ISSN 2304-0122 Ufa Mathematical Journal. Vol. 14. No 2 (2022). P. 67-77.

doi:10.13108/2022-14-2-67

TRIVIAL EXTENSIONS OF SEMIGROUPS AND SEMIGROUP C^*-ALGEBRAS

E.V. LIPACHEVA

Abstract. The object of the study in the paper is reduced semigroup C^*-algebras for left cancellative semigroups. Such algebras are a very natural object because it is generated by isometric shift operators belonging to the image of the left regular representation of a left cancellative semigroup. These operators act in the Hilbert space consisting of all square summable complex-valued functions defined on a semigroup. We study the question on functoriality of involutive homomorphisms of semigroup C^*-algebras, that is, the existence of the canonical embedding of semigroup C^*-algebras induced by an embedding of corresponding semigroups. In order to do this, we investigate the reduced semigroup C^*-algebras associated with semigroups involved in constructing normal extensions of semigroups by groups. At the same time, in the paper we consider one of the simplest classes of extensions, namely, the class of so-called trivial extensions. It is shown that if a semigroup L is a trivial extension of the semigroup S by means of a group G, then there exists the embedding of the reduced semigroup C^*-algebra $C_r^*(S)$ into the C^*-algebra $C_r^*(L)$ which is induced by an embedding of the semigroup S into the semigroup L.

In the work we also introduce and study the structure of a Banach $C_r^*(S)$-module on the underlying space of the reduced semigroup C^*-algebra $C_r^*(L)$. To do this, we use a topological grading for the C^*-algebra $C_r^*(L)$ over the group G. In the case when a semigroup L is a trivial extension of a semigroup S by means of a finite group, we prove the existence of the structure of a free Banach module over the reduced semigroup C^*-algebra $C_r^*(S)$ on the underlying Banach space of the semigroup C^*-algebra $C_r^*(L)$.

We give examples of extensions of semigroups and reduced semigroup C^*-algebras for a more complete characterization of the issues under consideration and for revealing connections with previous results.

Keywords: cancellative semigroup, normal extension of a semigroup, trivial extension of a semigroup, reduced semigroup C^*-algebra, embedding a semigroup C^*-algebra, Banach module, free module.

Mathematics Subject Classification: 46H25, 47L30, 20M15

Introduction

The paper is devoted to studying normal extensions of semigroups with cancellation and corresponding semigroup C^*-algebras.

The reduced semigroup C^*-algebra are operator algebras generated by left regular representations of semigroups with cancellation. First such algebras were studied in works by Coburn [1], [2] and Douglas [3]. They considered reduced semigroup C^*-algebras for semigroups being positive cones of ordered semigroups in an additive group of all real numbers. A

E.V. Lipacheva, Trivial extensions of semigroups and semigroup C^*-algebras.

The work is supported by the Kazan Federal University Strategic Academic Leadership Program ("Priorirty -2030").

Submitted December 2, 2021.

further studying of these algebras were made in works by Murphy [4], [5], Nica [6], Laca and Raeburn [7], Li [8] and others.

The present work is a continuation of the studies of reduced semigroup C^*-algebras initiated in [9]– [13]. We consider semigroup C^*-algebras corresponding to semigroups involved in constructing the extensions of the semigroups.

The theory of semigroup extensions plays an important role in studying the structures and characteristics of semigroups, in particular, of their cohomologies, see, for instance, [14]. In studies on semigroups, various types of extensions are considered: ideal extensions, [15], Schreier extensions [16], normal extensions [17], [18]. In [19] there was studied the action of the functor of Stone-Čech compactification on normal extensions of semigroups.

This paper is aimed on finding the relations between the extensions of the semigroups and corresponding semigroup C^*-algebras and completes the studies made in works [20]– [22]. We consider one of the simplest types of normal extensions, namely, trivial extensions. If L is a trivial extensions of S be means of a finite group, then the underlaying space of C^*-algebra $C_r^*(L)$ can be equipped with the structure of a free Banach module over the C^*-algebra $C_r^*(S)$. While proving this fact, we use a topological grading of C^*-algebra $C_r^*(L)$, the construction of which was described in [12]. We recall that the notion of a topologically graded C^*-algebra was introduced by Exel [23] in order to extend notions of harmonic analysis on a non-commutative case.

The paper consists of the Introduction and three sections. In Section 1 we recall needed facts from the theory of semigroup extensions, theory of semigroup C^*-algebras and Banach modules. Section 2 is devoted to the question on functoriality of morphisms of semigroup C^*-algebras, which raised in a general form in work [8] and was studied in [22] for reduced semigroup C^*-algebras constructed by semigroups, one of which is a normal extension of another. In Section 3 on the underlaying space of the C^*-algebra $C_r^*(L)$, we introduce and study the structure of Banach $C_r^*(S)$-module under the conditions that the semigroup L is a trivial extension of S by means of the group.

1. Preliminaries

Let S and L be discrete semigroups with a left cancellation and G be a group with a unit e. Assume we are given an injective homomorphism of semigroups $\tau : S \longrightarrow L$ and a surjective semigroup homomorphism $\sigma : L \longrightarrow G$. The triple (L, τ, σ) is called a *normal extension* of the semigroup S by means of the group G if $\tau(S)$ is the total preimage of the unit e under σ, that is,

$$\sigma^{-1}(e) = \tau(S).$$

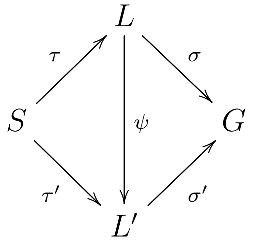
The semigroup L is also sometimes called an extension of the semigroup S by means of the group G. General definitions of the extensions of semigroups can be found in [17], [24].

