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HADAMARD TYPE OPERATORS IN SPACES OF HOLOMORPHIC FUNCTIONS ON A BALL

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Abstract. We study Hadamard type operators in the spaces of functions holomorphic in an open ball in $\mathbb{C}^N$ centered at the origin. These are continuous linear operators, for which each monomial is an eigenvector. We obtain a representation of Hadamard operators in the form of a multiplicative convolution. The proof of this representation employs essentially Fantappiè transformation realizing dual to the spaces of holomorphic functions and the holomorphy property of the characteristic function of a continuous linear operator in them. The applied method allows us to reduce the problem on representation of a Hadamard operator to the problem on holomorphic continuation of a function holomorphic at the point 0 into a given open ball in $\mathbb{C}^N$ with l_1-norm. We prove that the space of the Hadamard type operators from one mentioned space into another with the topology of the bounded convergence is linearly topologically isomorphic to the strong dual to the space of the germs of all functions holomorphic on a closed polydisk.

Keywords: Hadamard type operator, space of holomorphic functions.

Mathematics Subject Classification: 46E10, 47B91

1. Introduction

A natural interpretation of the Hadamard product of holomorphic functions in the operator theory is the notion of a Hadamard type operator. This is the name for continuous linear operators defined on a complex locally convex space containing all polynomials, for which all monomials are eigenfunctions. At present, there is a complete description of the Hadamard operators in the space of all functions holomorphic in an arbitrary simply connected domain in $\mathbb{C}$ [2], [3], [6], [18]. In the case of many complex variables a corresponding result, as a corollary of a more general description of the almost Hadamard type operators was obtained in [5] for the space of all entire functions in $\mathbb{C}^N$. For the spaces of functions holomorphic in domains in $\mathbb{C}^N$ different from $\mathbb{C}^N$ such description is absent. We mention a rather large number of such results for the spaces of real analytic, infinitely differentiable functions and distributions of both one and several variables [9]–[14], [19]–[23]. In the present work we study Hadamard type operators acting from the space $H(B_r)$ of all functions holomorphic in an open ball B_r of radius $r \in (0, \infty)$ centered at the point 0 in $\mathbb{C}^N$ into the space $H(B_R)$, $R \in (0, \infty)$. The main result of our paper is Theorem 2.1 in which we obtain a representation of Hadamard operators as a multiplicative convolution. A similar description holds also for in all earlier studied situations. An essential point in the proof of Theorem 2.1 is the employing of the Fantappiè transform, by means of which we realize a natural duality for the spaces of holomorphic functions of many variables. This allows us to reduce the problem on representing a Hadamard operator to the problem on holomorphic continuation of a function holomorphic in a vicinity of the point 0 into a given open ball in $\mathbb{C}^N$ with l_1-norm. We also employ the property of the holomorphy

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of a characteristic function of a continuous linear operator acting in the spaces of holomorphic functions.

The obtained representation is interpreted in terms of the Hadamard product of holomorphic functions. In Theorem 2.2, also by means of Fantappiè transform and the corresponding family of quotients defining it, we show that the space of Hadamard operators with a natural topology of bounded convergence is topologically isomorphic to a strong dual to the space of all functions holomorphic on a closed polydisk in $\mathbb{C}^N$ and to the Fréchet space holomorphic in an open ball in $\mathbb{C}^N$ with l_1-norm.

The facts from the theory of locally convex spaces, which we use here without citations, can be found in [8].

2. Description of operators of Hadamard type

2.1. Representation of Hadamard operator as multiplicative convolution. We fix $N \in \mathbb{N}$. We let

$$|z| := \left(\sum_{j=1}^N |z_j|^2\right)^{1/2}, \qquad \langle t, z\rangle := \sum_{j=1}^N t_j z_j, \qquad tz := (t_j z_j)_{j=1}^N, \qquad t, z \in \mathbb{C}^N,$$

$$L \cdot M := \{tz \,|\, t \in L,\ z \in M\}, \qquad uM := \{u\} \cdot M \quad \text{for sets} \quad L, M \subset \mathbb{C}^N, \quad u \in \mathbb{C}^N,$$

$$B_r := \{z \in \mathbb{C}^N \,|\, |z| < r\}, \qquad \overline{B}_r := \{z \in \mathbb{C}^N \,|\, |z| \leqslant r\},$$

