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ON TAYLOR COEFFICIENTS OF ANALYTIC FUNCTION RELATED WITH EULER NUMBER

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Abstract. We consider a classical construction of second remarkable limit. We pose a question on asymptotically sharp description of the character of such approximation of the number e. In view of this we need the information on behavior of the coefficients in the power expansion for the function $f(x) = e^{-1}(1+x)^{1/x}$ converging in the interval $-1 < x < 1$. We obtain a recurrent rule regulating the forming of the mentioned coefficients. We show that the coefficients form a sign-alternating sequence of rational numbers $(-1)^n a_n$, where $n \in \mathbb{N} \cup \{0\}$ and $a_0 = 1$, the absolute values of which strictly decay. On the base of the Faá di Bruno formula for the derivatives of a composed function we propose a combinatorial way of calculating the numbers a_n as $n \in \mathbb{N}$. The original function $f(x)$ is the restriction of the function $f(z)$ on the real ray $x > -1$ having the same Taylor coefficients and being analytic in the complex plane $\mathbb{C}$ with the cut along $(-\infty, -1]$. By the methods of the complex analysis we obtain an integral representation for a_n for each value of the parameter $n \in \mathbb{N}$. We prove that $a_n \to 1/e$ as $n \to \infty$ and find the convergence rate of the difference $a_n - 1/e$ to zero. We also discuss the issue on choosing the contour in the integral Cauchy formula for calculating the Taylor coefficients $(-1)^n a_n$ of the function $f(z)$. We find the exact values of arising in calculations special improper integrals. The results of the made study allows us to give a series of general two-sided estimates for the deviation $e-(1+x)^{1/x}$ consistent with the asymptotics of $f(x)$ as $x \to 0$. We discuss the possibilities of applying the obtained statements.

Keywords: Euler number, analytic function, Taylor coefficients, Faà di Bruno formula, integral representation, asymptotic behavior.

Mathematics Subject Classification: 30B10

1. Introduction

In the standard course of the mathematical analysis a base relation

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e \tag{1.1}$$

is proved, which forms a usual face of the theory of limits. The convergence rate of such approximation is discussed much less; we mention famous problem books [1, Part I, Ch. 4, Sect. 2, Probls. 170, 171], [2, Probls. 2.16, 2.17] and a recent paper [3]. Due to clear reasons the issue on the convergence rate of the number e by means (1.1) is of a natural interest. At the same time, we failed trying to find in the literature a complete description of the approximation picture including, for instance, asymptotic formulae for the deviation $e-(1+x)^{1/x}$ as $x \to 0$ supported by qualitative two-sided estimates. A small study made by the authors showed that the situation is rather curious. In this note we discuss some problems arising here.

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Formula (1.1) remains true while passing to the complex values of the variable. Indeed, using the standard notation for the principal branch of the logarithm

$$\ln\zeta = \ln|\zeta| + i\arg\zeta, \qquad \zeta \in \mathbb{C}\setminus(-\infty,\,-0], \quad \arg\zeta \in (-\pi,\,\pi),$$

we obtain that the function

$$f(z) \equiv \exp\left\{\frac{\ln(1+z)}{z} - 1\right\}, \tag{1.2}$$

for real $z = x > -1$ obviously coincides with the function $e^{-1}(1+x)^{1/x}$ and the limiting relation holds:

$$\lim_{z\to 0} f(z) = 1. \tag{1.3}$$

Formula (1.2) with the convention $f(0) = 1$ defines an analytic in the domain $D \equiv \mathbb{C}\setminus(-\infty,\,-1]$ function being a superposition of an entire function

$$\exp w = \sum_{n=0}^{\infty} \frac{w^n}{n!}, \qquad w \in \mathbb{C},$$

and an analytic in the domain D function

$$g(z) \equiv \frac{\ln(1+z)}{z} - 1.$$

The latter can be expanded in the unit circle:

$$g(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}\, z^n, \qquad |z| < 1. \tag{1.4}$$

Function (1.2) is also analytic in the unit circle and this is why

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, z^n = \sum_{n=0}^{\infty} (-1)^n\, a_n\, z^n, \qquad |z| < 1. \tag{1.5}$$

In view of (1.3) for real values of the variable we have:

$$f(x) = e^{-1}\,(1+x)^{\frac{1}{x}} = 1 + \sum_{n=1}^{\infty} (-1)^n\, a_n\, x^n, \qquad -1 < x < 1, \tag{1.6}$$

where $a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. As we shall see below, all numbers a_n are positive. This explains the form of writing the coefficients in (1.5), (1.6) stressing their sign alternation.

We note that the power series in (1.4) converges everywhere on the unit circumference except for the point $z = -1$. This property is not inherited by power series (1.5): it turns out, see Section 2, that its convergence domain is the open circle $|z| < 1$.

The study of Taylor coefficient of functions (1.2) is the main content of this work. In the next section we propose a way of calculating the coefficients in (1.5) and by means of this method we prove the decreasing of the sequence a_n. Then in Section 3 we prove an explicit combinatorial representation of the numbers a_n arising from the Faà di Bruno formula for the derivatives of a composed function. The fourth section is devoted to justifying a non-obvious property $a_n \to 1/e$ as $n \to \infty$. In order to do this, we employ the Cauchy formula for calculating the Taylor coefficients of an analytic function with a specially chose integration contour. We propose other options of choosing the contour leading to various integral representations of the numbers a_n for all $n \in \mathbb{N}$. Possible applications of the obtained results are discussed in concluding Section 5.

2. Recurrent formula for coefficients

We begin with deducing a useful recurrent relation for the numbers a_n.

Proposition 2.1. *The coefficients of power series* (1.5) *form a sign alternating sequence of rational numbers and*

$$a_0 = 1, \qquad a_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} \frac{k+1}{k+2}\, a_{n-k}, \qquad n \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}. \tag{2.1}$$

Proof. According to (1.2) we have:

$$f'(z) = f(z) \left(\frac{\ln(1+z)}{z} - 1 \right)', \qquad z \in D = \mathbb{C} \setminus (-\infty, -1]. \tag{2.2}$$

Since by (1.4), (1.5) in the circle $|z| < 1$ the representations hold

$$f'(z) = \sum_{n=0}^{\infty} (-1)^{n+1}\, a_{n+1}\, z^n,$$

$$\left(\frac{\ln(1+z)}{z} - 1 \right)' = \sum_{n=0}^{\infty} (-1)^{n+1}\, \frac{n+1}{n+2}\, z^n,$$

relation (2.2) gives the identity

$$\sum_{n=0}^{\infty} (-1)^{n+1}\, a_{n+1}\, z^n = \left(\sum_{n=0}^{\infty} (-1)^n\, a_n\, z^n \right) \left(\sum_{n=0}^{\infty} (-1)^{n+1}\, \frac{n+1}{n+2}\, z^n \right), \qquad |z| < 1.$$

