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MAXIMAL CONVERGENCE OF FABER SERIES IN WEIGHTED REARRANGEMENT INVARIANT SMIRNOV CLASSES

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Abstract. Let K be a bounded set on the complex plane $\mathbb{C}$ with a connected complement $K^- := \overline{\mathbb{C}}\backslash K$. Let $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$ and $\mathbb{D}^- := \overline{\mathbb{C}}\backslash\overline{\mathbb{D}}$. By φ we denote the conformal mapping of K^-onto $\{w \in \mathbb{C} : |w| > 1\}$ normalized by the conditions $\varphi(\infty) = \infty$ and $\lim_{z\to\infty} \varphi(z)/z > 0$. Let $\Gamma_R := \{z \in K^- : |\varphi(z)| = R > 1\}$ and $G_R := \operatorname{Int}\Gamma_R$. Let also $\Phi_k(z)$, $k = 0, 1, 2, \ldots$ be the Faber polynomials for K constructed via conformal mapping φ. As it is well known, if f is an analytic function in G_R, then the representation $f(z) = \sum\limits_{k=0}^{\infty} a_k(f)\Phi_k(z)$, $z \in G_R$ holds. The partial sums of Faber series play an important role in constructing approximations in complex plane and investigating properties of Faber series is one of the essential issue. In this work the maximal convergence of the partial sums of the partial sums of the Faber series of f in weighted rearrangement invariant Smirnov class $E_X(G_R, \omega)$ of analytic functions in G_R is studied. Here the weight ω satisfies the Muckenhoupt condition on Γ_R. The estimates are given in the uniform norm on K. The right sides of obtained inequalities involve the powers of the parameter R and $E_n(f, G)_{X.\omega}$ called the best approximation number of f in $E_X(G_R, \omega)$, defined as $E_n(f, G)_{X.\omega} := \inf\left\{\|f - P_n\|_{X(\Gamma,\omega)} : P_n \in \Pi_n\right\}$. Here Π_n is the class of algebraic polynomials of degree not exceeding n. These results given in this paper is a kind of generalisation of similar results obtained in the classical Smirnov classes.

Keywords: Maximal convergence, Banach function space, Faber series, weighted rearrangement invariant space.

Mathematics Subject Classification:30E10, 41A10, 41A30

1. Introduction

Banach function spaces include many important particular cases including Lebesgue and Orlicz spaces (see, [1, 2]). Earlier some theorems of approximation theory in the rearrangement of invariant Banach function spaces and Smirnov classes were proved in [3, 5, 6, 7, 8, 9]. The partial sums of Faber series are used in constructing approximation aggregates on complex plane generally. Faber series are used for solving many problems in mechanical science, such as the problems on the stress analysis on the piezoelectric plane in [10, 11]. As described below we investigate the maximal convergence property of the Faber series in the rearrangement invariant Smirnov classes. Some classical results of the series of Faber polynomials and their applications were considered comprehensively in [12] and [13]. Moreover, the distribution of zeros of Faber polynomials were investigated in [14]. The Faber series is defined as follows.

Let K be a bounded continuum with the connected complement $K^- := \overline{\mathbb{C}}\backslash K$. Let $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$, $\mathbb{T} := \partial\mathbb{D}$ and $\mathbb{D}^- := \overline{\mathbb{C}}\diagdown\overline{\mathbb{D}}$. Let also φ be conformal mappings of K^-

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onto $\mathbb{D}^-$ normalized by the conditions

$$\varphi(\infty)=\infty,\ \lim_{z\to\infty}\varphi(z)/z>0.$$

and let ψ be the inverse mappings of φ. We set

$$\Gamma_R:=\left\{z\in K^-:|\varphi(z)|=R\right\}\quad\text{and}\quad G_R:=\operatorname{Int}\Gamma_R\ ,R>1.$$

As it is known from [12], if a function is analytic on continuum K, then it has the Faber series expansion

$$f(z)=\sum_{k=0}^{\infty}a_k(f)\,\Phi_k(z),\qquad z\in K,\tag{1.1}$$

which converges absolutely and uniformly on K. Here $\Phi_k(z)$, $k=0,1,2,\ldots$, are Faber polynomials for K, which can be defined by the series representations

$$\frac{\psi'(t)}{\psi(t)-z}=\sum_{k=0}^{\infty}\frac{\Phi_k(z)}{t^{k+1}},\qquad z\in K,\quad |t|>1,$$

where the Faber coefficients $a_k(f)$, $k=0,1,2,\ldots$, are defined as

$$a_k(f):=\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{f(\psi(t))}{t^{k+1}}dt.$$