$$D_r := \{z \in \mathbb{C}^N \,|\, |z_j| < r, 1 \leqslant j \leqslant N\}, \qquad \overline{D}_r := \{z \in \mathbb{C}^N \,|\, |z_j| \leqslant r, 1 \leqslant j \leqslant N\},$$

$$U_r := \{z \in \mathbb{C}^N \,|\, \sum_{j=1}^N |z_j| < r\}, \qquad \overline{U}_r := \{z \in \mathbb{C}^N \,|\, \sum_{j=1}^N |z_j| \leqslant r\}, \qquad 0 < r < +\infty,$$

$$P_N := \{1, \ldots, N\}.$$

In what follows we shall use the sets of points with non-zero coordinates. We define $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and for $Q \subset \mathbb{C}^N$ we let $Q^{(0)} := Q \bigcap (\mathbb{C}^*)^N$.

Remark 2.1. *For all $r, \rho \in (0, \infty)$*
(i) $B_{r\rho} = \overline{D}_r \cdot B_\rho$;
(ii) the set $U_{r\rho}$ can be represented as the product of balls; one of them can be thinned out:

$$U_{r\rho} = B_r \cdot \overline{B}_\rho = B_r^{(0)} \cdot \overline{B}_\rho;$$

(iii) $\overline{U}_{r\rho} = \overline{B}_r \cdot \overline{B}_\rho$.

We define the quotients $p_t(z) := \frac{1}{1-\langle t,z\rangle}$ for $t, z \in \mathbb{C}^N$ such that $\langle t, z\rangle \neq 1$. For a set $Q \subset \mathbb{C}^N$, the dual for Q set Q^* is defined by the identity

$$Q^* := \{t \in \mathbb{C}^N \,|\, \langle t, z\rangle \neq 1 \,\text{for each}\ z \in Q\}$$

[1], [17, Sect. 1], [7, Ch. 3, Sect. 12, 4], [15, Ch. IV, Sect. 4.7].

Example 2.1. *Let $r \in (0, \infty)$.*
(i) According to [7, Ch. 3, Sect. 12, 4, Prop. 2^0]

$$B_r^* = \overline{B}_{1/r}, \qquad \overline{B}_r^* = B_{1/r}.$$

(ii) The identities hold $D_r^* = \overline{U}_{1/r}$, $\overline{D}_r^* = U_{1/r}$.
Identities in (ii) can be confirmed by straightforward calculations.

For a domain Q in $\mathbb{C}^N$, by $H(Q)$ we denote the space of all functions holomorphic in Q with the topology of uniform convergence on compact sets in Q. The symbol $H(\overline{D}_r)$ with $r \in (0, \infty)$ stands for the space of all germs of the functions holomorphic on $\overline{D}_r$. Let $(s_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence of positive numbers such that $s_n \to r$ and $H_c(D_{s_n})$ be the Banach space of all holomorphic in D_{s_n} and continuous on $\overline{D}_{s_n}$ functions with a norm $\max_{z\in\overline{D}_{s_n}} |f(z)|$, $n \in \mathbb{N}$. Then $H(\overline{D}_r) = \bigcup_{n\in\mathbb{N}} H_c(D_{s_n})$ and in $H(\overline{D}_r)$ we introduce a topology of inductive limit of the spaces $H_c(D_{s_n})$, $n \in \mathbb{N}$, with respect to their natural embeddings in $H(\overline{D}_r)$. This topology is independent of the choice of the sequence $(s_n)_{n\in\mathbb{N}}$ as above. For a locally convex space E the symbol E' denotes a topological dual to E space and E'_b is a strong dual space for E.

The Fantappiè transform of a functional $\nu \in H(\mathbb{C}^N)'$ is defined by the identity

$$\Phi(\nu)(t) := \widetilde{\nu}(t) := \nu(p_t).$$

According to [15, Ch. 4, Sect. 4.7], a function $\widetilde{\nu}$ is holomorphic at the point 0, that is, in some neighbourhood of the point 0.

By [17, Thm. 2.2], [1], [15, Thm. 4.7.8], the following lemma is true.