The product of the written power series is also a power series of form

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^{k+1}\, \frac{k+1}{k+2}\, (-1)^{n-k}\, a_{n-k} \right) z^n = \sum_{n=0}^{\infty} (-1)^{n+1} \left(\sum_{k=0}^{n} \frac{k+1}{k+2}\, a_{n-k} \right) z^n, \qquad |z| < 1.$$

Thus, in the unit circle we have:

$$\sum_{n=0}^{\infty} (-1)^{n+1}\, a_{n+1}\, z^n = \sum_{n=0}^{\infty} (-1)^{n+1} \left(\sum_{k=0}^{n} \frac{k+1}{k+2}\, a_{n-k} \right) z^n,$$

and in view of (1.3) this yields recurrent rule (2.1) for finding the coefficients of expansion (1.5). This rule shows that all numbers a_n are rational. The proof is complete. □

Calculations by formula (2.1) give several first coefficients

$$a_1 = \frac{1}{2}\, a_0 = \frac{1}{2}, \qquad a_2 = \frac{1}{2} \left(\frac{1}{2}\, a_1 + \frac{2}{3}\, a_0 \right) = \frac{1}{2} \left(\frac{1}{4} + \frac{2}{3} \right) = \frac{11}{24},$$

$$a_3 = \frac{1}{3} \left(\frac{1}{2}\, a_2 + \frac{2}{3}\, a_1 + \frac{3}{4}\, a_0 \right) = \frac{1}{3} \left(\frac{11}{48} + \frac{1}{3} + \frac{3}{4} \right) = \frac{7}{16},$$

$$a_4 = \frac{1}{4} \left(\frac{1}{2}\, a_3 + \frac{2}{3}\, a_2 + \frac{3}{4}\, a_1 + \frac{4}{5}\, a_0 \right) = \frac{1}{4} \left(\frac{7}{32} + \frac{11}{36} + \frac{3}{8} + \frac{4}{5} \right) = \frac{2447}{5760},$$

$$a_5 = \frac{1}{5} \left(\frac{1}{2}\, a_4 + \frac{2}{3}\, a_3 + \frac{3}{4}\, a_2 + \frac{4}{5}\, a_1 + \frac{5}{6}\, a_0 \right) = \frac{1}{5} \left(\frac{2447}{11520} + \frac{7}{24} + \frac{11}{32} + \frac{2}{5} + \frac{5}{6} \right) = \frac{959}{2304}.$$

Hence,

$$f(z) \equiv \exp\left\{ \frac{\ln(1+z)}{z} - 1 \right\} = 1 - \frac{1}{2}\, z + \frac{11}{24}\, z^2 - \frac{7}{16}\, z^3 + \frac{2447}{5760}\, z^4 - \frac{959}{2304}\, z^5 + \ldots, \qquad |z| < 1.$$

At the example of first coefficients we see that $a_0 > a_1 > a_2 > a_3 > a_4 > a_5$ since

$$a_0 = 1, \qquad a_1 = \frac{1}{2} = 0,5, \qquad a_2 = \frac{11}{24} = 0,458(3), \qquad a_3 = \frac{7}{16} = 0,4375,$$
$$a_4 = \frac{2447}{5760} = 0,4248263(8), \qquad a_5 = \frac{959}{2304} = 0,41623263(8).$$

As the index increases, whether the observed trend to a slow decreasing of arising numbers is kept? The answer is given in the following statement.

Proposition 2.2. *The numbers a_n given by recurrent rule* (2.1) *form a decreasing sequence, that is,*

$$d_n \equiv a_n - a_{n+1} > 0, \qquad n \in \mathbb{N}_0. \tag{2.3}$$

Proof. For accumulating the facts, we first calculated several first terms in the sequence defined in (2.3). We get:

$$d_0 = a_0 - a_1 = 1 - \frac{1}{2} = \frac{1}{2}, \quad d_1 = a_1 - a_2 = \frac{1}{2} - \frac{11}{24} = \frac{1}{24}, \quad d_2 = a_2 - a_3 = \frac{11}{24} - \frac{7}{16} = \frac{1}{48},$$
$$d_3 = a_3 - a_4 = \frac{7}{16} - \frac{2447}{5760} = \frac{73}{5760}, \qquad d_4 = a_4 - a_5 = \frac{2447}{5760} - \frac{959}{2304} = \frac{11}{1280}.$$

First numbers (2.3), the first differences of numbers (2.1), are positive and decrease:

$$d_0 > d_1 > d_2 > d_3 > d_4.$$

Indeed,

$$d_0 = \frac{1}{2} = 0,5, \qquad d_1 = \frac{1}{24} = 0,041(6), \qquad d_2 = \frac{1}{48} = 0,0208(3),$$
$$d_3 = \frac{73}{5760} = 0,0126736(1), \qquad d_4 = \frac{11}{1280} = 0,00859375.$$

Let us find a recurrent law of forming the numbers d_n. By (2.1) for $n \in \mathbb{N}_0$ we find:

$$\begin{aligned}
d_{n+1} = a_{n+1} - a_{n+2} &= \frac{1}{n+1} \sum_{k=0}^{n} \frac{k+1}{k+2}\, a_{n-k} - \frac{1}{n+2} \sum_{k=0}^{n+1} \frac{k+1}{k+2}\, a_{n+1-k} \\
&= \frac{1}{n+1} \sum_{k=0}^{n} \frac{k+1}{k+2}\, a_{n-k} - \frac{1}{n+2} \sum_{k=0}^{n} \frac{k+1}{k+2}\, a_{n+1-k} - \frac{1}{n+3} \\
&= \frac{1}{n+1} \sum_{k=0}^{n} \frac{k+1}{k+2} \big(a_{n-k} - a_{n+1-k}\big) + \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} \frac{k+1}{k+2}\, a_{n+1-k} - \frac{1}{n+3} \\
&= \frac{1}{n+1} \sum_{k=0}^{n} \frac{k+1}{k+2}\, d_{n-k} + \frac{1}{n+1} \left(\frac{1}{n+2} \sum_{k=0}^{n+1} \frac{k+1}{k+2}\, a_{n+1-k} - \frac{1}{n+3} \right) - \frac{1}{n+3} \\
&= \frac{1}{n+1} \sum_{k=0}^{n} \frac{k+1}{k+2}\, d_{n-k} + \frac{1}{n+1}\, a_{n+2} - \frac{n+2}{(n+1)(n+3)} \\
&= \frac{1}{n+1} \left(\sum_{k=0}^{n} d_{n-k} - \sum_{k=0}^{n} \frac{1}{k+2}\, d_{n-k} \right) + \frac{1}{n+1}\, a_{n+2} - \frac{n+2}{(n+1)(n+3)}.
\end{aligned}$$