In view of (1.1) we use the notation

$$R_n(f,z):=f(z)-\sum_{k=0}^{n}a_k(f)\,\Phi_k(z)=\sum_{k=n+1}^{\infty}a_k(f)\,\Phi_k(z),\quad z\in K.\tag{1.2}$$

The maximal convergence theorem estimates the rate of the convergence of $R_n(f,z)$ to zero in uniform norm on K in terms of parameter R and the best approximation number of analytic function f belonging to a given space. The results on maximal convergence properties of orthogonal polynomials can be found in [13]. The maximal convergence properties of the Faber series in the Smirnov-Orlicz classes were investigated in [16]. Later these results were extended to Smirnov classes with variable exponent by Israfilov et al. in [17, 18]. In this work we investigate the maximal convergence properties of the Faber series in the weighted rearrangement invariant Smirnov classes of analytic functions.

This work is organized as four sections. Necessary definitions and notations are given in second section. In the third section, some auxiliary results proved previously are formulated according to our notation used in this work. Finally, we state and prove the main results in the last section.

2. Preliminaries

Let $\mathcal{M}$ be the set of all measurable complex-valued functions on rectifiable Jordan curve Γ with respect to Lebesgue length measure $|d\tau|$ and let $\mathcal{M}^+$ be the subset of functions from $\mathcal{M}$ whose values lie in $[0,\infty]$. The characteristic function of a Lebesgue measurable set $E\subset\Gamma$ is denoted by χ_E.

A mapping $\rho:\mathcal{M}^+\to[0,\infty]$ is called a function norm if it satisfies the following properties for all measurable functions f,g,f_n $(n=1,2,\ldots)$, for all constants $a\geqslant 0$ and for all measurable sets $E\subset\Gamma$:

1. $\rho(f)=0\Leftrightarrow f=0$ *a.e.*, $\rho(af)=a\rho(f)$, $\rho(f+g)\leqslant\rho(f)+\rho(g)$,
2. If $0\leqslant g\leqslant f$ *a.e.*, then $\rho(g)\leqslant\rho(f)$,
3. If $0\leqslant f_n\nearrow f$ *a.e.*, then $\rho(f_n)\nearrow\rho(f)$,
4. If E has a finite Lebesgue measure, then $\rho(\chi_E)<\infty$,

5. If E has a finite Lebesgue measure, then

$$\int_E f(\tau)\,|d\tau| \leqslant C_E \rho(f)$$

where C_E is a positive constant depending on E and ρ does not depend on f.

Let ρ be a function norm. The set

$$X(\Gamma) = \{f \in \mathcal{M} : \rho(|f|) < \infty\}$$

is called a Banach function space generated by ρ and $X(\Gamma)$ becomes a Banach space equip with the norm $\|f\|_{X(\Gamma)} := \rho(|f|)$.

If ρ is a function norm, then associate norm of ρ is defined as

$$\rho' := \sup\left\{ \int_\Gamma f(\tau)\,g(\tau)\,|d\tau| : f \in \mathcal{M}^+, \rho(f) \leqslant 1 \right\}$$

for $g \in \mathcal{M}^+$ and ρ' is also itself a function norm. The Banach function space determined by ρ' is called the associate space of $X(\Gamma)$ and the associate space of $X(\Gamma)$ is denoted by $X'(\Gamma)$ in [1].

If $f \in X(\Gamma)$ and $g \in X'(\Gamma)$, then as it is known from Theorem 2.4 in [1], the Hölder inequality

$$\int_\Gamma |f(\tau)\,g(\tau)|\,|d\tau| \leqslant \|f\|_{X(\Gamma)} \|g\|_{X'(\Gamma)}$$

holds, where

$$\|f\|_{X(\Gamma)} := \sup\left\{ \int_\Gamma |f(\tau)\,g(\tau)|\,|d\tau| : g \in X'(\Gamma), \|g\|_{X'(\Gamma)} \leqslant 1 \right\},$$

$$\|g\|_{X'(\Gamma)} := \sup\left\{ \int_\Gamma |f(\tau)\,g(\tau)|\,|d\tau| : f \in X(\Gamma), \|f\|_{X(\Gamma)} \leqslant 1 \right\}.$$

Let $\mathcal{M}_0$ and $\mathcal{M}_0^+$ be classes of $a.e.$ finite functions in $\mathcal{M}$ and $\mathcal{M}^+$, respectively. The distribution function of f defined as

$$\mu_f(\lambda) := \text{mes}\{z \in \Gamma : |f(z)| > \lambda\}$$

for $\lambda \geqslant 0$. The pair of functions $f, g \in \mathcal{M}_0$ is called equimeasurable if $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \geqslant 0$.

