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APPLICATION OF GENERATING FUNCTIONS TO PROBLEMS OF RANDOM WALK

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Abstract. We consider a problem on determining the first hit time of the positive semi-axis under a homogenous discrete integer random walk on a line. More precisely, the object of our study is the graph of the generating function of the mentioned random variable. For the random walk with the maximal positive increment 1, we obtain the equation on the implicit generating function, which implies the rationality of the inverse generating function. In this case, we find the mathematical expectation and dispersion for the first hit time of a positive semi-axis under a homogenous discrete integer random walk on a line. We describe a general method for deriving systems of equations for the first hit time of a positive semi-axis under a homogenous discrete integer random walk on a line. For a random walk with increments -1, 0, 1, 2 we derive an algebraic equation for the implicit generating function. We prove that a corresponding planar algebraic curve containing the graph of generating function is rational. We formulate and prove several general properties of the generating function the first hit time of the positive semi-axis under a homogenous discrete integer random walk on a line.

Keywords: generating function, random walk.

Mathematics Subject Classification: 60G50

1. Introduction

1.1. Brief background. In this work we study a random walk. Following [1], we give a definition of this notion.

Definition 1.1. *A homogeneous discrete integer random walk on a straight line with parameters $p_1, a_1, \ldots, p_n, a_n$, where $p_i \in \mathbb{R}$, $p_i \geqslant 0$, $\sum_{i=1}^{n} p_i = 1$ and $a_i \in \mathbb{Z}$ is a discrete Markov process η_k, the propagator of which is equal to $P(\eta_{k+1} = n + a_i | \eta_k = n) = p_i$. The quantity $\xi = \min\{n \,|\, \eta_n > 0\}$ is called the first hit of the positive semi-axis. If $\eta_k < 0$, then one lets $\xi = \infty$.*

In what follows, if else is not said, by the random walk we always mean the homogeneous discrete integer random walk on the straight line.

The random walk, as the entire probability theory, appeared from gambling, where a_i are interpreted as a possible value of the payoff per step in conventional units, while p_i is the probability of this payoff. The game theory appeared first in XVIII century, but systematically it was presented during the Second World War in monograph [2]. At that time it had already separated from the probability theory. The game theory has many applications in economics, where the role of payoff is played by a profit that was mentioned in [2] as well as in work [3], brought the author the Nobel Prize in Economics. In the last work, one more application of game theory is indicated: in military affairs.

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Random walks have also applications in physics, namely, in the problem on diffusion and Brownian motion related to the physical kinetics; a model of one-dimensional Brownian motion is as follows: per time unit, a particle changes its coordinates by a_i with a probability p_i. In our case, a diffusion equation from [4] is rewritten as $\frac{\partial n}{\partial t} + v\frac{\partial n}{\partial x} = D\frac{\partial^2 n}{\partial x^2}$, where the flow velocity $v = \sum_{i=1}^{n} p_i a_i$ is the mathematical expectation of the coordinate increment per time unit, while the diffusion coefficient $D = \frac{1}{2}\sum_{i=1}^{n} p_i(a_i - v)^2$ is the half of its dispersion. This theory belongs to A. Einstein and M. Smolukhovsky.

We note that exactly statistical physics initiated a boost development of the theory of random processes in XX century. The foundations of the mathematical theory were exposed, for instance, in [5]. The first studied random process was a Wiener process being a continuous analog of the random walk. This and other processes were studied by A. Erlang, N. Wiener, A. Markov and others. More details can be found in [6]. The random walk is a particular case of a discrete homogeneous random walk with independent increments.

The topic of random walk is discussed in all books on probability theory. For instance, in book [1] so-called "trivial" ($n = 2; a_1 = 1; a_2 = 0$) and "symmetric" ($n = 2; a_1 = 1; a_2 = -1$) random walks were discussed. For the latter in [4] the diffusion equation was derived. "Trivial" corresponds to the number of successes in the Bernulli scheme, while "symmetric" does to an antagonistic game of two players with equal bet. In [1] the following statements were also formulated for the general case:

Theorem 1.1. *If $\sum_{i=1}^{n} p_i a_i = 0$, then with the probability 1 the sequence η_n contains infinitely many zeroes.*

Theorem 1.2. *If $\sum_{i=1}^{n} p_i a_i < 0$ (> 0), then with the probability 1 the sequence η_n contains only finitely many non-negative (respectively, non-positive) numbers.*

In textbooks [7] there were considered random walks on the straight line in the most general sense, not only integer and not necessarily with discrete increments. By means of the characteristic function of the increment certain statements are proved being the generalizations of Theorems 1.1 and 1.2. We restrict ourselves by a more narrow class: the increment can take only finitely many integer values. We shall consider the generating function of the variable ξ from Definition 1.1.