We take into consideration an obvious identity

$$\sum_{k=0}^{n} d_{n-k} = a_0 - a_{n+1} = 1 - a_{n+1}$$

and continue the calculations:

$$\begin{aligned} d_{n+1} &= \frac{1}{n+1}\left(1 - a_{n+1} - \sum_{k=0}^{n}\frac{1}{k+2}\,d_{n-k}\right) + \frac{1}{n+1}\,a_{n+2} - \frac{n+2}{(n+1)(n+3)} \\ &= \frac{1}{n+1} - \frac{n+2}{(n+1)(n+3)} - \frac{1}{n+1}\,\big(a_{n+1} - a_{n+2}\big) - \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{k+2}\,d_{n-k} \\ &= \frac{1}{(n+1)(n+3)} - \frac{1}{n+1}\,d_{n+1} - \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{k+2}\,d_{n-k}. \end{aligned}$$

As a result we have:

$$\frac{n+2}{n+1}\,d_{n+1} = \frac{1}{(n+1)(n+3)} - \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{k+2}\,d_{n-k}, \qquad n \in \mathbb{N}_0.$$

Thus, for the scalar sequence defined in (2.3) we obtain the recurrent rule:

$$d_0 = \frac{1}{2}, \qquad d_{n+1} = \frac{1}{n+2}\left(\frac{1}{n+3} - \sum_{k=0}^{n}\frac{1}{n+2-k}\,d_k\right), \qquad n \in \mathbb{N}_0. \tag{2.4}$$

For instance, in accordance with (2.4),

$$\begin{aligned} d_1 &= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2}\,d_0\right) = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{24}, \\ d_2 &= \frac{1}{3}\left(\frac{1}{4} - \frac{1}{3}\,d_0 - \frac{1}{2}\,d_1\right) = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{6} - \frac{1}{48}\right) = \frac{1}{48}, \end{aligned}$$

and this is supported by the above straightforward calculations.

We rewrite recurrent formula (2.4) in an equivalent form:

$$d_0 = \frac{1}{2}, \quad d_1 = \frac{1}{24}, \quad d_{n+1} = \frac{n+1}{2(n+2)^2(n+3)} - \frac{1}{n+2}\sum_{k=1}^{n}\frac{1}{n+2-k}\,d_k, \quad n \in \mathbb{N}. \tag{2.5}$$

Writing (2.5) is better than (2.4) adapted to proving property (2.3).

As $n = 1$, formula (2.5) gives

$$d_2 = \frac{1}{36} - \frac{1}{6}\,d_1 = \frac{1}{36} - \frac{1}{144} = \frac{1}{48},$$

and this coincides with the value calculated above twice in different ways. Let us prove the positivity of differences (2.3) by the induction in the index $n \in \mathbb{N}_0$. As $n = 0$ and $n = 1$ we respectively have $d_0 = 1/2 > 0$ and $d_1 = 1/24 > 0$. Assume that

$$d_k > 0, \qquad k = 2, \ldots, n, \tag{2.6}$$

for some $n \in \mathbb{N}$, $n \geqslant 2$. Then, as (2.5) shows, the estimate holds:

$$d_k < \frac{1}{2(k+1)(k+2)}, \qquad k = 1, \ldots, n. \tag{2.7}$$

Indeed, (2.7) for $k = 1$ becomes a true scalar inequality $1/24 < 1/12$, while for other indices $k = 2, \ldots, n$ by (2.5) in view of (2.6) we have

$$d_k = \frac{k}{2(k+1)^2(k+2)} - \frac{1}{k+1}\sum_{m=1}^{k-1}\frac{1}{k+1-m}\,d_m < \frac{k}{2(k+1)^2(k+2)} < \frac{1}{2(k+1)(k+2)}.$$

We need to show that $d_{n+1} > 0$. Applying (2.7) in (2.5), we obtain

$$d_{n+1} > \frac{n+1}{2(n+2)^2(n+3)} - \frac{1}{2(n+2)} \sum_{k=1}^{n} \frac{1}{(n+2-k)(k+1)(k+2)}. \tag{2.8}$$

We use a standard notation for harmonic numbers:

$$H_m \equiv \sum_{k=1}^{m} \frac{1}{k}, \qquad m \in \mathbb{N}.$$

The expansion into primitive fraction

$$\frac{1}{(n+2-k)(k+1)(k+2)} = \frac{1}{(n+3)(n+4)} \frac{1}{n+2-k} + \frac{1}{n+3} \frac{1}{k+1} - \frac{1}{n+4} \frac{1}{k+2}$$

shows that

$$\sum_{k=1}^{n} \frac{1}{(n+2-k)(k+1)(k+2)} = \frac{2H_{n+1}}{(n+3)(n+4)} + \frac{n^2-n-8}{2(n+2)(n+3)(n+4)}.$$

Substituting this relation into (2.8), we write the estimate

$$d_{n+1} > \frac{n+1}{2(n+2)^2(n+3)} - \frac{H_{n+1}}{(n+2)(n+3)(n+4)} - \frac{n^2-n-8}{4(n+2)^2(n+3)(n+4)},$$

and it is equivalent to

$$4(n+2)^2(n+3)(n+4)\, d_{n+1} > n^2 + 11n + 16 - 4(n+2)H_{n+1}.$$

The positivity of the right hand in the latter inequality for $n \geqslant 2$ is implied by the upper bound for the harmonic numbers:

$$H_m < \frac{m^2+9m+6}{4(m+1)}, \qquad m \geqslant 3.$$

The checking of such rather rough estimate is elementary; for instance, by the induction in m.

Finally, we have found out that assumption (2.6) implies $d_{n+1} > 0$. Hence, (2.3) holds. This shows the decreasing of the sequence a_n as $n \in \mathbb{N}_0$. The proof is complete. □

Proposition 2.2 indicates the existence of the limit of sequence (2.1):

$$\lim_{n\to\infty} a_n \equiv a \geqslant 0.$$

A bit unexpected is the fact that this limit is non-zero. Patient calculations lead us to a two-sided estimate

$$0.3433 < a < 0.3985. \tag{2.9}$$

Indeed, due to the decreasing of the sequence a_n, on the base of the facts we have we can immediately write $a < a_n < a_5 < 0.4163$, where $n > 5$. Let us try to low the upper bound and overcome the level 0.4. Owing to the monotonicity property proved in Proposition 2.2, we shall specify the upper bound continuing calculating the values a_n by formula (2.1) for indices $n \geqslant 6$. Reaching index $n = 9$, we find that

$$a_9 = \frac{123377159}{309657600} < 0.3985,$$

which implies the right inequality in (2.9). On the other hand, for all $n \in \mathbb{N}$ recurrent relation (2.5) in view of the initial condition $d_1 = 1/24$ gives

$$d_{n+1} \leqslant \frac{n+1}{2(n+2)^2(n+3)} - \frac{1}{24(n+1)(n+2)} = \frac{11n^2+19n+6}{24(n+1)(n+2)^2(n+3)}.$$