Definition 2.1. [1] *If $\rho(f) = \rho(g)$ for every pair of equimeasurable functions $f, g \in \mathcal{M}_0^+$ then the function norm ρ is called a rearrangement invariant function norm and the Banach function space generated by ρ is called a rearrangement invariant spaces.*

The function $f^*(a) := \inf\{\lambda : \mu_f(\lambda) \leqslant a\}$, $a \geqslant 0$, is called the decreasing rearrangement of the function $f \in \mathcal{M}_0$.

Let $|\Gamma|$ be the Lebesgue measure of Γ. We use the notation $([0,|\Gamma|], m)$ to indicate Lebesgue measure spaces over the interval $[0,|\Gamma|]$. By Luxemburg representation theorem [1] we obtain that there is a (not necessarily unique) rearrangement invariant function norm $\overline{\rho}$ over $([0,|\Gamma|], m)$ such that $\rho(f) = \overline{\rho}(f^*)$ for $f \in \mathcal{M}_0^+$. The rearrangement invariant space over $([0,|\Gamma|], m)$ generated by $\overline{\rho}$ is called Luxemburg representation of $X(\Gamma)$ and it is denoted by $\overline{X}$. We define the operator H_x on $([0,|\Gamma|], m)$ for each $x > 0$ as

$$(H_x f)(t) := \begin{cases} f(xt), & xt \in [0,|\Gamma|] \\ 0, & xt \notin [0,|\Gamma|] \end{cases}, \quad t \in [0,|\Gamma|], \quad f \in \mathcal{M}_0.$$

By [1] the operator $H_{1/x}$ is bounded on $\overline{X}$ with the operator norm

$$h_x(x) := \left\| H_{1/x} \right\|_{\mathcal{B}(\overline{X})}$$

where $B\left(\overline{X}\right)$ is the Banach algebra of bounded linear operators on $\overline{X}$.

The limits defined as

$$\alpha_X := \lim_{x\to 0} \frac{\log h_x(x)}{\log x}, \qquad \beta_X := \lim_{x\to\infty} \frac{\log h_x(x)}{\log x}$$

are called lower and upper Boyd indices of $X(\Gamma)$, respectively [1]. The Boyd indices satisfy $0 \leqslant \alpha_X \leqslant \beta_X \leqslant 1$. The Boyd indices are said to be nontrivial if they satisfy $0 < \alpha_X \leqslant \beta_X < 1$.

A function $\omega : \Gamma \to [0, \infty]$ is called a weight if ω is measurable and the preimage $\omega^{-1}(\{0, \infty\})$ has the zero measure. The weighted rearrangement invariant space is defined as

$$X(\Gamma, \omega) = \{ f \in \mathcal{M} : f\omega \in X(\Gamma) \}$$

which is equipped with the norm $\|f\|_{X(\Gamma,\omega)} := \|f\omega\|_{X(\Gamma)}$ where $X(\Gamma)$ is rearrangement invariant spaces.

Definition 2.2. [20] *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Let ω be weight function on Γ such that $\omega \in L^p_{loc}(\Gamma)$and $\omega \in L^q_{loc}(\Gamma)$. We say that ω satisfies the Muckenhoupt condition on Γ if*

$$\sup_{t\in\Gamma} \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon} \int\limits_{\Gamma(t,\varepsilon)} \omega(\tau)^p \, |d\tau| \right)^{1/p} \left(\frac{1}{\varepsilon} \int\limits_{\Gamma(t,\varepsilon)} \omega(\tau)^{-q} \, |d\tau| \right)^{1/q} < \infty$$

where $\Gamma(t, \varepsilon) := \{ \tau \in \Gamma : |\tau - t| < \varepsilon \}$ and $\varepsilon > 0$.

Let us we denote by $A_p(\Gamma)$ the set of all weight functions satisfying Muckenhoupt condition on Γ.

