_1\psi_1(a) + \gamma_2\psi_2(h^{-1}(a))$ exists. By (3.1), we see that

$$\widehat{Y}_a(a,\lambda) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \gamma_1^* \\ \gamma_2^* \end{pmatrix} K$$

for

$$\begin{pmatrix} \gamma_1 & \gamma_2 \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \gamma_1^* \\ \gamma_2^* \end{pmatrix} K = 0.$$

Hence,

$$\begin{pmatrix} I_n & M^*(a,\lambda) \end{pmatrix} \widehat{Z}^*(a,\lambda) J \widehat{Z}(a,\lambda) \begin{pmatrix} I_n \\ M(a,\lambda) \end{pmatrix} = 0,$$

that is, $\widehat{Y}_a^*(a,\lambda) J \widehat{Y}_a(a,\lambda) = 0$.

Vice versa, for some M we let

$$\widehat{Y}_a^*(a,\lambda) J \widehat{Y}_a(a,\lambda) = \begin{pmatrix} I_n & M^*(a,\lambda) \end{pmatrix} \widehat{Z}^*(a,\lambda) J \widehat{Z}(a,\lambda) \begin{pmatrix} I_n \\ M(a,\lambda) \end{pmatrix} = 0.$$

We let

$$\begin{pmatrix} \gamma_1 & \gamma_2 \end{pmatrix} = \begin{pmatrix} I_n & M^*(a,\lambda) \end{pmatrix} \widehat{Z}^*(a,\lambda) J$$

and we get the desired results. The proof is complete. □

We introduce Titchmarsh–Weyl circles.

Definition 3.2. *Let*

$$\mathcal{C}(a,\lambda) = \begin{pmatrix} I_n & M^*(a,\lambda) \end{pmatrix} \begin{pmatrix} \Theta_1 & \Theta_2^* \\ \Theta_2 & \Theta_3 \end{pmatrix} \begin{pmatrix} I_n \\ M(a,\lambda) \end{pmatrix} = 0, \tag{3.2}$$

where Θ_m are $n \times n$ matrices for $m = 1, 2, 3$ and

$$\begin{pmatrix} \Theta_1 & \Theta_2^* \\ \Theta_2 & \Theta_3 \end{pmatrix} = -\operatorname{sgn}(\operatorname{Im}\lambda)\, \widehat{Z}^*\left(a,\overline{\lambda}\right)(J/i)\, \widehat{Z}(a,\lambda). \tag{3.3}$$

Then $\mathcal{C}(a,\lambda)$ is called the Titchmarsh–Weyl circle for boundary value problem (2.1), (2.5), (2.6).

From the above definition we deduce that

$$\begin{aligned} \mathcal{C}(a,\lambda) &= \left(M_a + \Theta_3^{-1}\Theta_2\right)^* \Theta_4 \left(M_a + \Theta_3^{-1}\Theta_2\right) + \Theta_1 - \Theta_2^*\Theta_3^{-1}\Theta_2 \\ &= (M_a - \Theta_4) K_1^{-2} (M_a - \Theta_4) - K_2^2 = 0, \end{aligned}$$

where

$$\Theta_4 = -\Theta_3^{-1}\Theta_2, \qquad K_1^{-2} = \Theta_3^{-1}, \qquad K_2^2 = \Theta_2^*\Theta_3^{-1}\Theta_2 - \Theta_1.$$

