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ON INTEGRAL EQUATIONS OF FREDHOLM KIND IN BOHR SPACE OF ALMOST PERIODIC FUNCTIONS

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Abstract. In the present work we consider a question on extending the notion of the Fredholm integral equation or second kind integral equation, which allows one to consider the issue on existence of solutions in the space of almost periodic functions. Almost periodic functions are defined on the entire line. This is why it seems difficult to describe them by some characteristics on finite intervals.

The Fredholm equations are known to be closely related with first order differential equations. In some particular cases there were posed the questions on finding the solutions in various classes of almost periodic functions. In some known cases there are no solutions in the Bohr class for such equations with almost periodic coefficients.

There are known examples of almost periodic functions (in the Besicovitch sense), which can not be solutions for a rather wide class of differential equations. It is natural to expect that in the general case the integral equations are also not solvable in Bohr class of almost-periodic functions. This is why a more specific approach is needed for the problem in the space of almost-periodic functions.

Keywords: almost periodic functions, Bohr classes, Fredholm equation, integral equation, differential equation.

Mathematics Subject Classification:45B05

1. Introduction

In the present work we study the issue on an extension of the notion of the Fredholm integral equation, or second kind integral equation

$$\varphi(x) = f(x) + \lambda \int_a^b K(x,\xi)\varphi(\xi)d\xi \tag{1.1}$$

such that one could state that modified equation (1.1) is solvable in the Bohr class of almost periodic functions; here for simplicity the parameter λ takes real values. It is natural to impose certain condition on all functions involved in (1.1). The integration limits are to be chosen appropriately since the almost periodic functions are defined on the entire real line.

Before formulating the issue, we first make certain natural comments. It is known that equations of form (1.1) are closely related with first order differential equation

$$y' = f(x,y).$$

In the theory of almost periodic functions, various particular cases of this equations were considered and the questions on finding its solutions in various classes of almost periodic functions were treated and the solvability was studied. There are known cases, when in the Bohr

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class there are no solutions even for equations of form [1]

$$y' + A(x)y = f(x),$$

where the functions $f(x)$ and $A(x)$ are almost periodic.

There are well-known examples of Besicovitch almost periodic functions which can not solve a rather wide class of differential equations. For instance, in the well-known work [8], S.M. Voronin proved that the Riemann zeta function $y = y(t) = \zeta(\sigma + it)$, $t \in \mathbb{R}$, with $0.5 < \sigma < 1$ can not solve a differential equation of form

$$F(y, y', \dots, y^{(n)}) = 0,$$

where a function $F(x_1, x_2, \dots, x_{n+1})$ is continuous. The function $y(t) = \zeta(\sigma + it)$, $t \in \mathbb{R}$, is almost periodic in the Besicovitch sense, while for $\sigma > 1$ it is almost periodic in the Bohr sense. Indeed, there are many examples of such kind.

It is natural to expect that in a general case integral equations of form (1.1) can not be solvable in the usual sense in the Bohr class of almost periodic functions. Since the almost periodic functions are defined on the entire real line, instead of equation (1.1) with fixed a and b, we can consider the equation with a growing parameter T letting $a = 0$, $b = T > 0$:

$$\varphi(x) = f(x) + \lambda \int_0^T K(x, \xi)\varphi(\xi)d\xi.$$

However, in the space of the almost periodic functions a more specific approach to the problem is needed since we seek the functions defined on the entire real line. We replace the integral in the right hand side of (1.1) by an appropriate mean value and we write the integral equation as follows:

$$\varphi(x) = f(x) + \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x, \xi)\varphi(\xi)d\xi. \tag{1.2}$$

Speaking about integral Fredholm equation in the Bohr space of almost periodic function (and also in more general sense), we shall mean equation of form (1.2). The kernel $K(x, \xi)$ of the equation is also supposed to belong to the space of almost periodic functions of two variables in the corresponding class. The limit in the right hand side in (1.2) can be replaced by the upper limit. The main aim of the present paper is to prove that equation (1.2) has a solution for rather natural assumptions for the kernel and function $f(x)$. We shall apply known methods of the Fredhold theory with some modifications related with the features of the considered space. We also note that here we consider the case of a symmetric kernel. We shall establish a relation between equations (1.1) and (1.2) and in this way we shall reduce the question on studying equation (1.2) to similar questions for some equation of form (1.1).