Lemma 2.1. *For each $r > 0$, the transformation $\nu \mapsto \Phi(\nu)$ is a topological isomorphism of $H(\overline{D}_r)'_b$ onto $H(U_{1/r})$.*

We are going to prove a natural analog of a known result by G. Köthe [16, Thm. 19] for many complex variables; this result is about characteristic functions of continuous linear operators in the spaces of holomorphic functions of one complex variable. For $r, R \in (0,\infty)$ by $\mathcal{L}(H(B_r), H(B_R))$ we denote the space of all continuous linear operators from $H(B_r)$ into $H(B_R)$. For $A \in \mathcal{L}(H(B_r), H(B_R))$ we let

$$ch(A)(t,z) := A(p_t)(z), \qquad t \in \overline{B}_{1/r}, \qquad z \in B_R.$$

We introduce the orts $e^{(j)} := (\delta_{j,m})_{m=1}^N$, $j \in P_N$. In what follows for numbers $r, R \in (0,\infty)$ we fix strictly increasing sequences of positive numbers $(r_n)_{n\in\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}}$ such that $r_n \to r$ and $R_n \to R$.

Lemma 2.2. *Let $r, R \in (0,\infty)$. For each operator $A \in \mathcal{L}(H(B_r), H(B_R))$, its characteristic function $ch(A)$ possesses the following holomorphy property: for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that the function $ch(A)$ is holomorphic on $B_{1/r_m} \times B_{R_n}$.*

Proof. As in the one-dimensional case, this statement is a corollary of the continuity of A and of the properties of the function r_t. Since A is continuous from $H(B_r)$ into $H(B_R)$, then for each $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that the operator A can be uniquely continued to a continuous linear operator A_n from $H(B_{r_m})$ into $H(B_{R_n})$. We define a function $g_n(t,z) := A_n(p_t)(z)$, $t \in B_{1/r_m}$, $z \in B_{R_n}$. For all $t \in B_{1/r_m}$, $j \in P_N$, in the space $H(B_{r_m})$, that is, uniformly in u on each compact set B_{r_m}, there exists the limit

$$\lim_{h\in\mathbb{C},h\to 0} \frac{p_{t+he^{(j)}}(u) - p_t(u)}{h}$$

being equal to $\frac{u_j}{(1-\langle t,u\rangle)^2} =: s_{j,t}(u)$. Then for all $z \in B_{R_n}$, $t \in B_{1/r_m}$, $j \in P_N$ there exists a limit

$$\lim_{h\in\mathbb{C},h\to 0} \frac{g_n(t+he^{(j)},z) - g_n(t,z)}{h},$$

which is equal to $A_n(s_{j,t})(z)$. Therefore, for each $z \in B_{R_n}$ the function $g_n(t,z)$ is holomorphic in B_{1/r_m} in t. Moreover, for each $t \in B_{1/r_m}$ the function $g_m(t,z)$ is holomorphic in B_{R_m} in z. By Hartogs theorem g_n is holomorphic in $B_{1/r_m} \times B_{R_n}$. Since the function $ch(A)$ is equal to g_n on $\overline{B}_{1/r} \times B_{R_n}$, we continue $ch(A)$ holomorphically into $B_{1/r_m} \times B_{R_n}$. □

The main aim of the present work is to describe the operators of Hadamard type in the spaces of functions holomorphic in a ball. We let $f_\alpha(z) := z^\alpha := z_1^{\alpha_1}\cdots z_N^{\alpha_N}$, $\alpha \in \mathbb{N}_0^N$, $z \in \mathbb{C}^N$. An operator $A \in \mathcal{L}(H(B_r), H(B_R))$, $r, R \in (0,\infty)$, is called an operator of Hadamard type

(Hadamard operator) if all monomials f_α are its eigenfunctions, that is, for each $\alpha \in \mathbb{N}_0^N$ there exists $c_\alpha \in \mathbb{C}$, for which $A(f_\alpha) = c_\alpha f_\alpha$. By $\mathcal{L}_h(H(B_r), H(B_R))$ we denote the set of all operators of Hadamard type from $H(B_r)$ into $H(B_R)$. It is clear that $\mathcal{L}_h(H(B_r), H(B_R))$ is a subspace in $\mathcal{L}(H(B_r), H(B_R))$.