1.2. Formulation of problem and survey of related topics. We are interesting in the first hit time of the positive semi-axis. From the point of view of the game theory, this is a problem on continuing the game until the pure payoff. The age of this problem is to be estimated as approximately 300 years, since it was in fact studied by Huygens for the case of the "trivial" walk (he studied the first appearance of six when playing dice) and Moivre for the "symmetric" walk. The former obtained explicit formulae for the probability of the event that the game is over in n steps, see [6]. The physical interpretation of the problem is that from the positive side of the particle an adsorbing screen is installed.

As it follows from Theorems 1.1 and 1.2, the probability of the infinite value of the quantity ξ in Definition 1.1 is non-zero if and only if the expectation of the increment is negative: $\sum_{i=1}^{n} p_i a_i < 0$.

Since our random variable takes only natural values except for ∞, it is reasonable to consider the generating function.

Definition 1.2 ([8]). *A generating function of the quantity ξ is a function of a complex variable continuous on the unit circle and holomorphic inside denoted by $f_\xi(z)$ and defined by the sum of the series $\sum_{k=1}^{\infty} p(\xi = k)z^k$.*

It is obvious that $f_\xi(0) = 0$; $f'_\xi(0) = \sum_{a_i>0} p_i$; $f_\xi(1) = 1 - p(\xi = \infty)$, while in the case of the positive expectation of the increment the meaning of $f'_\xi(1)$ is the expectation of the time of the game. For the cases described in the above cases in [1] the following formulae were obtained: $f_\xi(z) = \frac{p_1 z}{1-p_2 z}$ for "trivial" and $f_\xi(z) = \frac{1-\sqrt{1-4p_1p_2z^2}}{2p_2 z}$ for "symmetric" random walk.

In work [9] there was considered a problem similar to the above formulated but for multi-dimensional periodic lattices. It arises in the physics of crystals. The generating functions are derived by means of the Green's function.

Monograph [10] is devoted to an important problem of the insurance economy, the problem of ruin. It considers random walks with continuous time (positive increments are equal to one and occur periodically, but negative increments are randomly distributed in value and time). Differential-integral equations were obtained for characteristic functions.

In [11], a connection between random walks on the half-line and orthogonal systems of polynomials was discussed. In contrast to our problem, the random walks here were non-homogeneous. In order to study them, the so-called Carlin-McGregor formula was used, which gives a representation for probabilities in the form of an integral with respect to some special measure.

In [12], random walks with an arbitrary distribution of increments and a small negative expectation of the increment were treated. An asymptotic estimate was obtained for the time distribution of the first passage of the level x. The generating function of moments of increment was employed.

In [13] the distribution of the number of records (values greater that all previous ones) over a fixed number of steps of a symmetric integer random walk were studied. The generating function of this distribution turned out to be transcendental.

The work [14] is devoted to a two-dimensional random walk in a quarter-plane, in which each step changes each coordinate by at most 1 and all possible variants of the step are equally probable. Equations for the generating function of three variables were obtained; these three variables were the number of steps, the number of hits of the horizontal half-line and the number of hits of the vertical half-line. For some cases, it was possible to find an algebraic solution, for others, a differential-algebraic solution was determined; a relation involved derivatives with respect to the variable reflecting the number of steps in the relation.

For a hexagonal lattice, some generating functions and explicit formulas for probabilities were obtained in [15].

For a multidimensional lattice and arbitrary increment distributions, it is possible to find the probability that 0 does not lie in the convex hull of several values. The formula contains coefficients specified by their generating function (see [16]).

In [17] a problem of the number of points visited exactly k times in n random walk steps was studied. The generating function was calculated by using the graph theory.

A special case of our problem was considered in [18] ($n = 3$, $a_1 = -1$, $a_2 = 0$, $a_3 = 1$). Explicit probability formulae containing hypergeometric functions were obtained, and a generating function was derived for the first hit time of an arbitrary point.

In [19], there was presented a method for solving our problem using multi-dimensional Galton-Watson branching processes. The calculations were made for the case from [18] and the case $n = 3$, $a_1 = -2$, $a_2 = -1$, $a_3 = 1$. There were obtained equations, which were special cases of our main result and it was proved that the generating functions were their smallest real solutions.