2. Preliminary fact and lemmata

We first recall the definition of almost periodic Bohr functions, see [9]. Let $f(x)$ be a real function defined on the entire real line; it is also possible to consider an arbitrary function $f : \mathbb{R} \to X$, where X is some normed space.

A number τ is called an ε-almost period if for each $x \in \mathbb{R}$ the inequality holds

$$|f(x + \tau) - f(x)| \leqslant \varepsilon.$$

Let $E \in \mathbb{R}$ be some countable subset in $\mathbb{R}$. The set E is called relatively dense if there exists $L > 0$ such that each interval of form

$$a < x < a + L, \quad a \in \mathbb{R}$$

of the length L contains at least one point from the set E.

A function $f(x)$ continuous on the entire real line $\mathbb{R}$ is called almost periodic if for each $\varepsilon > 0$ the set of all ε-almost periods of the function $f(x)$ is relatively dense in $\mathbb{R}$.

There exists a close relation between almost periodic and periodic functions of many variables.

Definition 2.1. *Let $F_1, F_2, \ldots$ be a sequence of continuous periodic functions $F_k : \mathbb{R}^n \to \mathbb{R}$. If $F_k \to F$ uniformly in $x \in \mathbb{R}^n$, then the limit of this sequence, the function F, is called a limit periodic function.*

In view of the definition it is clear that almost periodic functions are continuous. In the same way one defines a limit periodic function of countably many variables. We denote by $\mathbb{R}^\infty$ the set of all sequences of form $(\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots)$. Each function of finitely many variables of form $\Phi : \mathbb{R}^m \to \mathbb{R}$ can be treated as a function $\Phi' : \mathbb{R}^\infty \to \mathbb{R}$ of countably many variables by letting $\alpha_k = 0$ for all $k > m$.

Definition 2.2. *Assume that we are given a sequence of continuous periodic functions $F_k : \mathbb{R}^{m_k} \to \mathbb{R}$, $m_1 < m_2 < \cdots$, $m_k \to \infty$. If $F_k \to F$ uniformly in $x \in \mathbb{R}^\infty$, then the limit of this sequence, the function $F : \mathbb{R}^\infty \to \mathbb{R}$, is called a limit periodic function of countably many variables.*

It is convenient to introduce a metric in a space $\mathbb{R}^\infty$ of all real sequences, which makes this space a metric one. This metric is known as Tikhonov metric. Let $x, y \in \mathbb{R}^\infty$ and $x = (x_1, x_2, \ldots, x_k, \ldots)$. We introduce the Tikhonov metric by the identity [14]:

$$d(x, y) = \sum_{n=1}^{\infty} e^{1-n} |x_n - y_n| .$$

Without loss of generality we can extend all above definitions related with the infinite-dimensional case to the introduced metric space.

Substituting equal values $x = x_1 = \ldots = x_m$ instead of independent variables $x_1, \ldots, x_m$, we obtain a diagonal function $f(x) = F(x, \ldots, x)$ (or a function on the principal diagonal of the space). In the same way we define a diagonal function in the infinite-dimensional space. One of the main results by Bohr on almost periodic functions is as follows:

Lemma 2.1 (H. Bohr)**.** *Each almost periodic function is a diagonal function of some limit periodic function of finitely many or countably many variables.*

We make several remarks on almost periodic functions of two variables.

A pair of real numbers (τ, η) is called a pair of ε-almost periods for the function $f(x, y)$ if for each pair of real numbers (x, y) the relations hold:

$$|f(x + \tau, y) - f(x, y)| \leqslant \varepsilon, \qquad |f(x, y + \eta) - f(x, y)| \leqslant \varepsilon.$$

Definition 2.3. *A continuous on the entire plane $(x, y) \in \mathbb{R} \times \mathbb{R}$ function is called almost periodic if for each $\varepsilon > 0$ there exists $l = l(\varepsilon) > 0$ such that for each pair $(x, y) \in \mathbb{R} \times \mathbb{R}$ there exists at least one pair (τ, η) of ε-almost periods in the open square $(x, x+l) \times (y, y+l) \in \mathbb{R} \times \mathbb{R}$.*

It is well-known that an almost periodic function is uniformly approximated by trigonometric polynomials in the entire plane. A corresponding result reads as follows, see [3, Ch. 1, Sect. 12, Thm. 10].