As usually, $\mathbb{C}[z]$ is the space of all polynomials over $\mathbb{C}$ of variables $z_1, \dots, z_N$. We let $\partial_j f := \frac{\partial f}{\partial z_j}$, $j \in P_N$; $|\alpha| := \sum_{j=1}^{N} \alpha_j$ for $\alpha \in \mathbb{N}_0^N$; the latter notation coincides with the notation $|z|$ for $z \in \mathbb{C}^N$ but this produces no ambiguity. The subscript of the functional indicates the variables, which respect to which it acts.

Theorem 2.1. *For $r, R \in (0, \infty)$ the following statements are equivalent:*
(i) $A \in \mathcal{L}_h(H(B_r), H(B_R))$.
(ii) There exists a functional $\varphi \in H(\overline{D}_{r/R})'$ such that $A(f)(z) = \varphi_t(f(tz))$ for all $z \in B_R$, $f \in H(B_r)$.
For each $A \in \mathcal{L}_h(H(B_r), H(B_R))$ a functional $\varphi \in H(\overline{D}_{r/R})'$, for which $A(f)(z) = \varphi_t(f(tz))$, $z \in B_R$, $f \in H(B_r)$, is unique.

Proof. The implication (ii)⇒(i) is proved in a standard way. We fix a strictly decreasing sequence of numbers $(s_n)_{n\in\mathbb{N}}$, for which $s_n \to r/R$. First of all, for $f \in H(B_r)$, $z \in B_R$ the function $\varphi_t(f(tz))$ is well-defined since there exists $k \in \mathbb{N}$, for which $s_k|z| < r$ and then $z\overline{D}_{s_k} \subset B_r$. Moreover, $\varphi_t(f(tz))$ is holomorphic at each point $z \in B_R$. Indeed, we choose k for z as above. The identity

$$f(t(z + he^{(j)})) - f(tz) = \int_0^1 (\partial_j f)(t(z + \xi h e^{(j)}))ht_j d\xi, \quad t \in \overline{D}_{s_k},$$
$$h \in \mathbb{C}, \quad |h| < \frac{r - s_k|z|}{s_k},$$

implies that there exists a uniform in $t \in \overline{D}_{s_k}$ limit

$$\lim_{h\to 0} \frac{f(t(z + he^{(j)})) - f(tz)}{h},$$

which is equal to $t_j(\partial_j f)(tz)$. This implies that the function $\varphi_t(f(tz))$ is differentiable (in the complex sense) in each variable in B_R and hence, it is holomorphic in B_R. By the closed graph theorem, the linear operator A is continuous from $H(B_r)$ into $H(B_R)$. Since $A(f_\alpha) = \varphi(f_\alpha)f_\alpha$, $\alpha \in \mathbb{N}_0^N$, then A is an operator of Hadamard type.

(i)⇒(ii): Let $A(f_\alpha) = c_\alpha f_\alpha$, $c_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{N}_0^N$. We define a functional

$$\varphi(f) := \sum_{\alpha\in\mathbb{N}_0^N} \frac{c_\alpha}{\alpha!} f^{(\alpha)}(0), \quad f \in H(\mathbb{C}^N). \tag{2.1}$$

Let us prove that the series in (2.1) converges absolutely for each function $f \in H(\mathbb{C}^N)$. Due to the continuity of the operator A from $H(B_r)$ into $H(B_R)$ there exist $m \in \mathbb{N}$ and a constant $C > 0$ such that

$$\max_{|z|\leqslant R_1} |A(f)(z)| \leqslant C \max_{|z|\leqslant r_m} |f(z)|, \quad f \in H(B_r).$$

For $f := f_\alpha$ we obtain:

$$|c_\alpha| \max_{|z|\leqslant R_1} |z^\alpha| \leqslant C \max_{|z|\leqslant r_m} |z^\alpha| \leqslant C r_m^{|\alpha|}.$$

This implies: $|c_\alpha| \left(\frac{R_1}{\sqrt{N}}\right)^{|\alpha|} \leqslant C r_m^{|\alpha|}$ and

$$|c_\alpha| \leqslant C \left(\frac{r_m\sqrt{N}}{R_1}\right)^{|\alpha|}, \quad \alpha \in \mathbb{N}_0^N. \tag{2.2}$$

If $f \in H(\mathbb{C}^N)$, then

$$\lim_{|\alpha|\to\infty} \left(|f^{(\alpha)}(0)|/\alpha!\right)^{1/|\alpha|} = 0,$$

and hence, series (2.1) converges absolutely. By the Banach-Steinhaus theorem, a linear functional φ is continuous on $H(\mathbb{C}^N)$. It follows from (2.2) that the series $\sum_{\alpha\in\mathbb{C}^N} c_\alpha z^\alpha$ converges absolutely in some polydisk D_ρ, $\rho > 0$.