Lemma 3.1. *The inequality $\Theta_3 > 0$ holds true.*

Proof. From (2.7) and (3.3) we see that

$$\begin{aligned} \begin{pmatrix} \Theta_1 & \Theta_2^* \\ \Theta_2 & \Theta_3 \end{pmatrix} &= -\operatorname{sgn}(\operatorname{Im}\lambda) \begin{pmatrix} \varphi_1^*(x) & \varphi_2^*(h^{-1}(x)) \\ \psi_1^*(x) & \psi_2^*(h^{-1}(x)) \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} 0 & iI_n \\ -iI_n & 0 \end{pmatrix} \begin{pmatrix} \varphi_1(x) & \psi_1(x) \\ \varphi_2(h^{-1}(x)) & \psi_2(h^{-1}(x)) \end{pmatrix} \\ &= -\operatorname{sgn}(\operatorname{Im}\lambda) \begin{pmatrix} \widehat{\varphi}^*(J/i)\,\widehat{\varphi} & \widehat{\varphi}^*(J/i)\,\widehat{\psi} \\ i\widehat{\psi}^*(J/i)\,\widehat{\varphi} & \widehat{\psi}^*(J/i)\,\widehat{\psi} \end{pmatrix}. \end{aligned}$$

Hence,

$$\Theta_3 = -\operatorname{sgn}(\operatorname{Im}\lambda)\, \widehat{\psi}^*(J/i)\, \widehat{\psi}.$$

Straightforward calculations give:

$$2\operatorname{Im}\lambda\left(\int_{\omega_0}^{a}\psi^*W\psi d_{\omega,q}x\right)=\widehat{\psi}^*\,(J/i)\,\widehat{\psi}(a)-\widehat{\psi}^*\,(J/i)\,\widehat{\psi}\,(\omega_0)\,.$$

Since $\widehat{\psi}^*\,(J/i)\,\widehat{\psi}\,(\omega_0)=0$, we get the desired result. □

Lemma 3.2. *The inequality*

$$\Theta_2^*\Theta_3^{-1}\Theta_2-\Theta_1=\overline{\Theta_3}^{-1}>0$$

holds, where $\overline{\Theta_3}^{-1}:=\Theta_3^{-1}\left(\overline{\lambda}\right).$

Proof. It follows from (2.8) that $\widehat{Z}\,(x,\lambda)\,J\widehat{Z}^*\,(x,\lambda)=J$. Thus,

$$\begin{aligned}J=&\widehat{Z}^*\left(x,\overline{\lambda}\right)\left(-J\widehat{Z}\,(x,\lambda)\,J\widehat{Z}^*\,(x,\lambda)\,J\right)\widehat{Z}\left(x,\overline{\lambda}\right)\\=&-\left(\widehat{Z}^*\left(x,\overline{\lambda}\right)(J/i)\,\widehat{Z}\,(x,\lambda)\right)J\left(-\widehat{Z}^*\,(x,\lambda)\,(J/i)\,\widehat{Z}\left(x,\overline{\lambda}\right)\right),\end{aligned}$$

and

$$\begin{pmatrix}0&-I_n\\I_n&0\end{pmatrix}=-\begin{pmatrix}\Theta_1&\Theta_2^*\\\Theta_2&\Theta_3\end{pmatrix}\begin{pmatrix}0&-I_n\\I_n&0\end{pmatrix}\begin{pmatrix}\overline{\Theta_1}&\overline{\Theta_2}^*\\\overline{\Theta_2}&\overline{\Theta_3}\end{pmatrix},$$

since there is a sign change in the matrix when λ replaces $\overline{\lambda}$. Therefore,

$$\begin{aligned}0&=\Theta_1\overline{\Theta_2}-\Theta_2^*\overline{\Theta_1}, &\qquad -I_n&=\Theta_1\overline{\Theta_3}-\Theta_2^*\overline{\Theta_2},\\I_n&=\Theta_2\overline{\Theta_2}-\Theta_3\overline{\Theta_1}, &\qquad 0&=\Theta_2\overline{\Theta_3}-\Theta_3\overline{\Theta_2^*}.\end{aligned}$$

The last and second identities imply that

$$\overline{\Theta_3}^{-1}=\Theta_2^*\Theta_3^{-1}\Theta_2-\Theta_1.$$

This completes the proof. □

Corollary 3.1. $K_2=\overline{K_1}$

Theorem 3.2. *As* a *increases,* Θ_3, K_1 *and* K_2 *decrease.*

Proof. Since

$$\Theta_3=2\left|\operatorname{Im}\lambda\right|\left(\int_{\omega_0}^{a}\psi^*W\psi d_{\omega,q}x\right),$$

we get the desired results. □

Corollary 3.2. *The following limits exist*

$$\lim_{a\to\infty}K_1\,(a,\lambda)=K_0,\qquad \lim_{a\to\infty}K_2\,(a,\lambda)=\overline{K_0},$$

where $K_0\geqslant 0$ *and* $\overline{K_0}\geqslant 0$.