We obtain relations for generating functions $f_\xi(z)$ for general random walks by more elementary methods. As we can see, the random walk topic has been actively developed in recent decades, and our result will allow us to apply algebraic-geometric methods in it, in particular, methods of the birational geometry.

2. Main results

2.1. Case admitting a complete studying. The main result of the present work is the following statement.

Theorem 2.1. *Let in the definition of the random we have $n > 1$, $a_1 = 1$, $a_i \leqslant 0$, $1 < i \leqslant n$. Then for $w = f_\xi(z)$ the relation*

$$w = z \sum_{i=1}^{n} p_i w^{1-a_i} \tag{2.1}$$

holds. In other words, while substituting $w = f_\xi(z)$ into (2.1), we get the identity.

Proof. Let us obtain a recurrent formula for $c_k = p(\xi = k)$. It is clear that $c_1 = p_1$. Let $k > 1$. Then $\eta_1 = a_i$, $i > 1$ with probability p_i. In order to achieve a positive result 1, we need to pass through $-a_i$ intermediate values $a_i + 1, \ldots, 0$ and this is why remaining $k - 1$ steps are split into $1 - a_i$ terms, each of which can be interpreted as the first achievement of the payoff 1 in comparison with the previous step. Since all steps are independent, if the partition is of form $k - 1 = k_1 + \ldots + k_{1-a_i}$, the corresponding probability is equal to $\Pi_{j=1}^{1-a_i} c_{k_j}$. Summing over all possible first terms and partitions (the order of the terms in the partitions are obviously inessential), we obtain the relation

$$c_{k+1} = \sum_{i=2}^{n} \left(p_i \sum_{k_1,\ldots,k_{1-a_i}>0;\sum_{j=1}^{1-a_i} k_j = k} \Pi_{j=1}^{1-a_i} c_{k_j} \right). \tag{2.2}$$

It remains to substitute the Taylor series $\sum_{k=1}^{\infty} c_k z^k$ instead of w into relation (2.1), to take into consideration that $w^0 = 1$ and by equating the coefficients at the like powers of z, we confirm that recurrent relations (2.2) transform this relation into identity. □

For further studying we shall use such notions of the algebraic geometry as an algebraic curve and a rational curve. Following [20] and employing standard notation $C[\ldots]$ for the ring of polynomials of the set of variables inside the square brackets and $C(\ldots)$ for the field of rational functions of the set of variables inside the round brackets, we recall their definitions:

Definition 2.1. *A planar algebraic curve is the set of points C^2 satisfying the equation $P(x, y) = 0$, where x, y are Cartesian coordinates, $P \in C[x, y]$ is some polynomnial of two variables with complex coefficients. The degree of the polynomial is also called the degree of the corresponding algebraic curve. There are special titles for some powers, for instance, conic (degree 2), cubic (degree 3), quartic (power 4), quintic (power 5). If the polynomial is simple, that is, it is not factorized into non-constant factors, we say that the curve defined by this polynomial is irreducible, otherwise we say it consists of several components each being a curve defined by one of its simple factors.*

Definition 2.2. *An irreducible planar algebraic curve is called rational if there exist two rational functions of one variable with complex coefficients $x(t), y(t) \in C(t)$, the substitution of which into the equation of the curve turns it into the identity; t is called a rational parameter of the curve.*

Definition 2.3. *The set $\{(z, f(z)) | z \in D(f)\} \subset C^2$, where $D(f)$ is the domain of the function f is called the graph of the function f. If the function is meromorphic on its domain, we say that the graph is a smooth curve. At the same time, the straight line $\{(z_0 + t, f(z_0) + f'(z_0)t) \,|\, t \in C\}$ is called an oblique tangent line to the graph of the function f at the point $z_0 \in D(f)$ not being a pole of the derivative. If $f'(z)$ has a pole at the point z_0, then the straight line $\{(z_0, t) \,|\, t \in C\}$ is called a vertical tangential line to the graph of the function f at the point z_0.*

It follows from Theorem 2.1 that the graph of the function $f_\xi(z)$ is a subset of a planar algebraic curve of degree $2-\min\{a_2,\dots,a_n\}$ defined by equation (2.1). This curve is irreducible since for each value w at most one value z corresponds to (otherwise each component of the curve would give a correspondence $z(w)$) and it is rational with the rational parameter w. The function inverse to the generating one is rational with the denominator w.