We introduce an operator

$$S(f)(z) = \varphi_t(f(tz)), \qquad z \in \mathbb{C}^N, \qquad f \in H(\mathbb{C}^N).$$

If

$$f(z) = \sum_{\alpha\in\mathbb{N}_0^N} a_\alpha z^\alpha, \qquad z \in \mathbb{C}^N,$$

then for each $z \in \mathbb{C}^N$ the series $\sum_{\alpha\in\mathbb{N}_0^N} a_\alpha t^\alpha z^\alpha$ converges to $f(tz)$absolutely in t in $\mathbb{C}^N$ and uniformly on each compact set in $\mathbb{C}^N$. This is why

$$\varphi_t(f(tz)) = \sum_{\alpha\in\mathbb{N}_0^N} a_\alpha \varphi(f_\alpha) z^\alpha$$

for each $z \in \mathbb{C}^N$ and hence, $S(f) \in H(\mathbb{C}^N)$ for each function $f \in H(\mathbb{C}^N)$. The operator S is linear and continuous in $H(\mathbb{C}^N)$ and coincides with A on $\mathbb{C}[z]$, and therefore, on $H(\mathbb{C}^N)$. Let us show that the functional φ can be linearly and continuously continued on $H(\overline{D}_{r/R})$. In order to do this, in view of Lemma 2.1, we need to show that the function $\widetilde{\varphi}$ can be holomorphically continued into $U_{R/r}$.

There exists $M \geqslant r$ such that S can be uniquely continued to a continuous linear operator from $H(B_M)$ into $H(B_R)$ (we denote it again by S), while φ can be continued to a continuous linear functional on $H(\overline{D}_{M/R})$. The continued operator coincides with A on $H(B_M)$. By Statement (i) of this theorem, the operator $f \mapsto \varphi_t(f(tz))$ is linear and continuous from $H(B_M)$ into $H(B_R)$ and it coincides with S on $\mathbb{C}[z]$. Hence, $S(f)(z) = \varphi_t(f(tz))$, $z \in B_R$, $f \in H(B_M)$. By Lemma 2.2, for each $z \in B_R$ there exists $\rho(z) > 1/r$, for which the function $ch(A)(t,z)$ is holomorphic in t in $B_{\rho(z)}$. For each $t \in \overline{B}_{1/r}$ the function $ch(A)(t,z)$ is holomorphic in B_R in z. For each $z \in B_R$ there exists $\delta(z) \in (0, 1/M)$ such that if $|t| < \delta(z)$, then identities hold:

$$\widetilde{\varphi}(tz) = \varphi_u\left(\frac{1}{1-\langle tz, u\rangle}\right) = \varphi_u\left(\frac{1}{1-\langle t, uz\rangle}\right) = \mathrm{ch}(S)(t,z) = \mathrm{ch}(A)(t,z).$$

Thus, for each $z \in B_R^{(0)}$, the function $\widetilde{\varphi}$ can be holomorphically continued into a convex domain $zB_{\rho(z)}$ containing the point 0. Due to the principle of holomorphic continuation [4, Ch. 1, Sect. 6], $\widetilde{\varphi}$ can be holomorphically continued in $\bigcup_{z\in B_R^{(0)}} zB_{\rho(z)}$. Since $\bigcup_{z\in B_R^{(0)}} zB_{\rho(z)}$ contains the set $B_R^{(0)} \cdot \overline{B}_{1/r} = U_{R/r}$, see Remark 2.1, then $\widetilde{\varphi}$ is continued holomorphically in $U_{R/r}$. This implies Statement (ii).