Hence,

$$d_m \leqslant \frac{11m^2-3m-2}{24m(m+1)^2(m+2)}, \qquad m \geqslant 2.$$

But then for the indices $n \geqslant 2$ we obtain

$$a_{n+1} = 1 - \sum_{m=0}^{n} d_m = 1 - \left(\frac{1}{2} + \frac{1}{24} + \sum_{m=2}^{n} d_m\right) > \frac{11}{24} - \sum_{m=2}^{\infty} \frac{11m^2-3m-2}{24m(m+1)^2(m+2)} = \frac{4\pi^2-23}{48}.$$

The sum of the series is calculated by means of a famous result by Euler

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}$$

applied after the transformation

$$\frac{11m^2-3m-2}{24m(m+1)^2(m+2)} = \left(\frac{1}{m+1} - \frac{1}{m+2}\right) - \frac{1}{24}\left(\frac{1}{m} - \frac{1}{m+1}\right) - \frac{1}{2}\,\frac{1}{(m+1)^2}.$$

The found estimate

$$a_n > \frac{4\pi^2-23}{48} > 0.3433, \qquad n \geqslant 3,$$

which is true for all $n \in \mathbb{N}_0$, demonstrates the validity of the left inequality in (2.9).

As we shall show below, the exact value of the limit a is the number $1/e = 0.3678\ldots$ being of course in mentioned range (2.9). In our opinion, it is rather troublesome to derive an exact statement from recurrent formula (2.1). At least, the tools of the complex analysis turn out to be more effective in solving this problem. At the same time, approximate result (2.9) is very useful for understanding the situation in general. For instance, it implies the divergence of the power series in (1.5) on the circumference $|z| = 1$ despite f is a composition of two functions analytic respectively in the entire plane $\mathbb{C}$ and in the domain $\mathbb{C} \setminus (-\infty, -1]$, the power expansions of which converge at the points $z \neq -1$ of the mentioned circumference.

In Section 4 we return back to the issue on calculating the quantity $a = \lim\limits_{n\to\infty} a_n$, and now we discuss an alternative way of defining the sequence a_n of a combinatorial nature.

3. Faà di Bruno formula

The genesis of function (1.2) suggests an idea to apply one general result known as the Faà di Bruno formula for finding the coefficients in expansion (1.5), see, for instance, [4, Ch. 2, Sect. 8]. We mean here the rule of calculating the derivatives of a composed function. We employ the following version of its writing

$$\frac{1}{n!}\frac{d^n}{dz^n} h\big(g(z)\big) = \sum \frac{1}{m_1!\, m_2! \,\ldots\, m_n!}\, h^{(m_1+\ldots+m_n)}\big(g(z)\big) \prod_{j=1}^{n} \left(\frac{g^{(j)}(z)}{j!}\right)^{m_j}, \quad n \in \mathbb{N}, \quad (3.1)$$

where the summation is made over all sets $(m_1, m_2, \ldots, m_n)$ of the numbers $\mathbb{N}_0$ restricted for a given order of the derivative n by the condition

$$m_1 + 2m_2 + \ldots + nm_n = n. \qquad (3.2)$$

Many interesting facts on the Faà di Bruno formula can be found in a retrospective collection of Russian publications [5]–[7] and a detailed survey [8].

In our case

$$h(w) = \exp w, \qquad g(z) = \frac{\ln(1+z)}{z} - 1, \qquad f(z) = h\big(g(z)\big), \qquad h^{(m_1+\ldots+m_n)}\big(g(z)\big) = f(z).$$

By (1.3), (1.4) we have

$$f(0) = 1, \qquad \frac{g^{(j)}(0)}{j!} = \frac{(-1)^j}{j+1}, \quad j \in \mathbb{N}.$$

Taking into consideration these simple arguing, we substitute the point $z = 0$ into (3.1) and we obtain the formula

$$(-1)^n\, a_n = \frac{f^{(n)}(0)}{n!} = \sum \frac{1}{m_1!\, m_2!\, \ldots\, m_n!} \prod_{j=1}^{n} \left(\frac{(-1)^j}{j+1}\right)^{m_j}, \qquad n \in \mathbb{N},$$

for the coefficients of Taylor expansion (1.5). We note that relation (3.2) allows us to write

$$\prod_{j=1}^{n} \left(\frac{(-1)^j}{j+1}\right)^{m_j} = (-1)^n \prod_{j=1}^{n} \frac{1}{(j+1)^{m_j}}$$

for arbitrary $n \in \mathbb{N}$. Therefore, the following result is true.

Proposition 3.1. *For the coefficients in Taylor expansion* (1.5) *the representation*

$$a_n = \sum \frac{1}{m_1!\, m_2!\, \ldots\, m_n!\; 2^{m_1}\, 3^{m_2}\, \ldots\, (n+1)^{m_n}}, \qquad n \in \mathbb{N}, \tag{3.3}$$

holds with the summation by rule (3.2).

Let us test formula (3.3) by choosing the index $n = 4$. In this case, according to our above calculations based on recurrent formula (2.1), we show obtain for a_4 the value 2447/5760. Indeed, the equation

$$m_1 + 2m_2 + 3m_3 + 4m_4 = 4$$

has exactly five solutions $(m_1,\, m_2,\, m_3,\, m_4)$ with components in the set $\mathbb{N}_0$, namely,

$$(0,\,0,\,0,\,1), \qquad (1,\,0,\,1,\,0), \qquad (0,\,2,\,0,\,0), \qquad (2,\,1,\,0,\,0), \qquad (4,\,0,\,0,\,0).$$

Therefore, the sum

$$\sum \frac{1}{m_1!\, m_2!\, m_3!\, m_4!\; 2^{m_1}\, 3^{m_2}\, 4^{m_3}\, 5^{m_4}}$$

consists of five terms each being calculated by its set of integer components:

$$\begin{aligned}
(0,\,0,\,0,\,1) \quad &\Longrightarrow \quad \frac{1}{0!\,0!\,0!\,1!\;2^0\,3^0\,4^0\,5^1} = \frac{1}{5},\\
(1,\,0,\,1,\,0) \quad &\Longrightarrow \quad \frac{1}{1!\,0!\,1!\,0!\;2^1\,3^0\,4^1\,5^0} = \frac{1}{2\cdot 4} = \frac{1}{8},\\
(0,\,2,\,0,\,0) \quad &\Longrightarrow \quad \frac{1}{0!\,2!\,0!\,0!\;2^0\,3^2\,4^0\,5^0} = \frac{1}{2\cdot 9} = \frac{1}{18},\\
(2,\,1,\,0,\,0) \quad &\Longrightarrow \quad \frac{1}{2!\,1!\,0!\,0!\;2^2\,3^1\,4^0\,5^0} = \frac{1}{2\cdot 4\cdot 3} = \frac{1}{24},\\
(4,\,0,\,0,\,0) \quad &\Longrightarrow \quad \frac{1}{4!\,0!\,0!\,0!\;2^4\,3^0\,4^0\,5^0} = \frac{1}{24\cdot 16} = \frac{1}{384}.
\end{aligned}$$

As a result we have:

$$a_4 = \frac{1}{5} + \frac{1}{8} + \frac{1}{18} + \frac{1}{24} + \frac{1}{384} = \frac{2447}{5760}$$

and this is the desired fact.