Theorem 3.3. *As* $a\to\infty$, *the circles* $\mathcal{C}\,(a,\lambda)=0$ *are embedded.*

Proof. The interior of the circle is

$$-\operatorname{sgn}\left(\operatorname{Im}\lambda\right)\begin{pmatrix}I_n&M^*\,(a,\lambda)\end{pmatrix}\widehat{Z}^*\left(a,\overline{\lambda}\right)(J/i)\,\widehat{Z}\,(a,\lambda)\begin{pmatrix}I_n\\M\,(a,\lambda)\end{pmatrix}\leqslant 0.$$

By (3.2) we see that

$$\mathcal{C}\,(a,\lambda)=2\left|\operatorname{Im}\lambda\right|\left(\int_{\omega_0}^{a}Y_a^*WY_ad_{\omega,q}x\right)\pm\frac{1}{i}\left(M_a^*-M_a\right).$$

If M_a is in the circle at $a_2 \in I,\ a_2 > a$, then $\mathcal{C}(a,\lambda) \leqslant 0$ at the point a_2. At the point a_2, $\mathcal{C}(a,\lambda)$ is certainly smaller, and so $\mathcal{C}(a,\lambda)$ is in the circle at the point a_2 as well. Hence, the circles $\mathcal{C}(a,\lambda) = 0$ are embedded as $a \to \infty$. □

Theorem 3.4. *The following limit exists*

$$\lim_{a\to\infty} \mathcal{C}(a,\lambda) = \mathcal{C}^0.$$

Proof. From (3.2), we conclude that

$$\mathcal{C}(a,\lambda) = (M_a - D)^* K_1^{-2} (M_a - D) - K_2^2 = 0.$$

Therefore,

$$\left(K_1^{-1}(M_a - D)\overline{K_1^{-1}}\right)^* \left(K_1^{-1}(M_a - D)\overline{K_1^{-1}}\right) = I_n. \tag{3.4}$$

It follows from (3.4) that $U = K_1^{-1}(M_a - D)\overline{K_1^{-1}}$, where U is a unitary matrix, i.e., $U^*U = I_n$. Thus,

$$M_a(\lambda) = D + K_1 U \overline{K_1}. \tag{3.5}$$

As U ranges over the $n \times n$ unit sphere, $M_a(\lambda)$ ranges over a circle with center D.

Let D_1 be the center at $a' \in I$, D_2 be the center at $a'' \in I,\ a'' < a'$. By Theorem 3.7, we see that $\mathcal{C}(a'',\lambda) \subset \mathcal{C}(a',\lambda)$. By (3.5) we find that

$$M_{a'}(\lambda) = D_1 + K_1(a')U_1\overline{K_1(a')},$$

and

$$M_{a''}(\lambda) = D_2 + K_1(a'')U_2\overline{K_1(a'')}. \tag{3.6}$$

Since $\mathcal{C}(a'',\lambda) \subset \mathcal{C}(a',\lambda)$, we conclude that

$$M_{a''}(\lambda) = D_1 + K_1(a')V_1\overline{K_1(a')}, \tag{3.7}$$

where V_1 is a contraction. Subtracting (3.6) from (3.7) yields

$$D_1 - D_2 = K_1(a'')U_2\overline{K_1(a'')} - K_1(a')V_1\overline{K_1(a')}.$$

This gives:

$$V_1 = \left[D_1 - D_2 + K_1(a')V_1\overline{K_1(a')}\right].$$

We define a mapping Υ by the formula $\Upsilon(U_2) = V_1$. The mapping Υ is a continuous one from the unit ball into itself. Hence, it has a unique fixed point. Replacing U_2 and V_1 by U, we conclude that

$$\|D_1 - D_2\| = \left\|K_1(a'')U\overline{K_1(a'')} - K_1(a')U\overline{K_1(a')}\right\|$$

$$\leqslant \|K_1(a'')\| \left\|\overline{K_1(a'')} - \overline{K_1(a')}\right\| + \|K_1(a'') - K_1(a')\| \left\|\overline{K_1(a')}\right\|.$$

As a' and a'' approach a, K_1 and $\overline{K_1}$ have limits. The centers form a Cauchy sequence and converge.

Straightforward calculations give:

$$\Theta_2 = \pm\left(2\operatorname{Im}\lambda\left(\int_{\omega_0}^{a'} \psi^* W \varphi d_{\omega,q}x\right) - iI_n\right).$$

Thus, at a', the center

$$D = -\Theta_3^{-1}\Theta_2$$

$$= -\left(2\operatorname{Im}\lambda\left(\int\limits_{\omega_0}^{a'}\psi^* W_1 \psi d_{\omega,q}x\right)\right)^{-1}\left(2\operatorname{Im}\lambda\left(\int\limits_{\omega_0}^{a'}\psi^* W_1 \psi d_{\omega,q}x\right) - iI_n\right).$$

Hence, we obtain

$$\lim_{a'\to\infty} \mathcal{C}\left(a',\lambda\right) = \mathcal{C}^0.$$

The proof is complete. □

It is obvious that $M(\lambda) = D + K_1 U \overline{K_1}$ is well defined. As U ranges over the unit circle in $n \times n$ space, the limit circle or point $\mathcal{C}$ is covered.

Now we investigate the number of square-integrable solutions to (2.1).

Theorem 3.5. *Let M be a point inside $\mathcal{C}^0 \leqslant 0$. Let $\chi = \varphi + \psi M$. Then*

$$\chi \in L^2_{\omega,q,W}((\omega_0,\infty);\mathbb{C}^{2n}).$$

Proof. Since

$$\mathcal{C}\left(a,\lambda\right) = 2\left|\operatorname{Im}\lambda\right|\left(\int\limits_{\omega_0}^{a}\chi^* W \chi d_{\omega,q}x\right) \pm \frac{1}{i}\left[M - M^*\right] \leqslant 0,$$

we obtain

$$0 \leqslant \int\limits_{\omega_0}^{a}\chi^* W \chi d_{\omega,q}x \leqslant \frac{1}{2i\left|\operatorname{Im}\lambda\right|}\left[M - M^*\right].$$

As $a \to \infty$, the upper bound is fixed. The proof is complete. □

Lemma 3.3. *Let* $\operatorname{rank}\overline{K_1} = r$ *and* $S(U) = K_1 U \overline{K_1}$, *where U is unitary. Then we have the following relations:*
i) $\operatorname{rank} S(U) \leqslant r$,

ii) $\sup_U \operatorname{rank} S(U) = r$.

The proof follows clearly from the matrix theory.

Theorem 3.6. *Let $m = n + r$. For* $\operatorname{Im}\lambda \neq 0$, *there exists at least m square integrable solutions of (2.1), $n \leqslant m \leqslant 2n$.*

Proof. $\varphi + D\psi$ consists of n solutions in the space $L^2_{q,W}((\omega_0,a);\mathbb{C}^{2n})$. As U varies, $\psi\left(K_1 U \overline{K_1}\right)$ gives $m - n$ additional linearly independent solutions. By the reflection principles, the number of solutions is the same for $\operatorname{Im}\lambda < 0$ or $\operatorname{Im}\lambda > 0$. This completes the proof. □