Lemma 2.2. *Let a continuous function $f(x, y)$ be almost periodic in $\mathbb{R} \times \mathbb{R}$. Then for each $\varepsilon > 0$ we can find natural N_ε and L_ε such that*

$$\left| f(x, y) - \sum_{n=1}^{N_\varepsilon} \sum_{r=1}^{L_\varepsilon} a_{n,r} e^{2\pi i(\tau_n x + \mu_r y)} \right| < \varepsilon$$

for all $(x, y) \in \mathbb{R} \times \mathbb{R}$; here $a_{n,r}$, τ_n, μ_r are real numbers.

In particular, this lemma implies one more lemma.

Lemma 2.3. *Each almost periodic function $f(x,y)$ is a diagonal function of some limit periodic function of two finite or infinite system of variables.*

Let $\varphi(x_1,x_2,\ldots,x_n;y_1,y_2,\ldots,y_m)$ is a limit periodic function of two system of variables. This means that

$$\varphi(x_1,x_2,\ldots,x_n;y_1,y_2,\ldots,y_m)=\lim_{k\to\infty}G_k(x_1,x_2,\ldots,x_n;y_1,y_2,\ldots,y_m)$$

uniformly in $\mathbb{R}^{n+m}$, and $G_k(x_1,x_2,\ldots,x_n;y_1,y_2,\ldots,y_m)$ are periodic function. Then, taking the value of the function G_k on the principal diagonals of the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, we obtain an almost periodic function of two variables.

In the infinite-dimensional case the above said fact can be reformulated as follows: there exist two systems of variables $x_1,\ldots,x_m,\ldots$ and $y_1,\ldots,y_s,\ldots$ and a sequence of periodic in each variable functions $G_k(x_1,x_2,\ldots;y_1,y_2,\ldots)$, $k=1,\,2,\ldots$ such that

$$G(x_1,x_2,\ldots;y_1,y_2,\ldots)=\lim_{k\to\infty}G_k(x_1,x_2,\ldots;y_1,y_2,\ldots),$$

uniformly in the above introduced metric and

$$K(x,\xi)=G(x,x,\ldots;\xi,\xi,\ldots).$$

In the present work we consider a simplest case of symmetric kernel when both systems of variables are finite. Then the function

$$G(x_1,\ldots,x_m,y_1,\ldots,y_m)$$

is limit periodic of $2m$ variables and the periods in the pairs of the variables x_i and y_i coincide.

In work [9], H. Bohr studied mean values of form (1.2) on the base of a Kronecker theorem on uniform distribution (mod 1) of some curves in a multi-dimensional unit cube ([12]). The Kronecker theorem is as follows [7, App., Sect. 8, Thm. 1].

Lemma 2.4. *Let real numbers $\alpha_1,\alpha_2,\ldots,\alpha_N$ be linearly independent over the field of rational numbers, γ be some rectangular domain in the unit N-dimensional cube. Let $I_\gamma(T)$ be the measure of points $t\in(0,T)$, for which*

$$(\alpha_1 t,\alpha_2 t,\ldots,\alpha_N t)\in\gamma(mod 1).$$

Then

$$\lim_{T\to\infty}\frac{I_\gamma(T)}{T}=\Gamma,$$

where Γ stands for the volume of the domain γ.

We shall need the following definition from [10].

Definition 2.4. *A family F of functions f defined on a subset E of metric space X is called equicontinuous on E if for each $\varepsilon>0$ there exists $\delta>0$ such that $|f(x)-f(y)|<\varepsilon$ as $d(x,y)<\delta$, $x\in E$, $y\in E$, $f\in F$; here d denotes the distance in X.*

It is clear that each function in an equicontinuous family is uniformly continuous [10]. The following lemma was proved in [7, App., Sect. 8, Lm. 1].