As we see, a practical application of "explicit" formula (3.3) is complicated by a non-trivial summation over integer non-negative solutions of Diophantine equation (3.2) with n unknowns and the natural parameter n. As the parameter grows, the number of such solutions grows as well making representation (3.3) obscure. Of course, Proposition 3.1 has a peculiar combinatorial aesthetic, but it is not clear how extract from (3.3) an information about asymptotic behavior of the numbers a_n. This is why we aimed on seeking a "working" formula for a_n devoid of these disadvantages. In order to do this, we employ the technique of contour integration, which is often useful in such problems [9, Ch. I, Sect. 1.3].

4. Integral representation of coefficients

A key result of the section and the entire work is the following statement.

Proposition 4.1. *For the numbers a_n involved in expansion* (1.5) *the integral representation*

$$a_n = \frac{1}{e}\left(1 + \frac{1}{\pi}\int\limits_0^1 \frac{\sin(\pi\tau)}{\tau^{1-\tau}(1-\tau)^{\tau}}\,\tau^n\,d\tau\right), \qquad n \in \mathbb{N}, \tag{4.1}$$

holds.

Proof. We choose the numbers r, R so that $0 < r < 1 < 1 + r < R$. In the plane $\mathbb{C}$ we make a cut along the ray $(-\infty,\,-1]$ and construct a contour $\Gamma_{r,\,R}$ consisting of four parts written in the order of passing them:

- circumference (counterclockwise) $\gamma_R:\ z = Re^{i\theta}$, $\theta \in [-\pi,\,\pi]$;
- segment $l_{r,\,R}$ from the point $z_1 = -R$ till the point $z_2 = -(1+r)$ along the upper side of the cut;
- circumference (clockwise) γ_r^-, where $\gamma_r:\ z = -1 + re^{i\theta}$, $\theta \in [-\pi,\,\pi]$;
- segment $l_{r,\,R}^-$ from the point $z_2 = -(1+r)$ till the point $z_1 = -R$, along the lower side of the cut.

Each Taylor coefficient in expansion (1.5) is expressed via the contour integral by the Cauchy formula

$$(-1)^n\,a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int\limits_{\Gamma_{r,\,R}} \frac{f(z)}{z^{n+1}}\,dz, \qquad n \in \mathbb{N}. \tag{4.2}$$

We fix $n \in \mathbb{N}$ and calculate the integral in (4.2) as the sum

$$\frac{1}{2\pi i}\int\limits_{\Gamma_{r,\,R}} \frac{f(z)}{z^{n+1}}\,dz = I_{1,\,n}(R) + I_{2,\,n}(r,\,R) + I_{3,\,n}(r) + I_{4,\,n}(r,\,R), \tag{4.3}$$

where

$$I_{1,\,n}(R) \equiv \frac{1}{2\pi i}\int\limits_{\gamma_R} \frac{f(z)}{z^{n+1}}\,dz, \qquad I_{2,\,n}(r,\,R) \equiv \frac{1}{2\pi i}\int\limits_{l_{r,\,R}} \frac{f(z)}{z^{n+1}}\,dz,$$

$$I_{3,\,n}(r) \equiv \frac{1}{2\pi i}\int\limits_{\gamma_r^-} \frac{f(z)}{z^{n+1}}\,dz, \qquad I_{4,\,n}(r,\,R) \equiv \frac{1}{2\pi i}\int\limits_{l_{r,\,R}^-} \frac{f(z)}{z^{n+1}}\,dz.$$

We write the integral over the circumference γ_R as

$$I_{1,\,n}(R) = \frac{1}{2\pi i}\int\limits_{\gamma_R} \frac{f(z)}{z^{n+1}}\,dz = \frac{1}{2\pi R^n}\int\limits_{-\pi}^{\pi} e^{-in\theta}\,f\big(Re^{i\theta}\big)\,d\theta.$$

Since function (1.2) satisfies the estimate

$$\left|f\big(Re^{i\theta}\big)\right| = \left|\exp\left\{\frac{\ln(1+Re^{i\theta})}{Re^{i\theta}} - 1\right\}\right| \leqslant \exp\left\{\frac{\left|\ln(1+Re^{i\theta})\right|}{R} - 1\right\} \leqslant \exp\left\{\frac{\ln(1+R)+\pi}{R} - 1\right\}$$

holds for all $\theta \in [-\pi,\, \pi]$, then

$$\frac{1}{2\pi R^n}\left|\int_{-\pi}^{\pi} e^{-in\theta} f\big(Re^{i\theta}\big)\, d\theta\right| \leqslant \frac{1}{2\pi R^n}\int_{-\pi}^{\pi}\Big|f\big(Re^{i\theta}\big)\Big|\, d\theta \leqslant \frac{1}{eR^n}\, e^{\pi/R}\,(1+R)^{1/R}.$$

This is why for a given $n \in \mathbb{N}$ we have:

$$I_{1,\,n}(R) = \frac{1}{2\pi i}\int_{\gamma_R}\frac{f(z)}{z^{n+1}}\, dz \to 0, \qquad R \to +\infty. \tag{4.4}$$

Passing to the second integral in sum (4.3), we observe that at the points $z = x$ in the segment $l_{r,\,R}$ the function $g(z)$ is defined by the rule

$$g(z) \equiv \frac{\ln(1+z)}{z} - 1 = \frac{1}{x}\Big(\ln(-1-x) + \pi i\Big) - 1.$$

This is why

$$I_{2,\,n}(r,\,R) = \frac{1}{2\pi i}\int_{l_{r,\,R}}\frac{f(z)}{z^{n+1}}\, dz = \frac{1}{2\pi e i}\int_{-R}^{-(1+r)}\frac{1}{x^{n+1}}\,(-1-x)^{1/x}\, e^{\pi i/x}\, dx$$

$$= \frac{(-1)^{n+1}}{2\pi e i}\int_{1/R}^{1/(1+r)}\tau^{n+1}\Big(\frac{1}{\tau}-1\Big)^{-\tau} e^{-\pi\tau i}\,\frac{d\tau}{\tau^2} = \frac{(-1)^{n+1}}{2\pi e i}\int_{1/R}^{1/(1+r)}\frac{\tau^{\tau+n-1}}{(1-\tau)^{\tau}}\, e^{-\pi\tau i}\, d\tau.$$

Therefore, for a fixed $n \in \mathbb{N}$ the relation holds

$$I_{2,\,n}(r,\,R) = \frac{1}{2\pi i}\int_{l_{r,\,R}}\frac{f(z)}{z^{n+1}}\, dz \;\to\; \frac{(-1)^{n+1}}{2\pi e i}\int_{0}^{1}\frac{\tau^{\tau+n-1}}{(1-\tau)^{\tau}}\, e^{-\pi\tau i}\, d\tau, \qquad r \to 0+, \qquad R \to +\infty. \tag{4.5}$$