4. Boundary conditions in singular case

Theorem 4.1. *Let $\mathcal{Y}$ be a solution of the equation*

$$J\mathcal{Y}^{[h]}(x) = \left(\lambda_0 W + B\right)\mathcal{Y},$$

where $\operatorname{Im}\lambda_0 \neq 0$. *Then for all $\mathcal{Z} \in \mathcal{D}_{\max}$, the following limit*

$$A\left(\mathcal{Z}\right) = \lim_{x\to\infty} \widehat{\mathcal{Y}}^* J \widehat{\mathcal{Z}}$$

exists if and only if $\mathcal{Y} \in L^2_{\omega,q,W}((\omega_0,\infty);\mathbb{C}^{2n})$.

Proof. From the following equalities

$$J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x) = W(x)F(x),$$

and

$$J\mathcal{Y}^{[h]}(x) - B(x)\mathcal{Y}(x) = \lambda_0 W(x)\mathcal{Y}(x),$$

we obtain

$$\int_{\omega_0}^{x} \mathcal{Y}^*(x)W(x)\big(F(x) - \lambda_0\mathcal{Z}(x)\big)d_{\omega,q}x = \int_{\omega_0}^{x} \begin{pmatrix} \mathcal{Y}^*(x)\left(J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x)\right) \\ -\left(J\mathcal{Y}^{[h]}(x) - B(x)\mathcal{Y}(x)\right)^*\mathcal{Z}(x) \end{pmatrix} d_{\omega,q}x$$

$$= \int_{\omega_0}^{x} \mathcal{Y}^*(x)J\mathcal{Z}^{[h]}(x)d_{\omega,q}x - \int_{\omega_0}^{x} \left(J\mathcal{Y}^{[h]}(x)\right)^*\mathcal{Z}(x)d_{\omega,q}x$$

$$= \int_{\omega_0}^{x} \left(\mathcal{Y}_1^*(x)\left(-\frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{Z}_2(x)\right) + \mathcal{Y}_2^*(x)D_{\omega,q}\mathcal{Z}_1(x)\right)d_{\omega,q}x$$

$$- \int_{\omega_0}^{x} \left(\left(-\frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{Y}_2^*(x)\right)\mathcal{Z}_1(x) + D_{\omega,q}\mathcal{Y}_1^*(x)\mathcal{Z}_2(x)\right)d_{\omega,q}x$$

$$= \int_{\omega_0}^{x} \left(\mathcal{Y}_1^*(x)[\left(-\frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{Z}_2(x)\right) - D_{\omega,q}\mathcal{Y}_1^*(x)\mathcal{Z}_2(x)\right)d_{\omega,q}x$$

$$+ \int_{\omega_0}^{x} \left(\mathcal{Y}_2^*(x)D_{\omega,q}\mathcal{Z}_1(x) - \left(-\frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{Y}_2^*(x)\right)\mathcal{Z}_1(x)\right)d_{\omega,q}x.$$

Since

$$\begin{aligned} D_{\omega,q}\left(\mathcal{Y}_1^*(x)\mathcal{Z}_2\left(h^{-1}(x)\right)\right) &= \mathcal{Y}_1^*(x)D_{\omega,q}\mathcal{Z}_2\left(h^{-1}(x)\right)D_{\omega,q}\left(h^{-1}(x)\right) + D_{\omega,q}\mathcal{Y}_1^*(x)\mathcal{Z}_2(x) \\ &= \mathcal{Y}_1^*(x)\left(\frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{Z}_2(x)\right) + D_{\omega,q}\mathcal{Y}_1^*(x)\mathcal{Z}_2(x) \end{aligned}$$

and

$$\begin{aligned} D_{\omega,q}\left(\mathcal{Y}_2^*\left(h^{-1}(x)\right)\mathcal{Z}_1(x)\right) &= (D_{\omega,q}\mathcal{Y}_2^*\left(h^{-1}(x)\right)D_{\omega,q}\left(h^{-1}(x)\right)\mathcal{Z}_1(x) + \mathcal{Y}_2^*(x)D_{\omega,q}\mathcal{Z}_1(x) \\ &= \left(\frac{1}{q}D_{-\omega q^{-1},q^{-1}}\mathcal{Y}_2^*(x)\right)\mathcal{Z}_1(x) + \mathcal{Y}_2^*(x)D_{\omega,q}\mathcal{Z}_1(x). \end{aligned}$$