Let $G \subset \mathbb{C}$ be a Jordan domain bounded by rectifiable curve Γ. We denote by $L^1(\Gamma)$, $1 \leqslant p < \infty$, the set off all measurable complex-valued functions f defined on Γ such that $|f|$ is Lebesgue integrable with respect to arc length on Γ. If there exists a sequence $(G_\nu)_{\nu=1}^{\infty} \subset G$ of domains G_ν, the boundary of which is a rectifiable Jordan curve $(\Gamma_\nu)_{\nu=1}^{\infty}$ such that the domain G_ν contains each compact subset G^* of G for $\nu \geqslant \nu_0$ for some $\nu_0 \in \mathbb{N}$ and

$$\limsup_{\nu\to\infty} \int\limits_{\Gamma_\nu} |f(z)| \, |dz| < \infty,$$

then we say that f belongs to the Smirnov class $E^1(G)$. Each function $f \in E^1(G)$ has the nontangential boundary value almost everywhere $(a.e.)$ on Γ and the boundary function belongs to $L^1(\Gamma)$ [15].

Definition 2.3. *Let ω be weight function on Γ. The class of analytic functions*

$$E_X(G, \omega) := \left\{ f \in E^1(G) : \; f \in X(\Gamma, \omega) \right\}$$

is called a rearrangement invariant Smirnov class.

The best approximation number of f in $E_X(G, \omega)$ is defined as

$$E_n(f, G)_{X.\omega} := \inf \left\{ \|f - P_n\|_{X(\Gamma,\omega)} : P_n \in \Pi_n \right\}$$

where Π_n is the class of algebraic polynomials of degree not exceeding n.

3. Auxiliary results

The direct theorems of approximation theory in the weighted rearrangement invariant Smirnov class were proved in [4]. For the sake of clarity, in this section we formulate these results and some notations in our terms.

We recall that $\Gamma_R := \{z \in K^- : |\varphi(z)| = R\}$ and $G_R := \operatorname{Int}\Gamma_R$, $R > 1$, where K is a bounded continuum with connected complement $K^- = \overline{\mathbb{C}} \setminus K$. If $\omega \in X(\Gamma_R)$ and $\omega^{-1} \in X'(\Gamma_R)$ for $R > 1$, then by Hölder inequality we have

$$L^\infty(\Gamma_R) \subset X(\Gamma_R, \omega) \subset L^1(\Gamma_R).$$

Since Γ_R with $R > 1$ is a analytic curve, by [19] there are positive constants such that

$$0 < c_1 \leqslant \left|\psi'(\zeta)\right| \leqslant c_2 < \infty, \quad |\zeta| = R,$$
$$0 < c_3 \leqslant \left|\varphi'(z)\right| \leqslant c_4 < \infty, \quad z \in \Gamma_R. \tag{3.1}$$

Let us we define the functions $f_0(w) := f \circ \psi(Rw)$ and $\omega_0(w) := \omega \circ \psi(Rw)$ for $w \in \mathbb{T}$.

If $f \in X(\Gamma_R, \omega)$ for $R > 1$, then by (3.1) we have $f_0 \in X(\mathbb{T}, \omega_0)$. The Cauchy type integral for a given $f_0 \in L^1(\mathbb{T})$ is defined as

$$f_0^+(w) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_0(\tau)}{\tau - w} d\tau, \quad w \in \mathbb{D},$$

which is analytic in $\mathbb{D}$.

Lemma 3.1. [4, Lm. 1] *If Boyd indices α_X and β_X are nontrivial and*

$$\omega_0 \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T}),$$

then $f^+ \in E_X(\mathbb{D}, \omega_0)$ for each $f \in X(\mathbb{T}, \omega_0)$.

Given $f \in X(\mathbb{T}, \omega_0)$, we define the mean value operator as

$$\sigma_h(f)(w) := \frac{1}{2h}\int_{-h}^{h} f\left(we^{it}\right) dt, 0 < h < \pi \text{ and} w \in \mathbb{T}.$$

If the Boyd indices α_X and β_X are nontrivial and $\omega_0 \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$, then for each $f \in X(\mathbb{T}, \omega_0)$ the inequality

$$\|\sigma_h(f)\|_{X(\mathbb{T},\omega_0)} \leqslant c\,\|f\|_{X(\mathbb{T},\omega_0)}$$

follows from Lemma 2.2 proved in [3].

If $R > 1$, then by (3.1) we have the conditions $\omega \in A_{1/\alpha_X}(\Gamma_R) \cap A_{1/\beta_X}(\Gamma_R)$ and $\omega_0 \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$ are equivalent. Lemma 3.1 implies that the nontangential boundary value of f_0^+ belongs to $X(\mathbb{T}, \omega_0)$. Consequently, we can give the following definition.