We write the integral over the circumference γ_r^- as

$$I_{3,\,n}(r) = \frac{1}{2\pi i}\int_{\gamma_r^-}\frac{f(z)}{z^{n+1}}\, dz = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f\big(-1+re^{i\theta}\big)\, re^{i\theta}}{\big(-1+re^{i\theta}\big)^{n+1}}\, d\theta.$$

Since

$$\begin{aligned} f\big(-1+re^{i\theta}\big)\, re^{i\theta} &= \frac{re^{i\theta}}{e}\ \exp\frac{\ln(re^{i\theta})}{-1+re^{i\theta}} \\ &= \frac{1}{e}\ \exp\left\{\frac{\ln r + i\theta}{-1+re^{i\theta}} + \ln r + i\theta\right\} = \frac{1}{e}\ \exp\frac{re^{i\theta}(\ln r + i\theta)}{-1+re^{i\theta}} \end{aligned}$$

for all $\theta \in [-\pi,\, \pi]$, then as $r \to 0+$ the integrand

$$\frac{f\big(-1+re^{i\theta}\big)\, re^{i\theta}}{\big(-1+re^{i\theta}\big)^{n+1}}$$

tends to $(-1)^{n+1}\,(1/e)$ uniformly in $\theta \in [-\pi,\, \pi]$. Applying the known result on an integral depending on a parameter, see, for instance, [10, Ch. I, Sect. 6], for each $n \in \mathbb{N}$ we obtain that

$$I_{3,\,n}(r) = \frac{1}{2\pi i}\int_{\gamma_r^-}\frac{f(z)}{z^{n+1}}\, dz \;\to\; \frac{(-1)^n}{e}, \qquad r \to 0+\,. \tag{4.6}$$

Finally, we take into consideration that at the points $z = x$ of the segment $l^-_{r,R}$ the function $g(z)$ is defined as

$$g(z) \equiv \frac{\ln(1+z)}{z} - 1 = \frac{1}{x}\Big(\ln(-1-x) - \pi i\Big) - 1.$$

Hence,

$$I_{4,n}(r,\,R) = \frac{1}{2\pi i}\int\limits_{l^-_{r,R}} \frac{f(z)}{z^{n+1}}\,dz = -\frac{1}{2\pi e i}\int\limits_{-R}^{-(1+r)} \frac{1}{x^{n+1}}\,(-1-x)^{1/x}\,e^{-\pi i/x}\,dx$$

$$= \frac{(-1)^n}{2\pi e i}\int\limits_{1/R}^{1/(1+r)} \tau^{n+1}\left(\frac{1}{\tau}-1\right)^{-\tau} e^{\pi\tau i}\,\frac{d\tau}{\tau^2} = \frac{(-1)^n}{2\pi e i}\int\limits_{1/R}^{1/(1+r)} \frac{\tau^{\tau+n-1}}{(1-\tau)^\tau}\,e^{\pi\tau i}\,d\tau.$$

Therefore, for a fixed $n \in \mathbb{N}$ we have

$$I_{4,n}(r,\,R) = \frac{1}{2\pi i}\int\limits_{l^-_{r,R}} \frac{f(z)}{z^{n+1}}\,dz \to \frac{(-1)^n}{2\pi e i}\int\limits_0^1 \frac{\tau^{\tau+n-1}}{(1-\tau)^\tau}\,e^{\pi\tau i}\,d\tau, \quad r\to 0+, \quad R\to +\infty. \tag{4.7}$$

Applying limiting relations (4.4)–(4.7) in (4.3), we obtain that for an arbitrary index $n \in \mathbb{N}$ the integral

$$\frac{1}{2\pi i}\int\limits_{\Gamma_{r,R}} \frac{f(z)}{z^{n+1}}\,dz$$

tends to the quantity

$$\frac{(-1)^{n+1}}{2\pi e i}\int\limits_0^1 \frac{\tau^{\tau+n-1}}{(1-\tau)^\tau}\,e^{-\pi\tau i}\,d\tau + \frac{(-1)^n}{e} + \frac{(-1)^n}{2\pi e i}\int\limits_0^1 \frac{\tau^{\tau+n-1}}{(1-\tau)^\tau}\,e^{\pi\tau i}\,d\tau,$$

which is obviously equal to

$$\frac{(-1)^n}{e}\left(1 + \frac{1}{\pi}\int\limits_0^1 \frac{\sin(\pi\tau)}{\tau^{1-\tau}(1-\tau)^\tau}\,\tau^n\,d\tau\right),$$

if $r \to 0+$ and $R \to +\infty$. Passing to limit in (4.2), we confirm the validity of formula (4.1). The proof is complete. □

Taking exact values for the first four numbers a_n starting with a_1, see Section 2, we obtain by (4.1) a series of delicate identities

$$\int\limits_0^1 \frac{\tau^\tau}{(1-\tau)^\tau}\,\sin(\pi\tau)\,d\tau = \frac{\pi(e-2)}{2}, \qquad \int\limits_0^1 \frac{\tau^{1+\tau}}{(1-\tau)^\tau}\,\sin(\pi\tau)\,d\tau = \frac{\pi(11e-24)}{24},$$

$$\int\limits_0^1 \frac{\tau^{2+\tau}}{(1-\tau)^\tau}\,\sin(\pi\tau)\,d\tau = \frac{\pi(7e-16)}{16}, \qquad \int\limits_0^1 \frac{\tau^{3+\tau}}{(1-\tau)^\tau}\,\sin(\pi\tau)\,d\tau = \frac{\pi(2447e-5760)}{5760},$$

which can be continued if it is needed. But the main destination of Proposition 4.1 is another: now it is easy to obtain a general impression on the asymptotic behavior of the Taylor coefficients of function (1.2).

Proposition 4.2. *A two-sided estimate*

$$\frac{1}{e}\left(1+\frac{2}{\pi(n+1)}\right) < a_n < \frac{1}{e}\left(1+\frac{1}{n+1}\right), \qquad n \in \mathbb{N}, \tag{4.8}$$

holds and it implies the order relation

$$a_n - \frac{1}{e} \asymp \frac{1}{n}, \qquad n \to \infty. \tag{4.9}$$

Proof. We write integral representation (4.1) in a compact form

$$a_n = \frac{1}{e}\left(1 + \int\limits_0^1 \varphi(\tau)\,\tau^n\,d\tau\right), \qquad n \in \mathbb{N}, \tag{4.10}$$

where

$$\varphi(\tau) \equiv \frac{\sin(\pi\tau)}{\pi}\,\frac{1}{\tau^{1-\tau}(1-\tau)^{\tau}}, \qquad \tau \in (0,\,1). \tag{4.11}$$

Function (4.11) under the natural convention

$$\varphi(0) \equiv \lim_{\tau\to 0+} \varphi(\tau) = 1, \quad \varphi(1) \equiv \lim_{\tau\to 1-0} \varphi(\tau) = 1$$

is continuous on $[0,\,1]$ and possesses the properties:

1) $\varphi(\tau) = \varphi(1-\tau)$, $\tau \in [0,\,1]$,
2) $\varphi(\tau)$ decreases on $[0,\,1/2]$ and increases on $[1/2,\,1]$,
3) $\min\limits_{0\leqslant\tau\leqslant 1} \varphi(\tau) = \varphi(1/2) = 2/\pi$, $\quad \max\limits_{0\leqslant\tau\leqslant 1} \varphi(\tau) = \varphi(0) = \varphi(1) = 1$.