Hence,

$$\begin{aligned} &\int_{\omega_0}^{x} \mathcal{Y}^*(x)W(x)\left(F(x) - \lambda_0\mathcal{Z}(x)\right)d_{\omega,q}x \\ &= \int_{\omega_0}^{x} D_{\omega,q}\left\{\mathcal{Y}_2^*\left(h^{-1}(x)\right)\mathcal{Z}_1(x) - \mathcal{Y}_1^*(x)\mathcal{Z}_2\left(h^{-1}(x)\right)\right\}d_{\omega,q}x \\ &= \widehat{\mathcal{Y}}^*J\widehat{\mathcal{Z}}(x) - \widehat{\mathcal{Y}}^*J\widehat{\mathcal{Z}}\left(\omega_0\right). \end{aligned} \tag{4.1}$$

If $\mathcal{Y} \in L^2_{\omega,q,W}((\omega_0,\infty);\mathbb{C}^{2n})$, then as $x \to \infty$, the integral in (4.1) converges, and the limit

$$\lim_{x\to\infty}(\widehat{\mathcal{Y}}^*J\widehat{\mathcal{Z}})(x)$$

exists. And vice versa, suppose that the integral in (4.1) converges for all

$$\mathcal{Z}, F \in L^2_{\omega,q,W}((\omega_0,\infty);\mathbb{C}^{2n}).$$

By the Hahn–Banach theorem on existence of a linear bounded functional and the Riesz representation theorem, we see that

$$\mathcal{Y} \in L^2_{\omega,q,W}((\omega_0,\infty);\mathbb{C}^{2n}).$$

The proof is complete. □

Suppose that λ_0 is fixed, where $\operatorname{Im}\lambda_0 \neq 0$.

Definition 4.1. *Let*

$$M_a\left(\overline{\lambda}\right) = \overline{D} + \overline{K_1} U K_1$$

be on the limit circle. Let

$$\chi\left(x,\overline{\lambda_0}\right) = \varphi\left(x,\overline{\lambda_0}\right) + \psi\left(x,\overline{\lambda_0}\right) M\left(\overline{\lambda_0}\right) \in L^2_{\omega,q,W}((\omega_0,\infty);\mathbb{C}^{2n})$$

and let $\chi\left(x,\overline{\lambda_0}\right)$ satisfies the equation

$$J\mathcal{Z}^{[h]}(x) = \left(\lambda_0 W(x) + B(x)\right)\mathcal{Z}(x).$$

Then we define $S_{\lambda_0}(\mathcal{Z})$ by the formula

$$S_{\lambda_0}(\mathcal{Z}) = \lim_{x\to\infty} \widehat{\chi}^*\left(x,\lambda_0\right) J\widehat{\mathcal{Z}}(x)$$

for all $\mathcal{Z} \in \mathcal{D}_{\max}$.

5. Self-adjoint operator

Here we define a self-adjoint operator. We suppose that the number of solutions of (2.1) is m. Then we define the operator L by the rule

$$\begin{aligned} &L : \mathcal{D} \to L^2_{\omega,q,W}((\omega_0,\infty);\mathbb{C}^{2n}),\\ &\mathcal{Z} \to L\mathcal{Z} = F \quad \text{if and only if} \quad \Gamma(\mathcal{Z}) = WF, \end{aligned}$$

where

$$\mathcal{D} := \left\{\mathcal{Z} \in \mathcal{D}_{\max} : \Sigma\widehat{\mathcal{Z}}(\omega_0) = 0 \quad \text{and} \quad S_{\lambda_0}(\mathcal{Z}) = 0,\ \operatorname{Im}\lambda_0 \neq 0\right\}.$$

The following theorem holds true.