Let X be a Banach space and X^* be its dual space. We define the dual of X^* by setting $X^{**} := (X^*)^*$. Let $J(X)$ be the image of X in the canonical mapping $J : X \to X^{**}$. A Banach space X is said to be reflexive if $J(X) = X^{**}$ (see, [2, p. 21]). If the rearrangement invariant space X is reflexive, then X is called reflexive rearrangement invariant space.

Definition 3.1. *Let Boyd indices α_X and β_X are nontrivial and $\omega \in A_{1/\alpha_X}(\Gamma_R) \cap A_{1/\beta_X}(\Gamma_R)$ with $R > 1$. Let $X(\mathbb{T})$ be a reflexive rearrangement invariant space. The function*

$$\Omega^{\nu}_{\Gamma_R,X,\omega}(f,\delta) := \sup_{\substack{i=1,2,\dots,r\\ 0<h_i\leqslant\delta}} \left\|\prod_{i=1}^{\nu}(I - \sigma_{h_i})\left(f_0^+\right)\right\|_{X(\mathbb{T},\omega_0)}, \quad \nu = 1, 2, \dots, \ and \delta > 0,$$

is called ν a modulus of smoothness of $f \in E_X(G_R, \omega)$.

Since $X(\mathbb{T})$ is a reflexive rearrangement invariant space, the Boyd indices α_X and β_X of which are nontrivial, where $\omega_0 \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$, the set of continuous functions is dense in $X(\mathbb{T}, \omega_0)$, see [3, Lm. 2]. Hence, it is guaranteed that

$$\lim_{\delta \to 0} \Omega^{\nu}_{\Gamma_R, X, \omega}(f, \delta) = 0.$$

Lemma 3.2. [4, Cor. 1] *Let $X(\mathbb{T})$ be a reflexive rearrangement invariant space. Let Boyd indices α_X and β_X be nontrivial and $\omega \in A_{1/\alpha_X}(\Gamma_R) \cap A_{1/\beta_X}(\Gamma_R)$ with $R > 1$. If $f \in E_X(G_R, \omega)$, then there is a positive constant c such that the inequality*

$$\|f - P_n(\cdot, f)\|_{X(\Gamma_R, \omega)} \leqslant c\Omega^{\nu}_{\Gamma_R, X, \omega}\left(f, \frac{1}{n+1}\right), \quad \nu = 1, 2, \dots,$$

holds for each $n = 1, 2, \dots$, where $P_n(\cdot, f)$ is the nth partial sum of the Faber series of f.

It is known that [12]

$$E_k(\psi(\zeta)) = \frac{1}{2\pi i} \int\limits_{|\tau|=r} \tau^k F(\tau, \zeta)\, d\tau, \quad |\zeta| \geqslant r > 1 \tag{3.2}$$

and Lebedev's results

$$\frac{1}{2\pi i} \int\limits_{|\tau|=r} |F(\tau, \zeta)|\, |d\tau| \leqslant \sqrt{\frac{r^2}{r^4 - 1} \ln \frac{r^2}{r^2 - 1}}, \quad |\zeta| \geqslant r > 1, \tag{3.3}$$

where

$$F(\tau, \zeta) := \frac{\psi'(\tau)}{\psi(\tau) - \psi(\zeta)} - \frac{1}{\tau - \zeta}, \qquad |\tau| > 1, \quad |\zeta| > 1.$$

4. Main results

Our main results are as follows.

Theorem 4.1. *Let the Boyd indices α_X and β_X be nontrivial and*

$$\omega \in A_{1/\alpha_X}(\Gamma_R) \cap A_{1/\beta_X}(\Gamma_R)$$

with $R > 1$. If $f \in E_X(G_R, \omega)$, then

$$|R_n(f, z)| \leqslant \frac{c}{R^{n+1}(R-1)} E_n(f, G_R)_{X.\omega} \sqrt{n \ln n}, \quad z \in K,$$

holds for each $n = 1, 2, \dots$, where c is a positive constant independent of n.

Combining Theorem 4.1 and Lemma 3.2, we have the following corollary.

Corollary 4.1. *Let $X(\mathbb{T})$ be a reflexive rearrangement invariant space. Let the Boyd indices α_X and β_X be nontrivial and $\omega \in A_{1/\alpha_X}(\Gamma_R) \cap A_{1/\beta_X}(\Gamma_R)$ with $R > 1$. If $f \in E_X(G_R, \omega)$, then*

$$|R_n(f, z)| \leqslant \frac{c}{R^{n+1}(R-1)} \Omega^{\nu}_{\Gamma_R, X, \omega}\left(f, \frac{1}{n+1}\right) \sqrt{n \ln n}, \quad z \in K,$$

holds for any $n = 1, 2, \dots$, and $\nu = 1, 2, \dots$, where c is a positive constant independent of n.