The first property is obvious, the third property is implied by the second one. This is why it is sufficient to confirm the decreasing of the function $\varphi(\tau)$ as $0 \leqslant \tau \leqslant 1/2$. This can be done by the standard tools of the analysis on the base of the relations

$$\eta(\tau) \equiv \ln\varphi(\tau) = \ln\sin(\pi\tau) - \ln\pi - (1-\tau)\ln\tau - \tau\ln(1-\tau), \quad \tau \in (0,\,1/2], \quad \eta(0) = 0,$$

$$\eta'(\tau) = \pi\cot(\pi\tau) + \ln\tau - \frac{1-\tau}{\tau} - \ln(1-\tau) + \frac{\tau}{1-\tau}, \qquad \tau \in (0,\,1/2], \quad \eta'(0+) = -\infty,$$

$$\eta''(\tau) = -\frac{\pi^2}{\sin^2(\pi\tau)} + \frac{1}{\tau} + \frac{1}{\tau^2} + \frac{1}{1-\tau} + \frac{1}{(1-\tau)^2}, \qquad \tau \in (0,\,1/2], \quad \eta''(0+) = +\infty.$$

More precisely, by means of the estimate

$$\sin s > s - \frac{s^3}{6}, \qquad s > 0,$$

we prove the inequality $\eta''(\tau) > 0$ valid for all $\tau \in (0,\,1/2]$ and implying the increasing of η' on the segment $(0,\,1/2]$ from $\eta'(0+) = -\infty$ till $\eta'(1/2) = 0$, and hence, the decreasing of η on $[0,\,1/2]$ from $\eta(0) = 0$ till $\eta(1/2) = -\ln(\pi/2)$.

As a results for each $n \in \mathbb{N}$ we have a two-sided estimate

$$\frac{2}{\pi(n+1)} = \frac{2}{\pi}\int\limits_0^1 \tau^n\,d\tau < \int\limits_0^1 \varphi(\tau)\,\tau^n\,d\tau < \int\limits_0^1 \tau^n\,d\tau = \frac{1}{n+1}.$$

Applying it in (4.10), we obtain (4.8), (4.9). The proof is complete. □

We demonstrate a quality of general estimate (4.8) by numerical calculations. We successively substitute the values $n = 1, 2, 3, 4$ into (4.8) and we write:

$$0.4849\ldots = \frac{1}{e}\left(1+\frac{1}{\pi}\right) < a_1 = \frac{1}{2} < \frac{3}{2e} = 0.5518\ldots,$$
$$0.4459\ldots = \frac{1}{e}\left(1+\frac{2}{3\pi}\right) < a_2 = \frac{11}{24} = 0.458(3) < \frac{4}{3e} = 0.4905\ldots,$$
$$0.4264\ldots = \frac{1}{e}\left(1+\frac{1}{2\pi}\right) < a_3 = \frac{7}{16} = 0.4375 < \frac{5}{4e} = 0.4598\ldots,$$
$$0.4147\ldots = \frac{1}{e}\left(1+\frac{2}{5\pi}\right) < a_4 = \frac{2447}{5760} = 0.4248263(8) < \frac{6}{5e} = 0.4414\ldots,$$

with a nice approximation, especially from below.

We also note that (4.10), (4.11) easily implies a recurrent relation

$$a_{n+1} = a_n - \int_0^1 \psi(\tau)\,\tau^n\,d\tau, \qquad n \in \mathbb{N}, \tag{4.12}$$

where a positive function ψ is defined by the formula

$$\psi(\tau) \equiv \frac{1-\tau}{e}\,\varphi(\tau) = \frac{\sin(\pi\tau)}{\pi e}\left(\frac{1-\tau}{\tau}\right)^{1-\tau}, \qquad \tau \in (0,\,1). \tag{4.13}$$

Thus, we have found another way of justifying Proposition 2.2.

We also note that the choice of the contour in the proof of Proposition 4.1 seems to be optimal in certain sense but initially there is a large experimental margin. Omitting the details, for comparison we provide a result arising in the case when for an appropriate combination of the parameters r, R the contour $\widetilde{\Gamma}_{r,R}$ in the domain D consists of four parts:

– arc of the circumference (counterclockwise)

$$\widetilde{\gamma}_R:\ z = Re^{i\theta}, \quad \theta \in \big[-\pi + \arccos(1/R),\, \pi - \arccos(1/R)\big]\,;$$

– vertical segment from the point $-1 + i\sqrt{R^2-1}$ till the point $-1 + ir$;
– semi-circumference (clockwise) $\widetilde{\gamma}_r^-$, where $\widetilde{\gamma}_r:\ z = -1 + re^{i\theta}$, $\theta \in [-\pi/2,\,\pi/2]$;
– vertical segment from the point $-1 - ir$ till the point $-1 - i\sqrt{R^2-1}$.

Such approach seems promising but this is just an illusion: the formula

$$(-1)^n\,a_n = \frac{1}{2\pi i}\int_{\widetilde{\Gamma}_{r,R}} \frac{f(z)}{z^{n+1}}\,dz, \qquad n \in \mathbb{N}, \tag{4.14}$$

after appropriate transformation of the contour integral followed by the passing to the limit as $r \to 0+$, $R \to +\infty$ is reduced to

$$a_n = \frac{1}{e}\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} \mathrm{Re}\left\{\frac{1}{(1-ix)^{n+1}}\,\exp\frac{\ln(ix)}{ix-1}\right\}dx\right), \qquad n \in \mathbb{N}, \tag{4.15}$$

with a converging improper integral. The integrand in (4.15) is given by the expression

$$\frac{1}{(1+x^2)^{\frac{n+1}{2}}}\,\exp\left(\frac{\frac{\pi}{2}x - \ln x}{1+x^2}\right)\cos\left(\frac{\frac{\pi}{2} + x\ln x}{1+x^2} - (n+1)\arctan x\right).$$