Theorem 5.1. *If $J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x) = W(x)F(x)$, $W\mathcal{Z} = 0$ implies $\mathcal{Z} = 0$, then the set $\mathcal{D}$ is dense in $L^2_{\omega,q,W}((\omega_0,\infty);\mathbb{C}^{2n})$.*

Proof. Suppose that $\mathcal{D}$ is not dense in $L^2_{\omega,q,W}((\omega_0,\infty);\mathbb{C}^{2n})$. Then there exists

$$G \in L^2_{\omega,q,W}((\omega_0,\infty);\mathbb{C}^{2n})$$

such that G is orthogonal to the set $\mathcal{D}$. Let $\mathcal{Y}$ satisfy $\mathcal{Y} \in \mathcal{D}$,

$$J\mathcal{Y}^{[h]}(x) - B(x)\mathcal{Y}(x) = \overline{\lambda_0}W(x)\mathcal{Y}(x) + W(x)G(x)$$

for $\operatorname{Im}\lambda_0 \neq 0$. Then for $\mathcal{Z} \in \mathcal{D}$, we see that

$$\begin{aligned} 0 = (\mathcal{Z}, G) &= \int_{\omega_0}^{\infty} G^* W \mathcal{Z} d_{\omega,q} x \\ &= \int_{\omega_0}^{\infty} \left(J\mathcal{Y}^{[h]}(x) - B(x)\mathcal{Y}(x) - \overline{\lambda_0}W(x)\mathcal{Y}(x)\right)^* \mathcal{Z} d_{\omega,q} x \end{aligned}$$

$$= \int_{\omega_0}^{\infty} \mathcal{Y}^* \left(J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x) - \lambda_0 W(x)\mathcal{Z}(x) \right) d_{\omega,q}x.$$

We define

$$J\mathcal{Z}^{[h]}(x) - B(x)\mathcal{Z}(x) - \lambda_0 W(x)\mathcal{Z}(x) = W(x)F(x).$$

Then we have

$$0 = (F, \mathcal{Y}) = \int_{\omega_0}^{\infty} \mathcal{Y}^* W F d_{\omega,q}x. \tag{5.1}$$

Since F is arbitrary, we take $F = \mathcal{Y}$. By (5.1), we see that $\mathcal{Y} = 0$ which yields $WG = 0$ and $G = 0$ in $L^2_{\omega,q,W}((\omega_0, \infty); \mathbb{C}^{2n})$. The proof is complete. □

Define

$$(L - \lambda I)^{-1} = \int_{\omega_0}^{\infty} G(\lambda, x, t) W(t) F(t) d_{\omega,q}t, \tag{5.2}$$

where $\operatorname{Im} \lambda \neq 0$ and

$$G(\lambda, x, t) = \begin{cases} \chi(x, \lambda) \psi^*(t, \lambda), & \omega_0 \leqslant t \leqslant x < \infty, \\ \psi(x, \lambda) \chi^*(t, \lambda), & \omega_0 \leqslant x \leqslant t < \infty. \end{cases}$$

The following theorem holds.