Proof of Theorem 4.1. Let $z \in \Gamma_r$ and $1 < r < R$. Let the Boyd indices α_X and β_X be nontrivial and $\omega \in A_{1/\alpha_X}(\Gamma_R) \cap A_{1/\beta_X}(\Gamma_R)$. Let P_n be the best approximating polynomial of degree at most n to $f \in E_X(G_R, \omega)$. Since f is analytic function on G_R, then we have the Faber coefficients

$$a_k(f) = \frac{1}{2\pi i} \int\limits_{|t|=R} \frac{f(\psi(t))}{t^{k+1}} dt, \quad k = 1, 2, \dots$$

Taking into account $P_n \in \Pi_n$ and applying the Cauchy integral formula for derivatives, we have

$$\frac{1}{2\pi i}\int\limits_{\mathbb{T}} P_n(\psi(t))\left[\sum_{k=n+1}^{\infty}\frac{\Phi_k(z)}{t^{k+1}}\right]dt = 0. \tag{4.1}$$

Using (1.2) and (4.1) respectively we obtain (see, [12])

$$R_n(f,z) = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} f(\psi(t))\left[\sum_{k=n+1}^{\infty}\frac{\Phi_k(z)}{t^{k+1}}\right]dt$$

and

$$R_n(f,z) = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} \left[f(\psi(t)) - P_n(\psi(t))\right]\left[\sum_{k=n+1}^{\infty}\frac{\Phi_k(z)}{t^{k+1}}\right]dt. \tag{4.2}$$

The $\Phi_k(z)$, $k = 1, 2, \ldots$, are the polynomial part of Laurent series expansion of $[\varphi(z)]^k$ such that

$$\Phi_k(z) = [\varphi(z)]^k + E_k(z), \quad z \in K^-,$$

where E_k is an analytic function on K^-. Therefore we get

$$\sum_{k=n+1}^{\infty}\frac{\Phi_k(z)}{t^{k+1}} = \sum_{k=n+1}^{\infty}\frac{[\varphi(z)]^k}{t^{k+1}} + \sum_{k=n+1}^{\infty}\frac{E_k(z)}{t^{k+1}},$$

and by (4.2) we obtain

$$\begin{aligned}|R_n(f,z)| \leqslant &\frac{1}{2\pi}\int\limits_{|t|=R} |f(\psi(t)) - P_n(\psi(t))|\left|\sum_{k=n+1}^{\infty}\frac{\zeta^k}{t^{k+1}}\right||dt| \\ &+ \frac{1}{2\pi}\int\limits_{|t|=R} |f(\psi(t)) - P_n(\psi(t))|\left|\sum_{k=n+1}^{\infty}E_k(\psi(\zeta))\frac{1}{t^{k+1}}\right||dt|.\end{aligned}$$

Denoting

$$I_1 := \frac{1}{2\pi}\int\limits_{|t|=R} |f(\psi(t)) - P_n(\psi(t))|\left|\sum_{k=n+1}^{\infty}\frac{\zeta^k}{t^{k+1}}\right||dt|$$

and

$$I_2 := \frac{1}{2\pi}\int\limits_{|t|=R} |f(\psi(t)) - P_n(\psi(t))|\left|\sum_{k=n+1}^{\infty}E_k(\psi(\zeta))\frac{1}{t^{k+1}}\right||dt|,$$

we get

$$|R_n(f,z)| \leqslant I_1 + I_{2.} \tag{4.3}$$

Since $z \in \Gamma_r$ and $\varsigma \in \Gamma_R$ for $1 < r < R$ then $|\varphi(z)| = r$ and $|\varphi(\varsigma)| = R$. Thus $|R - r| \leqslant |\varphi(\varsigma) - \varphi(z)|$ implies that

$$\frac{1}{|\varphi(\varsigma) - \varphi(z)|} \leqslant \frac{1}{R - r}. \tag{4.4}$$