Let $A \in \mathcal{L}_h(H(B_r), H(B_R))$ and $\varphi \in H(\overline{D}_{r/R})'$ be a functional, for which

$$A(f)(z) = \varphi_t(f(tz)), \qquad z \in B_R, \qquad f \in H(B_r).$$

Due to the identities $A(f_\alpha)=\varphi(f_\alpha)f_\alpha$, $\alpha\in\mathbb{N}_0^N$, and the density of the set of all polynomials in $H(B_r)$ such functional φ is unique. $\square$

Remark 2.2. *(i) Let φ be a functional defined by the identity (2.1). Then there exists $\varepsilon>0$ such that $\widetilde{\varphi}(z)=\sum\limits_{\alpha\in\mathbb{N}_0^N}\frac{c_\alpha|\alpha|!}{\alpha!}z^\alpha$ if $z\in D_\varepsilon$; the series converges absolutely in D_ε and the function $\widetilde{\varphi}$ is holomorphic in D_ε.*

(ii) Let us independently separate a statement established in the proof of the previous theorem. Let $A\in\mathcal{L}_h(H(B_r),H(B_R))$ and $A(f_\alpha)=c_\alpha f_\alpha$, $\alpha\in\mathbb{N}_0^N$. Then the series $\sum\limits_{\alpha\in\mathbb{N}_0^N}c_\alpha z^\alpha$ converges absolutely in some polydisk D_ρ, $\rho>0$, and a holomorphic at 0 function $\sum\limits_{\alpha\in\mathbb{N}_0^N}c_\alpha\frac{|\alpha|!}{\alpha!}z^\alpha$ is holomorphically continued into $U_{R/r}$.

Let us interpret the previous results in terms of the Hadamard product of holomorphic functions. By H_0 we denote the space of the germs of all functions holomorphic at the point 0. For functions

$$b(z)=\sum_{\alpha\in\mathbb{N}_0^N}b_\alpha z^\alpha,\qquad c(z):=\sum_{\alpha\in\mathbb{N}_0^N}c_\alpha z^\alpha$$

from H_0 their Hadamard product is defined by the identity

$$(b*c)(z):=\sum_{\alpha\in\mathbb{N}_0^N}b_\alpha c_\alpha z^\alpha.$$

If the series $\sum\limits_{\alpha\in\mathbb{N}_0^N}b_\alpha z^\alpha$ converges absolutely in the polydisk D_r, and the series $\sum\limits_{\alpha\in\mathbb{N}_0^N}c_\alpha z^\alpha$ converges absolutely in D_ρ, where $r,\rho>0$, then the series $\sum\limits_{\alpha\in\mathbb{N}_0^N}b_\alpha c_\alpha z^\alpha$ converges absolutely in $D_{r\rho}$, and hence, the function $b*c$ is holomorphic in $D_{r\rho}$. It follows from the Cauchy integral formula that if $f_n\in H(D_r)$, $n\in\mathbb{N}$, and $f_n\to0$ in $H(D_r)$, then for each $c\in H(D_\rho)$ we also have $f_n*c\to0$ in $H(D_{r\rho})$.

Corollary 2.1. *Let $r,R\in(0,\infty)$, a function $c(z):=\sum\limits_{\alpha\in\mathbb{N}_0^N}c_\alpha z^\alpha$ be holomorphic at the point 0 (the series converges absolutely in some polydisk D_ε, $\varepsilon>0$). The following statements are equivalent:*

(i) For each holomorphic in B_r function $b(z)=\sum\limits_{\alpha\in\mathbb{N}_0}b_\alpha z^\alpha$ the Hadamard product $b*c$ is holomorphically continued into B_R.

(ii) The function $\sum\limits_{\alpha\in\mathbb{N}_0^N}\frac{c_\alpha|\alpha|!}{\alpha!}z^\alpha$ is holomorphically continued into $U_{R/r}$.

Proof. (i)⇒(ii): Let A be an operator mapping a function $b\in H(B_r)$ into a holomorphic continuation $b*c$ into B_R. Due to the uniqueness of the holomorphic continuation the operator A is well-defined. It is clear that the operator A from $H(B_r)$ into $H(B_R)$ is linear. Let us show that the graph of A is closed. Let $f_n\in H(B_r)$, $n\in\mathbb{N}$, $f_n\to0$ in $H(B_r)$ and $A(f_n)\to g$ in $H(B_R)$. There exists $\rho\in(0,R)$ such that $f_n*c\to0$ in $H(D_\rho)$. Hence, $g\equiv0$ on $\overline{D}_\rho$ and this is why $g\equiv0$ in B_R. By the theorem on closed graph, $A\in\mathcal{L}(H(B_r),H(B_R))$. Since $A(f_\alpha)=c_\alpha f_\alpha$ for each $\alpha\in\mathbb{N}_0^N$, the operator A is Hadamard. By Remark 2.2 statement (ii) holds.