Theorem 5.2. *L is a self-adjoint operator.*

Proof. Let $L\mathcal{Z} - \lambda_0 \mathcal{Z} = F$ and $L^*\mathcal{Z} - \overline{\lambda_0}\mathcal{Z} = H$ ($\operatorname{Im} \lambda_0 \neq 0$). Then

$$\begin{aligned} \left((L - \lambda_0 I)^{-1} F, H \right) &= \int_{\omega_0}^{\infty} H^*(x) W(x) \left(\int_{\omega_0}^{\infty} G(\lambda_0, x, t) W(t) F(t) d_{\omega,q}t \right) d_{\omega,q}x \\ &= \int_{\omega_0}^{\infty} \left(\int_{\omega_0}^{\infty} (G(\lambda_0, x, t))^* W(x) H(x) d_{\omega,q}x \right)^* W(t) F(t) d_{\omega,q}t \\ &= \int_{\omega_0}^{\infty} \left(\int_{\omega_0}^{\infty} (G(\overline{\lambda_0}, t, x) W(x) H(x) d_{\omega,q}x \right)^* W(t) F(t) d_{\omega,q}t \\ &= \int_{\omega_0}^{\infty} \left(\int_{\omega_0}^{\infty} G(\overline{\lambda_0}, x, t) W(t) H(t)\, d_{\omega,q}t \right)^* W(x) F(x) d_{\omega,q}x \\ &= \left(F, \left(L - \overline{\lambda_0} I \right)^{-1} H \right), \end{aligned}$$

due to $G\left(\overline{\lambda_0}, t, x\right) = (G(\lambda_0, x, t))^*$.

Since

$$\left((L - \lambda_0 I)^{-1} F, H \right) = \left(F, \left(L^* - \overline{\lambda_0} I \right)^{-1} H \right),$$

we see that

$$\left(L - \overline{\lambda_0} I \right)^{-1} = \left(L^* - \overline{\lambda_0} I \right)^{-1}.$$

We thus get $L = L^*$. The proof is complete. □

Theorem 5.3. *Let* $\operatorname{Im}\lambda_0 \neq 0$. *The operator* $(L-\lambda_0 I)^{-1}$ *defined by the formula (5.2) is a bounded operator and*

$$\left\|(L-\lambda_0 I)^{-1}\right\| \leqslant \frac{1}{|\operatorname{Im}\lambda_0|}.$$

Proof. Let $(L-\lambda_0 I)\,\mathcal{Z} = F$. Then

$$\begin{aligned}(\mathcal{Z},F)-(F,\mathcal{Z}) &= (\mathcal{Z},(L-\lambda_0 I)\,\mathcal{Z}) - ((L-\lambda_0 I)\,\mathcal{Z},\mathcal{Z}) \\ &= \left(\lambda_0-\overline{\lambda_0}\right)(\mathcal{Z},\mathcal{Z}).\end{aligned}$$

Using Cauchy-Schwartz inequality, we obtain

$$2\,|\operatorname{Im}\lambda_0|\,\|\mathcal{Z}\|^2 \leqslant 2\,\|\mathcal{Z}\|\,\|F\|.$$

Hence,

$$\left\|(L-\lambda_0 I)^{-1}F\right\| \leqslant \frac{1}{|\operatorname{Im}\lambda_0|}\,\|F\|$$

yields the result. □

Theorem 5.4. *Let*

$$\chi(x,\lambda_0) = \varphi(x,\lambda_0) + \psi(x,\lambda_0)\,M(\lambda_0),$$

where $\operatorname{Im}\lambda_0 \neq 0$. *Then we have*

$$\lim_{x\to\infty} \widehat{\chi}^*(x,\lambda_0)\,J\widehat{\chi}(x,\lambda_0) = 0.$$

Proof. Since

$$\begin{aligned}\widehat{\chi}^*(x,\lambda_0)\,J\widehat{\chi}(x,\lambda_0) &= \begin{pmatrix} I_n & M^*(\lambda_0)\end{pmatrix} \widehat{\mathcal{Z}}^*(x,\lambda_0)\,J\widehat{\mathcal{Z}}(x,\lambda_0)\begin{pmatrix} I_n \\ M(\lambda_0)\end{pmatrix} \\ &= \begin{pmatrix} I_n & M^*(\lambda_0)\end{pmatrix} J \begin{pmatrix} I_n \\ M(\lambda_0)\end{pmatrix} = 0,\end{aligned}$$

we get the desired result. The proof is complete. □

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