Lemma 2.5. *Let a curve $\gamma(t)$ be uniformly distributed modulo 1 in the space $\mathbb{R}^n$. Let D be a closed subdomain in the unit cube measurable in the Jordan sense, Φ be a family of complex-valued continuous functions defined on D. If Φ is uniformly bounded and equicontinuous, then*

$$\lim_{T\to\infty}T^{-1}\int\limits_0^T f(\{\gamma(t)\})dt=\int\limits_D f\,dx_1\ldots dx_N$$

uniformly in $f \in \Phi$, where in the left hand side the integration is taken over $t \in (0,T)$, for which

$$\gamma(t) \in D(mod 1)$$

and

$$\{\gamma(t)\} = (\{\gamma_1(t)\}, \dots, \{\gamma_N(t)\}).$$

In particular, Lemma 2.5 is true for an arbitrary continuous function f on D. The following lemma is useful for applications [10, Ch. 7, Thms. 7.24, 7.25].

Lemma 2.6. *Let K be a compact set.*

a) if $\{f_n\}$ is a uniformly converging sequence of functions continuous on K, then $\{f_n\}$ is equicontinuous on K;

b) if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then $\{f_n\}$ contains a uniformly converging subsequence and is uniformly bounded on K.

The following lemma is known as Hurwitz theorem [13, Ch. 3, Thm. 3.45].

Lemma 2.7. *Let $f_1(z), f_2(z), \dots$ be a sequence of functions analytic in some domain D enveloped by a simple closed contour and let $f_n(z) \to f(z)$ uniformly in D. Assume that the function $f(z)$ is not identically zero. Then a point z_0 lying inside D is a zero of the function $f(z)$ if and only if in D there exists a sequence of points z_1, $z_2 \dots$ converging to z_0 such that z_n is the zero of the function $f_n(z)$ as $n > n_0 = n_0(z_0)$.*

3. Main results

The main results of the present paper are summarized in two theorems. To formulate and prove these theorems, we first make some preliminary remarks. Let the kernel of the integral, or, more precisely, limit integral equation (1.2) is a uniform limit periodic function of two variables in the Bohr sense. Suppose that equation (1.2) possesses an almost periodic solution $\varphi(x)$. Substituting into (1.2), we consider the identity:

$$\varphi(x) - f(x) = \lambda \lim_{T\to\infty} \frac{1}{T} \int_0^T K(x,\xi)\varphi(\xi)d\xi. \tag{3.1}$$

The right hand side of the identity is almost periodic in x and has almost periods coinciding with the almost periods of the function $K(x,\xi)$ in x (and in ξ by the symmetry). This is why, representing $K(x,y)$ as a diagonal function of some limit periodic function, we can write:

$$K(x,y) = \lim_{T\to\infty} K_m(x,x,\dots,x;y,y,\dots,y), \tag{3.2}$$

where $K_m(x_1,x_2,\dots,x_{s(m)};y_1,y_2,\dots,y_{s(m)})$ is a periodic function with periods $\lambda_1,\dots,\lambda_{s(m)}$, $\lambda_1,\dots,\lambda_{s(m)}$. Since the functions $K_m(x_1,x_2,\dots,x_{s(m)};y_1,y_2,\dots,y_{s(m)})$ are periodic with the above mentioned periods and continuous, they have the following Fourier expansions:

$$\begin{aligned} &K_m(x_1,x_2,\dots,x_{s(m)};y_1,y_2,\dots,y_{s(m)}) \\ &\sim \sum_{i_1=-\infty}^{\infty}\cdots\sum_{i_{s(m)}=-\infty}^{\infty}\sum_{j_1=-\infty}^{\infty}\cdots\sum_{j_{s(m)}=-\infty}^{\infty} a_{i_1\cdots j_m} \\ &\times e^{2\pi i(i_1\theta_1 x_1+\cdots+i_{s(m)}\theta_{s(m)}x_{s(m)}+j_1\theta_1 y_1+\cdots+j_{s(m)}\theta_{s(m)}y_{s(m)})}, \end{aligned} \tag{3.3}$$

where $\theta_1,\dots,\theta_{s(m)}$ are inverse values of the periods.