(ii)⇒ (i): We observe that the series $\sum\limits_{\alpha\in\mathbb{N}_0^N}\frac{c_\alpha|\alpha|!}{\alpha!}z^\alpha$ converges absolutely in some polydisk D_δ, $\delta>0$, and the function $d(z)=\sum\limits_{\alpha\in\mathbb{N}_0^N}\frac{c_\alpha|\alpha|!}{\alpha!}z^\alpha$ is holomorphic in D_δ. Let d be holomorphically continued into $U_{R/r}$. By Lemma 2.1, the functional $\varphi:=\Phi^{-1}(d)$ is linear and continuous on

$H(\overline{D}_{r/R})$, while by Theorem 2.1 the operator $A(f)(z) := \varphi_t(f(tz))$ is linear and continuous from $H(B_\rho)$ into $H(B_{(\rho R)/r})$ for each $\rho > 0$. We take a function $b(z) = \sum\limits_{\alpha\in\mathbb{N}_0^N} b_\alpha z^\alpha$ holomorphic in B_r. There exists $\rho \in (0, r)$, for which the latter series converges absolutely in the space $H(B_\rho)$. Treating the operator A as that from $H(B_\rho)$ into $H(B_{(\rho R)/r})$ (we denote it by A_0), we obtain that

$$A_0(b)(z) = A_0\left(\sum_{\alpha\in\mathbb{N}_0} b_\alpha f_\alpha\right)(z) = \sum_{\alpha\in\mathbb{N}_0} b_\alpha A_0(f_\alpha) z^\alpha = (b * c)(z), \quad z \in B_{(\rho R)/r}.$$

A holomorphic in B_R function $A(b)$ (now we treat A as an operator from $H(B_r)$ into $H(B_R)$) is a holomorphic continuation $b * c$ into B_R. □

2.2. On topological isomorphism. Let $r, R \in (0, \infty)$. The symbol $\mathcal{L}_h(H(B_r), H(B_R))_b$ denotes the space $\mathcal{L}_h(H(B_r), H(B_R))$ with the topology of uniform convergence on the family $\mathcal{B}(H(B_r))$ of all bounded subsets in $H(B_r)$. The set of seminorms

$$q_{T,n}(A) := \sup_{f\in T} \max_{|z|\leqslant R_n} |A(f)(z)|,\ T \in \mathcal{B}(H(B_r)), \quad n \in \mathbb{N},$$

is a fundamental system of continuous seminorms in $\mathcal{L}_h(H(B_r), H(B_R))_b$. We introduce the set $T_0 := \{p_u \mid u \in \overline{B}_{1/r}\}$. Since

$$\sup_{|z|\leqslant r_n} |p_u(z)| = \sup_{|z|\leqslant r_n} \frac{1}{|1 - \langle u, z\rangle|} \leqslant \frac{1}{1 - r_n/r}$$

for all $u \in \overline{B}_{1/r}$, $n \in \mathbb{N}$, the set T_0 is bounded in $H(B_r)$.

Let $(\delta_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence of positive numbers such that $\delta_n \to R/r$. The sequence of norms $\max\limits_{z\in\overline{U}_{\delta_n}} |f(z)|$, $n \in \mathbb{N}$, defines the topology of the Fréchet space $H(U_{R/r})$. We fix a strictly decreasing sequence of numbers $(s_n)_{n\in\mathbb{N}}$ such that $s_n \to r/R$ and we let

$$\|f\|_n := \max_{z\in\overline{D}_{s_n}} |f(z)|, \quad f \in H_c(D_{s_n}), \quad n \in \mathbb{N}.$$

We also define

$$\|\varphi\|_n^* := \sup_{\|f\|_n\leqslant 1} |\varphi(f)|, \quad \varphi \in H(\overline{D}_{r/R})', \quad n \in \mathbb{N}.$$

A sequence $(\|\cdot\|_n^*)_{n\in\mathbb{N}}$ is a fundamental sequence of continuous seminorms in the Fréchet space $H(\overline{D}_{r/R})'_b$. For $\varphi \in H(\overline{D}_{r/R})'$ we let $A_\varphi(f)(z) := \varphi_t(f(tz))$, $z \in B_R$, $f \in H(B_r)$.