It is unlikely possible to derive from here Proposition 4.2 by a simple analysis. The comparison of (4.15) with (4.1) is very instructive. But as a kind of compensation, we obtain an opportunity possibility of an exact calculation for a countable set of exotic improper integrals like

$$\int_0^{+\infty} \frac{1}{1+x^2} \exp\left(\frac{\frac{\pi}{2}x - \ln x}{1+x^2}\right) \cos\left(\frac{\frac{\pi}{2} + x\ln x}{1+x^2} - 2\arctan x\right) dx = \frac{\pi(e-1)}{2},$$

$$\int_0^{+\infty} \frac{1}{(1+x^2)^{3/2}} \exp\left(\frac{\frac{\pi}{2}x - \ln x}{1+x^2}\right) \cos\left(\frac{\frac{\pi}{2} + x\ln x}{1+x^2} - 3\arctan x\right) dx = \frac{\pi(11e-12)}{24}.$$

Completing the section, for small r and large R we propose to replace the contour $\widetilde{\Gamma}_{r,R}$ in Cauchy integral formula (4.14) by a modified contour of the form:

– arc of circumference (counterclockwise)

$$\gamma_{r,R}: \ z = Re^{i\theta}, \quad \theta \in \Big[-\pi + \arccos\big((1+r)/R\big), \, \pi - \arccos\big((1+r)/R\big)\Big] \,;$$

– vertical segment from the point $-(1+r) + i\sqrt{R^2 - (1+r)^2}$ till the point $-(1+r)$;
– arc of circumference (clockwise) γ_r^-, where $\gamma_r: \ z = -1 + re^{i\theta}$, $\theta \in [-\pi, \pi]$;
– vertical segment from the point $-(1+r)$ till the point $-(1+r) - i\sqrt{R^2 - (1+r)^2}$.

It would be interesting to reproduce then mutatis mutandis the scheme of the proof of Proposition 4.1 and to compare arising in this way an integral representation for the numbers a_n with formula (4.15).

5. Concluding remarks

The obtained results can be applied in the initial problem on asymptotic behavior of the deviation $e - (1+x)^{1/x}$ as $x \to 0$. Indeed, on the base of the power expansion

$$(1+x)^{\frac{1}{x}} = e - \frac{e}{2}x + \frac{11e}{24}x^2 - \frac{7e}{16}x^3 + \ldots = e\sum_{n=0}^{\infty} (-1)^n a_n x^n, \qquad -1 < x < 1,$$

equivalent to (1.6) and a known property of Leibnitz type series (taking into consideration Proposition 2.2) we have a series of two-sided estimates

$$e\sum_{n=1}^{2N} (-1)^{n-1} a_n x^n < e - (1+x)^{\frac{1}{x}} < e\sum_{n=1}^{2N+1} (-1)^{n-1} a_n x^n, \tag{5.1}$$

acting for all $x \in (0, 1)$ for each $N \in \mathbb{N}$. For instance, choosing $x = 1/m$ and the first value $N = 1$, by (5.1) we obtain that

$$e\left(\frac{1}{2m} - \frac{11}{24m^2}\right) < e - \left(1 + \frac{1}{m}\right)^m < e\left(\frac{1}{2m} - \frac{11}{24m^2} + \frac{7}{16m^3}\right)$$

for all $m \in \mathbb{N}$, $m \geqslant 2$. We independently confirm that as $m = 1$ the written inequality holds. Therefore,

$$\frac{12m-11}{24m^2}e < e - \left(1+\frac{1}{m}\right)^m < \frac{24m^2 - 22m + 21}{48m^3}e, \qquad m \in \mathbb{N}. \tag{5.2}$$

Letting $x = 1/m$ in (5.1) and increasing the value of the parameter N, we can form arbitrarily large reserves of two-sided inequalities successively specifying (5.2).

We compare (5.2) with the known two-sided estimate

$$\frac{e}{2m+2} < e - \left(1+\frac{1}{m}\right)^m < \frac{e}{2m+1}, \qquad m \in \mathbb{N}, \tag{5.3}$$

taken from [1, Part I, Ch. 4, Sect. 2, Probl. 170]. By obvious inequalities

$$\frac{12m-11}{24m^2} > \frac{1}{2m+2}, \qquad m \geqslant 12,$$
$$\frac{24m^2-22m+21}{48m^3} < \frac{1}{2m+1}, \qquad m \geqslant 2,$$

new estimate (5.2) is better than classical estimate (5.3) as $m \geqslant 12$ and the upper bound in (5.2) is better than the upper bound in (5.3) for all natural values of the variable starting from $m = 2$. Moreover, (5.2) supports the asymptotic formula

$$e - \left(1+\frac{1}{m}\right)^m = \frac{e}{2m} - \frac{11e}{24m^2} + O\left(\frac{1}{m^3}\right), \qquad m \to \infty,$$

while (5.3) gives a weaker result

$$e - \left(1+\frac{1}{m}\right)^n = \frac{e}{2m} + O\left(\frac{1}{m^2}\right), \qquad m \to \infty.$$

In a recent note [3], there was provided an inequality

$$e - \left(1+\frac{1}{m}\right)^m \left(1+\frac{1}{2m+1}\right) < \frac{1}{2(2m+1)^2}$$

valid for all $m \in \mathbb{N}$. We rewrite it as

$$e - \left(1+\frac{1}{m}\right)^m < \frac{2e(2m+1)+1}{4(2m+1)(m+1)}, \qquad m \in \mathbb{N}. \tag{5.4}$$

It is easy to check that estimate (5.4) is stronger than the upper bound in (5.3) for each $m \in \mathbb{N}$ but is weaker than the upper bound in (5.2) as $m \geqslant 11$. We do not continue discussing of similar examples demonstrating the effectiveness of result (5.1).

Of course, a general asymptotic expansion

$$e - (1+x)^{\frac{1}{x}} = e \sum_{n=1}^{p} (-1)^{n-1} a_n x^n + O\left(x^{p+1}\right), \qquad x \to 0+, \quad p \in \mathbb{N},$$

holds and for calculating its coefficients we can use both recurrent rule (2.1) and representations (3.3), (4.1). It would be useful to specify order relation (4.9) by providing exact law of tending to zero for the quantity

$$\alpha_n \equiv a_n - 1/e = \frac{1}{\pi e} \int_0^1 \frac{\sin(\pi\tau)}{\tau^{1-\tau}(1-\tau)^\tau} \tau^n \, d\tau, \qquad n \to \infty.$$

A problem on asymptotic behavior of first differences (2.3) is of certain interest and according to (4.12), they can be written as the moments

$$d_n = \int_0^1 \psi(\tau)\, \tau^n \, d\tau, \qquad n \in \mathbb{N},$$

of a positive on (0, 1) function (4.13).

There are reasons to believe that bounding property (5.1) can be extended from the interval $x \in (0, 1)$ to the ray $x > 0$. A simplest arguing for such conjecture is the fact that the two-sided inequality

$$e - \frac{e}{2}x < (1+x)^{\frac{1}{x}} < e$$

holds for all $x > 0$. The issue is also important because of unexpected approximation effects discussed in [11], [12].

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