We proceed to the diagonal function in (3.3) substituting equal values $x = x_1 = \ldots = x_{s(m)}$ and $y = y_1 = \ldots = y_{s(m)}$. We let

$$K_m(x,x,\dots,x;y,y,\dots,y) = K_m(x_1,x_2,\dots,x_{s(m)};y_1,y_2,\dots,y_{s(m)})|_{x=x_1=\cdots=x_{s(m)},y=y_1=\cdots=y_{s(m)}}.$$

In the same way we define functions $H_m(x, x, \ldots, x)$. Returning back to (3.1), we replace the kernel and solution of the equation by its expressions via limit periodic functions introduced above:

$$K(x, \xi) = G(x, x, \ldots; \xi, \xi, \ldots), \qquad \varphi(\xi) = (\xi, \xi, \ldots),$$

where

$$G(x, x, \ldots; \xi, \xi, \ldots) = \lim_{T\to\infty} G_m(x, x, \ldots, x; \xi, \xi, \ldots, \xi),$$
$$\varphi(\xi) = (\xi, \xi, \ldots) = \lim_{T\to\infty} H_m(\xi, \xi, \ldots, \xi).$$

The right hand sides of these identities involve limit periodic functions of $2s(m)$ and $s(m)$ variables, respectively. We obtain the integral

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T G(x, x, \ldots; \xi, \xi, \ldots) H(\xi, \xi, \ldots) d\xi.$$

Since two functions under the integral are uniform limits of periodic functions, we can replace the limit periodic functions by periodic ones with an error uniformly tending to zero as $m \to \infty$:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T G_m(x, x, \ldots; \xi, \xi, \ldots) H_m(\xi, \xi, \ldots) d\xi. \tag{3.4}$$

As it is known [2], the reciprocals $\theta_1, \ldots, \theta_{s(m)}$ of the periods can be assumed to be linearly independent; otherwise we express in the rational way one via the others, we can replace the periodic function K_m by another one depending on less number of independent variables, which on the principal diagonal gives the same periodic function. We consider a curve $\gamma(t)$ defined by the identity $\gamma(t) = \left(2\pi t\lambda_1^{-1}, \ldots, 2\pi t\lambda_{s(m)}^{-1}\right)$. Then latter integral (3.1) is also the integral in Lemma 2.5 taken over this curve uniformly distributed in the cube $\Delta_{s(m)} = \Delta$ with sides equalling to 1. Applying Lemma 2.5, we obtain:

$$\lim_{T\to\infty} T^{-1} \int_0^T f(\{\gamma(t)\}) dt = \int_\Delta f dy_1 \ldots dy_{s(m)},$$

where $N = n$, $f(y_1, \ldots, y_{s(m)}) = G_m(x_1, \ldots, x_{s(m)}; y_1, \ldots, y_{s(m)}) H_m(y_1, \ldots, y_{s(m)})$. In the first two $s(m)$ components the values of the variables are arbitrary and if it is needed, we can replace them by the value x to obtain the vector $(x, \ldots, x)$. Thus, our integral equation turns into a usual Fredholm equation:

$$H_k(\bar{x}) - f_k(\bar{x}) = \lambda \int_\Delta G_k(\bar{x}, \bar{\xi}) H_k(\bar{\xi}) d\bar{\xi}, \qquad \bar{x} \in \Delta. \tag{3.5}$$

Therefore, assuming the existence of solutions, we arrive at equation (3.5). This is why the solvability of equation (3.5) is a necessary condition for the existence of solution to equation (1.2).

Theorem 3.1. *Let equation* (3.5) *be solvable for all considered k and has a periodic solution with periods* 1 *in all variables. If the sequence of solutions is uniformly bounded in the unit cube, then equation* (1.2) *possesses an almost periodic solution.*