Differentiating relation (2.1) and substituting $w = z = 1$, we find the expectation of the duration of the game. It is equal to $\frac{1}{M}$. Here $M = \sum_{i=1}^{n} p_i a_i$ is the expectation of the payoff in one step. This result is a particular case of the following fact mentioned in [1].

Theorem 2.2. *The expectation of the sum of random number of independent identically distributed terms is equal to the product the expectation of the number of the terms by the expectation of each term.*

Finding the second derivative at the same point

$$\frac{d^2w}{dz^2} = \frac{1}{\frac{dz}{dw}} \frac{d}{dw}\left(\frac{1}{\frac{dz}{dw}}\right), \tag{2.3}$$

we can also calculate the dispersion of the duration of the game, which turns out to be equal to $\frac{D}{M^3}$, where M is the same as above and $D = \sum_{i=1}^{n} p_i(a_i - M)^2$ is the dispersion of the payoff in one step.

2.2. General case. It is interesting to consider the problem in the general case. Let us formulate a few rather obvious statements.

Theorem 2.3. *if $a_i \leqslant 0$, then $f_\xi(z) = 0$.*

Proof. In the mentioned case all η_i in Definition 1.1 are non-positive and hence, we surely have $\xi = \infty$. This is why $P(\xi = n) = 0$ for each natural n and therefore, $f_\xi(z) = 0$. □

Theorem 2.4. *If $a_i \geqslant 0$, then the quantity ξ has the same expectation as for the "trivial" walk with $\tilde{p_2} = p_i$ for i such that $a_i = 0$ and $\tilde{p_2} = 0$ if all a_i satisfy $a_i > 0$; $\tilde{p_1} = 1 - \tilde{p_2}$.*

Proof. In this case all η_i obey $\eta_i \geqslant 0$ and this is why

$$\xi = \min\{n|\eta_n > 0\} = \min\{n|\eta_n \neq 0\}.$$

This corresponds to the "trivial" random walk with the probability $\tilde{p_1}$ equalling to the sum of probabilities of all non-zero values of increments and $\tilde{p_2}$ equalling to the probability of the zero increment of the zero increment (if it exists). □

In what follows we assume that there exist a_i of different signs.

Theorem 2.5. *If the greatest common divisor of all a_i is equal to $d > 1$, then the random walk with the parameters $n, p_i, \tilde{a}_i = a_i/d$ (let us call it reduced) has the same distribution of the first hit time of the positive axis as the initial one.*

Proof. It is sufficient to observe that for the initial process all η_n obey $\eta_n \in d\mathbb{Z}$ and the linear change of the coordinates $x' = x/d$ on the straight line transforms this lattice into $\mathbb{Z}$ and does our walk into the reduced one. This changes does not influence the positivity of η_n and this is why ξ is mapped into itself. □

This means that without loss of generality we can regard the set a_i being coprime.

We consider an example not covered by Theorem 2.1: $n = 4$, $a_1 = -1$, $a_2 = 0$, $a_3 = 1$, $a_4 = 2$.

Theorem 2.6. *The recurrent relations in this example read as follows:*

$$\begin{cases} c_k = c_k^{(1)} + c_k^{(2)}, \\ c_1^{(1)} = p_3, \\ c_1^{(2)} = p_4, \\ c_{k+1}^{(1)} = p_1(c_k^{(2)} + \sum_{j=1}^{k-1} c_j^{(1)} c_{k-j}^{(1)}) + p_2 c_k^{(1)}, \\ c_{k+1}^{(2)} = p_1 \sum_{j=1}^{k-1} c_j^{(1)} c_{k-j}^{(2)} + p_2 c_k^{(2)}. \end{cases}$$

Proof. In this example at the moment of the end of the game two cases are possible: $\eta_n = 1$ and $\eta_n = 2$. Further arguing is similar to the proof of Theorem 2.1. For instance, if $\eta_1 = -1$, then we can achieve the value 1 either by achieving first the value 0 or avoiding 0, while the value 2 can be achieved only via the value 0. □

Corollary 2.1. *The generating function satisfies the system of equations*

$$\begin{cases} f_\xi(z) = f_1(z) + f_2(z), \\ f_1(z) = z(p_1(f_2(z) + f_1^2(z)) + p_2 f_1(z) + p_3), \\ f_2(z) = z(p_1 f_1(z) f_2(z) + p_2 f_2(z) + p_4). \end{cases}$$