We know that $1/\omega \in X'(\Gamma)$ by Theorem 2.1 in [21]. Hence, by (3.1), Hölder inequality and (4.4) we have

$$I_1 = \frac{1}{2\pi}\int\limits_{|t|=R} |f(\psi(t)) - P_n(\psi(t))|\left|\sum_{k=n+1}^{\infty}\frac{\zeta^k}{t^{k+1}}\right||dt|$$

$$
\begin{aligned}
&= \frac{1}{2\pi} \int\limits_{\Gamma_R} |f(\varsigma) - P_n(\varsigma)| \left| \sum_{k=n+1}^{\infty} \frac{[\varphi(z)]^k}{[\varphi(\varsigma)]^{k+1}} \right| |\varphi'(\varsigma)| \, |d\varsigma| \\
&\leqslant \frac{c}{2\pi} \int\limits_{\Gamma_R} |f(\varsigma) - P_n(\varsigma)| \, \omega(\varsigma) \frac{1}{\omega(\varsigma)} \left| \sum_{k=n+1}^{\infty} \frac{[\varphi(z)]^k}{[\varphi(\varsigma)]^{k+1}} \right| |d\varsigma| \\
&\leqslant \frac{c}{2\pi} \|(f - P_n)\,\omega\|_{X(\Gamma)} \left\| \frac{1}{\omega(\varsigma)} \sum_{k=n+1}^{\infty} \frac{[\varphi(z)]^k}{[\varphi(\varsigma)]^{k+1}} \right\|_{X'(\Gamma)} \\
&= \frac{c}{2\pi} \|f - P_n\|_{X(\Gamma,\omega)} \left\| \frac{1}{\omega(\varsigma)} \frac{[\varphi(z)]^{n+1}}{[\varphi(\varsigma)]^{n+1} (\varphi(\varsigma) - \varphi(z))} \right\|_{X'(\Gamma)} \\
&= \frac{c}{2\pi} \|f - P_n\|_{X(\Gamma,\omega)} \left\| \left| \frac{1}{\omega(\varsigma)} \right| \frac{|\varphi(z)|^{n+1}}{|\varphi(\varsigma)|^{n+1} |\varphi(\varsigma) - \varphi(z)|} \right\|_{X'(\Gamma)} \\
&\leqslant \frac{c}{2\pi} E_n(f, G_R)_{X.\omega} \left\| \left| \frac{1}{\omega(\varsigma)} \right| \frac{r^{n+1}}{R^{n+1}(R - r)} \right\|_{X'(\Gamma)} \\
&= \frac{c}{2\pi} E_n(f, G_R)_{X.\omega} \frac{r^{n+1}}{R^{n+1}(R - r)} \|1/\omega(\varsigma)\|_{X'(\Gamma)} \\
&\leqslant \frac{c}{2\pi} E_n(f, G_R)_{X.\omega} \frac{r^{n+1}}{R^{n+1}(R - r)}.
\end{aligned}
$$

Therefore,

$$
I_1 \leqslant \frac{c}{2\pi} E_n(f, G_R)_{X.\omega} \frac{r^{n+1}}{R^{n+1}(R - r)}. \tag{4.5}
$$

On the other hand by (3.2) and applying Fubini's theorem we have

$$
\begin{aligned}
I_2 &= \frac{1}{2\pi} \int\limits_{|t|=R} |f(\psi(t)) - P_n(\psi(t))| \left| \sum_{k=n+1}^{\infty} E_k(\psi(\zeta)) \frac{1}{t^{k+1}} \right| |dt| \\
&= \frac{1}{2\pi} \int\limits_{|t|=R} |f(\psi(t)) - P_n(\psi(t))| \left| \sum_{k=n+1}^{\infty} \frac{1}{2\pi} \int\limits_{|\tau|=r} \frac{\tau^k}{t^{k+1}} F(\tau,\zeta) \, |d\tau| \right| |dt| \\
&= \frac{1}{2\pi} \int\limits_{|t|=R} |f(\psi(t)) - P_n(\psi(t))| \left\{ \frac{1}{2\pi} \int\limits_{|\tau|=r} \left| \sum_{k=n+1}^{\infty} \frac{\tau^k}{t^{k+1}} \right| |F(\tau,\zeta)| \, |d\tau| \right\} |dt| \\
&\leqslant \frac{1}{2\pi} \int\limits_{|t|=R} |f(\psi(t)) - P_n(\psi(t))| \left\{ \frac{1}{2\pi} \int\limits_{|\tau|=r} \left| \frac{\tau^{n+1}}{t^{n+1}(t - \tau)} \right| |F(\tau,\zeta)| \, |d\tau| \right\} |dt| \\
&\leqslant \frac{1}{2\pi} \int\limits_{|\tau|=r} |\tau|^{n+1} |F(\tau,\zeta)| \left\{ \frac{1}{2\pi} \int\limits_{|t|=R} \frac{|f(\psi(t)) - P_n(\psi(t))|}{|t|^{n+1} |t - \tau|} |dt| \right\} |d\tau|
\end{aligned}
$$

$$=\frac{r^{n+1}}{2\pi R^{n+1}}\int\limits_{|\tau|=r}|F(\tau,\zeta)|\left\{\frac{1}{2\pi}\int\limits_{|t|=R}|f(\psi(t))-P_n(\psi(t))|\frac{1}{|t-\tau|}|dt|\right\}|d\tau|.$$