Proof. For each k, we continue the found solutions $H_k(\bar{x})$ of equation (3.5) periodically in all variables of the vector $\bar{x} \in \Delta$ and obtain functions defined in the entire space $\mathbb{R}^{s(m)}$; the obtained functions are again denoted by $H_k(\bar{x})$. We make a permutation replacing x_i by the number $\theta_i x_i$. Then we obtain a sequence of functions

$$Y_k(\bar{x}) = H_k(\theta_1 x_1, \dots, \theta_{s(m)} x_{s(m)}),$$

which are continuous and have periods $\lambda_1, \dots, \lambda_{s(m)}$. Let us prove that this sequence converges uniformly to some limit periodic function $Y(\bar{x})$ and the corresponding diagonal function is almost periodic and solves equation (1.2). Since there exists a uniform limit

$$\tilde{K}(\bar{x}, \bar{y}) = \lim_{T\to\infty} G_m(x_1, x_2, \dots, x_{s(m)}; y_1, y_2, \dots, y_{s(m)}), \quad \bar{x}, \bar{y} \in \Delta,$$

by Lemma 2.1 this sequence is equicontinuous. Let us prove that the sequence of solutions is also equicontinuous. By conditions, the set of solutions is uniformly bounded:

$$|Y_k(\bar{x})| \leqslant L, \quad \bar{x} \in \Delta, \quad k \geqslant 1,$$

with some constant L. Then

$$|H_k(\bar{x}) - H_k(\bar{x}')| \leqslant |f_k(\bar{x}) - f_k(\bar{x}')| + L\,|\lambda| \int_{\Delta} \left|G_k(\bar{x}, \bar{\xi}) - G_k(\bar{x}', \bar{\xi})\right| d\bar{\xi} \leqslant \varepsilon(1 + L\,|\lambda|),$$

as soon as $|\bar{x} - \bar{x}'| < \delta$. This relation and the arbitrariness of ε implies the needed statement. Then by Lemma 2.6 the sequence of solutions contains a uniformly converging subsequence. Passing uniformly to the limit over such subsequence, we obtain that the limit periodic function defined by this subsequence gives a some almost periodic function on the principal diagonal of the space. Then this solution solves equation (1.2). The proof is complete. □

We proceed to studying the solvability of equation (3.5) and we are going to establish the conditions under which (3.5) has a sequence of solutions uniformly bounded in k.

We introduce analogues of known notions from the theory of Fredholm integral equations [11], [13]. The analogues of the Fredholm functions $D(\lambda)$ and $D(x, y; \lambda)$ are defined as follows.

Let

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{b_n \lambda^n}{n!},$$

where

$$b_n = (-1)^n \lim_{T\to\infty} \frac{1}{T^n} \times \int_0^T \cdots \int_0^T \begin{vmatrix} K(x_1, \xi_1) & K(x_1, \xi_2) & \cdots & K(x_1, \xi_n) \\ K(x_2, \xi_1) & K(x_2, \xi_2) & \cdots & K(x_2, \xi_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_n, \xi_1) & K(x_n, \xi_2) & \cdots & K(x_n, \xi_n) \end{vmatrix} d\xi_1 d\xi_2 \cdots d\xi_n.$$

In the same way we let

$$D_k(x, y; \lambda) = \lambda D(\lambda) K(x, y) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{Q_n(x, y)\lambda^{n+1}}{n!}, \quad x, y \in \mathbb{R},$$

where

$$Q_n(x, y) = -\lim_{T\to\infty} \frac{n}{T^n} \int_0^T \cdots \int_0^T P_n(x, \xi, \xi_1, \dots, \xi_{n-1}) K(\xi, y) d\xi d\xi_1 \cdots d\xi_{n-1},$$

and

$$[P_n(x, \xi, \xi_1, \dots, \xi_{n-1}) = \begin{vmatrix} K(x, \xi) & K(x, \xi_1) & \cdots & K(x, \xi_{n-1}) \\ K(\xi_1, \xi) & K(\xi_1, \xi_1) & \cdots & K(\xi_1, \xi_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ K(\xi_{n-1}, \xi) & K(\xi_{n-1}, \xi_1) & \cdots & K(\xi_{n-1}, \xi_{n-1}) \end{vmatrix}.$$

By an arguing similar to that in [11], [13], one can show the the analogues of the Fredholm functions are entire functions and by the symmetricity of the kernel the equation $D(\lambda)=0$ has only real roots [1]. In our case these roots play an important role for equation (1.2). Here we consider only the case when λ is a real number not being a root of the equation $D(\lambda)=0$.