Making certain transformations, we obtain that the graph of the function $f_\xi(z)$ is a subset of a planar algebraic curve, quantic:

$$Aw^3z^2 + Bw^2z^2 + Cw^2z + Dwz^2 + Ewz + Fz^2 + w + Gz = 0, \tag{2.4}$$

where

$$A = p_1^2, \qquad B = p_1(2p_2 + p_3), \qquad C = -2p_1, \qquad D = p_2 + p_1(p_3 - p_2 + 3p_4),$$
$$E = p_1 - p_2 - 1, \qquad F = (p_3 + p_4)(1 - p_1) - p_1 p_4, \qquad G = -(p_3 + p_4).$$

Theorem 2.7. *If $p_1 > 0$ and $p_4 > 0$, then quintic (2.4) is irreducible and rational.*

Proof. By the change of the coordinates $\left(t = \frac{P(w)z+Q(w)}{w-1}, w\right)$, where P, Q are some polynonmials and this quintic is transformed into a conic of form $t^2 = kw + l$ with some coefficients k, l; the polynomials and the coefficients depend on the values p_i, and if $p_1 > 0$ and $p_4 > 0$, then $P \neq 0$ and $k \neq 0$. At the same time, it possesses only finitely many singular points with respect to the aforementioned change of coordinates and this is why it is irreducible since the number of the components in old and new coordinates coincide and the conic $t^2 = kw + l$ is irreducible. It also follows from the change that t is a rational parameter of our curve. □

In the general case the idea is the same, but we deal with more equations, they involve more terms of higher degree and despite from this system we can derive a polynomial relation for z, $w = f(z)$, but the corresponding polynomial has a higher degree even for a_i with small absolute values. The issue whether the corresponding planar algebraic curve is rational remains open. We can say only the following:

Theorem 2.8. *The planar algebraic curve containing the graph of the function $f_\xi(z)$ for random walk with parameters $p_1, a_1, \ldots, p_n, a_n$ possesses the following properties:*

1) The linear part of the curve is of the form $w - z\sum_{a_i>0} p_i$, then the terms of degree 2 and higher come;

2) The curve passes through the point $w = z = 1$;

3) As $M > 0$, there is an oblique tangential line $w - 1 = k(z - 1)$ at this point, where $\frac{1}{M} \leqslant k \leqslant \frac{\max\{a_1,\ldots,a_n\}}{M}$, while in the "critical" case $M = 0$ this is a vertical tangential line $z = 1$. Here M is the same as in Section 2.1.

Proof. 1) Linearizing system of equations (2.1) (the linear part is responsible for the "recursion base"), we obtain

$$\begin{cases} f_\xi(z) = \sum_{j=1}^{max\{a_1,\ldots,a_n\}} f_j(z), \\ f_j(z) = zp_{i_j}, \end{cases}$$

where i_j are chosen so that $a_{i_j} = j$ and if there is no such index, we suppose $p_{i_j} = 0$. Substituting all equations of the system into the first one, we obtain the linearized equation of the curve.

2) We consider the problem with fixed a_i and varying p_i. The equation of the curve is a polynomial of $z, w, p_1, \ldots, p_n$. We substitute $w = z = 1$. We obtain a polynomial $Q(p_1, \ldots, p_n)$ of n of real variables equalling zero on the polyhedron $\{p_i \geqslant 0, \sum_i p_i a_i \geqslant 0\}$ in the hyperplane $\sum_i p_i = 1$ since in this case $f_\xi(1) = 1$, see Section 1.2. This implies that inside the orthogonal projection of this polyhedron on the hyperplane $p_1 = 0$ all the derivatives of a polynomial of $(n-1)$ variables obtained by substituting $p_1 = 1 - p_2 - \ldots - p_n$ into Q vanish. This is why it vanishes identically and hence $Q = 0$ on the entire hyperplane $\sum_i p_i = 1$. This means that for all p_i the point $w = z = 1$ lies on the corresponding curve.

3) It follows from Theorem 2.2 that the product of M by the expectation of the quantity ξ making sense for $M > 0$ and equalling in this case to $f_\xi'(1)$ is equal to the expectation of η_n at the moment of the end of the game. The latter quantity takes values from 1 till $\max\{a_1, \ldots, a_n\}$, and therefore, its expectation is between these values. This gives an estimate for the slope of the tangential line to the graph of the function $f_\xi(z)$ at the point $(1, f_\xi(1) = 1)$, which is equal to the derivative of this function at this point. This tangential line is one of the tangential lines to our curve at the given point. The case $M = 0$ is obtained by passing to the limit as $M \to +0$ (as it was mentioned above, the equation of the curve is polynomial and hence, depends continuously on p_i). □

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