Changing the variables and by (3.1), Hölder inequality, (3.3), (4.4), we get

$$\begin{aligned}
I_2\leqslant&\frac{r^{n+1}}{2\pi R^{n+1}}\int\limits_{|\tau|=r}|F(\tau,\zeta)|\left\{\frac{1}{2\pi}\int\limits_{|t|=R}|f(\varsigma)-P_n(\varsigma)|\frac{|\varphi'(\varsigma)|}{|\varphi(\varsigma)-\varphi(z)|}|d\varsigma|\right\}|d\tau|\\
\leqslant&c\frac{r^{n+1}}{4\pi^2 R^{n+1}}\int\limits_{|\tau|=r}|F(\tau,\zeta)|\left\{\frac{1}{2\pi}\int\limits_{|t|=R}|f(\varsigma)-P_n(\varsigma)|\,\omega(\varsigma)\,\frac{1}{\omega(\varsigma)}\frac{|\varphi'(\varsigma)|}{|\varphi(\varsigma)-\varphi(z)|}|d\varsigma|\right\}|d\tau|\\
\leqslant&\frac{cr^{n+1}}{4\pi^2 R^{n+1}}\int\limits_{|\tau|=r}|F(\tau,\zeta)|\,\|(f-P_n)\,\omega\|_{X(\Gamma)}\left\|\frac{1}{\omega(\varsigma)}\frac{|\varphi'(\varsigma)|}{|\varphi(\varsigma)-\varphi(z)|}\right\|_{X'(\Gamma)}|d\tau|\\
\leqslant&\frac{cr^{n+1}}{4\pi^2 R^{n+1}}\|f-P_n\|_{X(\Gamma,\omega)}\left\|\frac{1}{\omega(\varsigma)}\frac{1}{R-r}\right\|_{X'(\Gamma)}\int\limits_{|\tau|=r}|F(\tau,\zeta)|\,|d\tau|\\
=&\frac{cr^{n+1}}{4\pi^2 R^{n+1}(R-r)}E_n(f,G_R)_{X.\omega}\left\|\frac{1}{\omega(\varsigma)}\right\|_{X'(\Gamma)}\int\limits_{|\tau|=r}|F(\tau,\zeta)|\,|d\tau|\\
\leqslant&\frac{cr^{n+1}}{4\pi^2 R^{n+1}(R-r)}E_n(f,G_R)_{X.\omega}\int\limits_{|\tau|=r}|F(\tau,\zeta)|\,|d\tau|\\
\leqslant&\frac{cr^{n+1}}{2\pi R^{n+1}(R-r)}E_n(f,G_R)_{X.\omega}\sqrt{\frac{r^2}{r^4-1}\ln\frac{r^2}{r^2-1}}.
\end{aligned}$$

Thus we get

$$I_2\leqslant\frac{cr^{n+1}}{2\pi R^{n+1}(R-r)}E_n(f,G_R)_{X.\omega}\sqrt{\frac{r^2}{r^4-1}\ln\frac{r^2}{r^2-1}}.\qquad(4.6)$$

and combining (4.3), (4.5) and (4.6) we have

$$\begin{aligned}
|R_n(f,z)|\leqslant&\frac{c}{2\pi}E_n(f,G_R)_{X.\omega}\frac{r^{n+1}}{R^{n+1}(R-r)}\\
&+\frac{cr^{n+1}}{2\pi R^{n+1}(R-r)}E_n(f,G_R)_{X.\omega}\sqrt{\frac{r^2}{r^4-1}\ln\frac{r^2}{r^2-1}}.
\end{aligned}$$

Finally, setting $z\in K$ and $r:=1+\frac{1}{n}$, we obtain the desired inequality

$$|R_n(f,z)|\leqslant\frac{c}{R^{n+1}(R-1)}E_n(f,G_R)_{X.\omega}\sqrt{n\ln n}.$$

□

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