Theorem 3.2. *Let λ be a real number and $D(\lambda)\neq 0$. Then equation* (1.2) *has a unique almost periodic solution and it is given by the formula*

$$\varphi(x)=f(x)+\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T f(\xi)\frac{D(x,\xi;\lambda)}{D(\lambda)}d\xi. \tag{3.6}$$

Proof. Reproducing the arguing of the proof of Theorem 3.1, we consider equation (3.5) for a fixed k. As it is known, [11], the techniques from the theory of system of linear equations gives an expression for the Fredholm function $D_k(\lambda)$:

$$D_k(\lambda)=1+\sum_{n=1}^{\infty}\frac{a_n\lambda^n}{n!},$$

and

$$a_n=(-1)^n\int\limits_\Delta\cdots\int\limits_\Delta\begin{vmatrix} G_k(\bar{x}_1,\bar{\xi}_1) & G_k(\bar{x}_1,\bar{\xi}_2) & \cdots & G_k(\bar{x}_1,\bar{\xi}_n)\\ G_k(\bar{x}_2,\bar{\xi}_1) & G_k(\bar{x}_2,\bar{\xi}_2) & \cdots & G_k(\bar{x}_2,\bar{\xi}_n)\\ \vdots & \vdots & \ddots & \vdots\\ G_k(\bar{x}_n,\bar{\xi}_1) & G_k(\bar{x}_n,\bar{\xi}_2) & \cdots & G_k(\bar{x}_n,\bar{\xi}_n)\end{vmatrix}d\bar{\xi}_1d\bar{\xi}_2\cdots d\bar{\xi}_n;$$

the integration is made over the unit cube Δ. The integral in the right hand side of the latter identity has a multiplicity $ns(k)$. The Fredholm function $D_k(\bar{x},\bar{y},\lambda)$ is represented by a series converging in the entire complex plane [11]:

$$D_k(\bar{x},\bar{y};\lambda)=\lambda D(\lambda)K(\bar{x},\bar{y})+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{Q_n(\bar{x},\bar{y})\lambda^{n+1}}{n!},\quad \bar{x},\bar{y}\in\Delta,$$

where

$$Q_n(\bar{x},\bar{y})=-n\int\limits_\Delta\cdots\int\limits_\Delta P_nK(\bar{\xi},\bar{y})d\bar{\xi}d\bar{\xi}_1\cdots d\bar{\xi}_{n-1},$$
$$d\bar{\xi}_j=d\xi_{j1}\cdots d\xi_{js(k)},\quad \xi_{ji}\in\Delta,$$

and

$$P_n=P_n(\bar{x},\bar{\xi},\bar{\xi}_1,\ldots,\bar{\xi}_{n-1})=\begin{vmatrix} G_k(\bar{x},\bar{\xi}) & G_k(\bar{x},\bar{\xi}_1) & \cdots & G_k(\bar{x},\bar{\xi}_{n-1})\\ G_k(\bar{\xi}_1,\bar{\xi}) & G_k(\bar{\xi}_1,\bar{\xi}_1) & \cdots & G_k(\bar{\xi}_1,\bar{\xi}_{n-1})\\ \vdots & \vdots & \ddots & \vdots\\ G_k(\bar{\xi}_{n-1},\bar{\xi}) & G_k(\bar{\xi}_{n-1},\bar{\xi}_1) & \cdots & G_k(\bar{\xi}_{n-1},\bar{\xi}_{n-1})\end{vmatrix}.$$

Let us prove that in each closed domain in the complex plane the limiting relations

$$D(z)=\lim_{k\to\infty}D_k(z),\qquad D(x,y,z)=\lim_{k\to\infty}D_k(x,\ldots,x,y,\ldots,y;z)$$

hold uniformly in z. Without loss of generality we can suppose that $|z|\leqslant M$ for sufficiently large M. Since the function $K(x,y)$ is almost periodic, it is bounded on the entire real line, that is, there exists a positive constant L such that $|K(x,y)|\leqslant L$. Let $\varepsilon>0$ be an arbitrary positive number and N be a natural number, which we shall define below. Let us estimate an Nth remainder of the series for $D(\lambda)$. By the Hadamard inequality [11], [13] we have:

$$|b_n|\leqslant L^n n^{n/2}.$$

Hence,

$$\left|\sum_{n=N+1}^{\infty}\frac{b_n\lambda^n}{n!}\right|\leqslant\sum_{n=N+1}^{\infty}\frac{(|\lambda|\,L\sqrt{n})^n}{n!}\leqslant\sum_{n=N+1}^{\infty}\left(\frac{eLM}{\sqrt{N}}\right)^n.$$