Theorem 2.2. *(i) The mapping $\chi(\varphi) := A_\varphi$ is a linear topological isomorphism of $H(\overline{D}_{r/R})'_b$ onto $\mathcal{L}_h(H(B_r), H(B_R))_b$.*

(ii) The space $\mathcal{L}_h(H(B_r), H(B_R))_b$ is linearly topologically isomorphic to $H(U_{R/r})$.

Proof. (i): By Theorem 2.1, the linear mapping χ is bijective. We fix a set $T \in \mathcal{B}(H(B_r))$ and $n \in \mathbb{N}$. There exist $k \in \mathbb{N}$ and $m \in \mathbb{N}$, for which $R_n s_k \leqslant r_m$, and therefore, $B_{R_n} \cdot \overline{D}_{s_k} = B_{R_n s_k} \subset B_{r_m}$. Then for each $\varphi \in H(\overline{D}_{r/R})'$

$$q_{T,n}(A_\varphi) = \sup_{f\in T} \max_{|z|\leqslant R_n} |\varphi_t(f(tz))| \leqslant \|\varphi\|_k^* \sup_{f\in T} \sup_{|z|\leqslant R_n} \max_{t\in\overline{D}_{s_k}} |f(tz)| \leqslant \left(\sup_{f\in T} \max_{|u|\leqslant r_m} |f(u)|\right) \|\varphi\|_k^*.$$

This implies that $\chi : H(\overline{D}_{r/R})'_b \to \mathcal{L}_h(H(B_r), H(B_R))_b$ is continuous.

Let us show that the mapping $\chi^{-1} : \mathcal{L}_h(H(B_r), H(B_R))_b \to H(\overline{D}_{r/R})'_b$ is continuous. We fix $k \in \mathbb{N}$. Since by Lemma 2.1 the Fantappiè transform Φ is a topological isomorphism $H(\overline{D}_{r/R})'_b$ onto $H(U_{R/r})$, then there exist $m \in \mathbb{N}$ and a constant $C > 0$, for which

$$\|\varphi\|_k^* \leqslant C \max_{v\in\overline{U}_{\delta_m}} |\widetilde{\varphi}(v)|, \qquad \varphi \in H(\overline{D}_{r/R})'.$$

We choose $n \in \mathbb{N}$ such that $\delta_m \leqslant R_n/r$. Since

$$q_{T_0,n}(A_\varphi) = \sup_{|u|\leqslant 1/r} \max_{|z|\leqslant R_n} \left|\varphi_t\left(\frac{1}{1-\langle u, tz\rangle}\right)\right|$$
$$= \sup_{|u|\leqslant 1/r} \max_{|z|\leqslant R_n} \left|\varphi_t\left(\frac{1}{1-\langle t, zu\rangle}\right)\right| = \sup_{|u|\leqslant 1/r} \max_{|z|\leqslant R_n} |\widetilde{\varphi}(zu)|$$

for each $\varphi \in H(\overline{D}_{r/R})'$ and $\overline{B}_{1/r} \cdot \overline{B}_{R_n} = \overline{U}_{R_n/r} \supset \overline{U}_{\delta_m}$, then for each $\varphi \in H(\overline{D}_{r/R})'$

$$q_{T_0,n}(A_\varphi) \geqslant \max_{v\in \overline{U}_{\delta_m}} |\widetilde{\varphi}(v)|.$$

Hence, for each $\varphi \in H(\overline{D}_{r/R})'$ the inequality holds

$$\|\varphi\|_k^* \leqslant C q_{T_0,n}(A_\varphi).$$

Thus, the mapping $\chi^{-1} : \mathcal{L}_h(H(B_r), H(B_R))_b \to H(\overline{D}_{r/R})'_b$ is continuous.

(ii): A topological isomorphism of $\mathcal{L}_h(H(B_r), H(B_R))_b$ onto $H(U_{R/r})$ is the map $\Phi\chi^{-1}$. □

Since in the proof of the continuity of χ^{-1} in the previous theorem it is sufficient to choose one bounded in $H(B_r)$ set T_0, we arrive at the corollary.

Corollary 2.2. *The space $\mathcal{L}_h(H(B_r), H(B_R))_b$ is a Fréchet space with a fundamental sequence of continuous prenorms $q_{T_0,n}$, $n \in \mathbb{N}$.*

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