Then taking N large enough, we get the inequality

$$\left|\sum_{n=N+1}^{\infty}\frac{b_n\lambda^n}{n!}\right|\leqslant\frac{1}{4}\varepsilon.$$

According to the above obtained relations

$$|D(\lambda)-D_k(\lambda)|\leqslant\frac{1}{2}\varepsilon+\sum_{n=1}^{N}\frac{|b_n-a_n|\,M^n}{n!}\leqslant\frac{1}{2}\varepsilon+eM^N\max_{n\leqslant N}|b_n-a_n|\,.$$

Now we observe that we open the determinant in the expression for b_n and then integrate term by term the obtained sum, by the above arguing we see that for sufficiently large k the expression for a_n is arbitrarily close to b_n, that is, as $n\leqslant N$, the relation $\lim\limits_{k\to\infty}a_n\to b_n$ holds uniformly on the principal diagonal. This is why for sufficiently large k we have:

$$|D(\lambda)-D_k(\lambda)|\leqslant\varepsilon,$$

and this proves the uniform convergence

$$D(z)=\lim_{k\to\infty}D_k(z)$$

in the circle $|z|\leqslant M$. In the same way we prove the uniform convergence

$$D(x,y;z)=\lim_{k\to\infty}D_k(x,\dots,x,y,\dots,y;z).$$

To complete the proof of Theorem 3.2, we need to establish the unique solvability of equation (1.2) and to prove the formula for its solution.

Let λ be a real number such that $D(\lambda)\neq0$. We take a closest to the number λ zero λ_0 of the function $D(\lambda)$ and $|\lambda_0-\lambda|=r$. By the Hurwitz theorem [4, Lm. 2.7], in the neighbourhood $|\lambda-z|\leqslant r/2$ of the number λ there can be at most finitely many zeroes of the functions $D_k(\lambda)$. This is why for sufficiently large k we have: $D_k(\lambda)\neq0$, at the same time, $D_k(\lambda)$ is sufficiently close to $D(\lambda)$. For such values k, equation (3.5) possesses a unique solution defined by the formula:

$$H_k(\bar{x})=f_k(\bar{x})+\int\limits_{\Delta}f_k(\bar{\xi})\frac{D_k(\bar{x},\bar{\xi};\lambda)}{D_k(\lambda)}d\bar{\xi},\quad d\bar{\xi}=d\xi_1\cdots d\xi_{s(k)},\quad \xi_j\in\Delta_j.$$

Since the convergence in k is uniform, by the above estimates and arguing we get the uniform boundedness of these solutions. Then by Theorem 3.1, equation (1.2) has an almost periodic solution. Substituting the vector $\bar{x}'=(x,x,\dots,x)$ instead of the variables $\bar{x}$ under the integral, we obtain

$$H_k(\bar{x}')=f_k(\bar{x}')+\int\limits_{\Delta}f_k(\bar{\xi})\frac{D_k(\bar{x}',\bar{\xi};\lambda)}{D_k(\lambda)}d\bar{\xi}=f_k(\bar{x}')+\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T f_k(\bar{\xi}')\frac{D_k(\bar{x}',\bar{\xi}';\lambda)}{D_k(\lambda)}d\xi.$$

By the uniform convergence, we can pass to limit in k under the integral interchanging also the order of passing to the limit. Indeed, deducting the latter expression from the right hand side of (3.6), we get:

$$f(x)-f_k(\bar{x}')+\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T\left(f(\xi)\frac{D(x,\xi;\lambda)}{D(\lambda)}-f_k(\bar{\xi}')\frac{D_k(\bar{x}',\bar{\xi}';\lambda)}{D_k(\lambda)}\right)d\xi.$$

For sufficiently large k, the absolute value of this difference is arbitrarily small. This proves the needed statement. We therefore obtain solution (3.6) to equation (1.2). The uniqueness of the solution is implied by that for equation (3.5). The proof is complete. □

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