Let a set X be such that $X \subset L \setminus \tau(S)$ and $X \cap \sigma^{-1}(g) = \{x_g\}$ for each $g \in G$, $g \neq e$. We shall say that an extension (L, τ, σ) of the semigroup S *is generated by the set* X if each element $y \in L \setminus \tau(S)$ is uniquely represented as $y = \tau(a)x_g$ for some $a \in S$ and $g \in G$. In this case each subset $\sigma^{-1}(g)$, $g \neq e$, reads as

$$\sigma^{-1}(g) = \tau(S)x_g := \{\tau(a)x_g \mid a \in S\}.$$

We note that the extensions possessing generating sets are Schreier extensions, see [14].

Two extensions (L, τ, σ) and (L', τ', σ') of the semigroup S by means of the group G are called *equivalent* if there exists an isomorphism of semigroups $\psi : L \longrightarrow L'$ making the giagram



commutative.

We consider the Cartesian product $S \times G$ of the semigroup S and the group G. This is a semigroup with the multiplication operation

$$(a, g) \cdot (b, h) = (ab, gh), \tag{1.1}$$

where $a, b \in S$, $g, h \in G$. An extension of form $(S \times G, \tau, \sigma)$, where $\tau(a) = (a, e)$ and $\sigma(a, g) = g$ for all $a \in S$, $g \in G$, or any equivalent to it is called *trivial* extension of the semigroup S by means of the group G.

Let us recall the definition of the reduced semigroup C^*-algebra. Let P be a discrete semigroup with a left cancellation. We introduce a Hilbert space on this semigroup: this is the space $l^2(P)$ of square integrable complex-valued functions on P. We denote by e_p, $p \in P$, the function of the space $l^2(P)$ defined by the formula

$$e_p(q) := \begin{cases} 1, & \text{if } p = q\,; \\ 0, & \text{if } p \neq q\,, \end{cases}$$

where $q \in P$. Then the set of the functions $\{e_p \mid p \in P\}$ is an orthonormalized basis in the Hilbert space $l^2(P)$.

In the algebra of all bounded linear operators $B(l^2(P))$ on the space $l^2(P)$ we consider C^*-subalgebra $C_r^*(P)$ generated by the set of isometries $\{T_p \mid p \in P\}$, where $T_p(e_q) = e_{pq},\ p, q \in P$. It is called a *reduced semigroup C^*-algebra*. The unit element of this algebra is denoted by I.

If $P = \mathbb{N}$ is an additive semigroup of natural numbers, then the reduced semigroup C^*-algebra $C_r^*(\mathbb{N})$ is called a *Toeplitz algebra* and is denoted by the symbol $\mathcal{T}$.

We are going to describe a dense subalgebra in the C^*-algebra $C_r^*(P)$. For each element $p \in P$ we consider the symbols T_p^{-1} and T_p^1. We denote by $\mathcal{F}(P)$ a free semigroup of words formed by the letters in the alphabet $\{T_p^{-1}, T_p^1 \mid p \in P\}$. The semigroup $\mathcal{F}(P)$ is an involutive semigroup. An arbitrary element of this semigroup is a word *(monomial)* of form

$$V_{\overline{p}} := T_{p_1}^{i_1} T_{p_2}^{i_2} \dots T_{p_k}^{i_k}, \tag{1.2}$$

where $\overline{p} = (p_1, \dots, p_k)$ is an element of a k-multiple Cartesian product $P^{\times k} := P \times \dots \times P$, $i_1, \dots, i_k \in \{-1, 1\}$, $k \in \mathbb{N}$. The number k in writing (1.2) is called a *length* of the monomial.

The involution operation on the semigroup $\mathcal{F}(P)$ is defined by the formula

$$V_{\overline{p}}^* = T_{p_k}^{-i_k} T_{p_{k-1}}^{-i_{k-1}} \dots T_{p_1}^{-i_1}.$$

Each monomial $V_{\overline{p}}$ defines a bounded linear operator $\widehat{V_{\overline{p}}}$ on the Hilbert space $l^2(P)$ as follows:

$$\widehat{T_p^1} := T_p, \quad \widehat{T_p^{-1}} := T_p^*,$$

and for each monomial $V_{\overline{p}}$ of form (1.2) we let

$$\widehat{V_{\overline{p}}} := \widehat{T_{p_1}^{i_1}} \widehat{T_{p_2}^{i_2}} \dots \widehat{T_{p_k}^{i_k}}. \tag{1.3}$$

We call $\widehat{V_{\overline{p}}}$ an *operator monomial*.

Finite linear combinations of operators (1.3)

$$A = \sum_{i=1}^{m} \alpha_i \widehat{V}_{\overline{p}_i} \tag{1.4}$$

form a dense involutive subalgebra in the C^*-algebra $C_r^*(P)$, which we denote by $\mathcal{P}(P)$.

In what follows we recall necessary definitions from book [25] related with modules. By module we mean a left module.

Let $\mathfrak{A}$ be a unital C^*-algebra. A module $\mathfrak{M}$ over the C^*-algebra $\mathfrak{A}$ is called *Banach $\mathfrak{A}$-module* if it is a Banach space with the norm satisfying the inequality: $\|A \cdot M\| \leqslant \|A\|\|M\|$, where $A \in \mathfrak{A}$, $M \in \mathfrak{M}$.

An element M in a Banach $\mathfrak{A}$-module $\mathfrak{M}$ is called *cyclic* if the identity holds:

$$\mathfrak{M} = \mathfrak{A} \cdot M := \{A \cdot M \mid A \in \mathfrak{A}\}.$$

A Banach module possessing a cyclic element is called a *cyclic module*.

Let E be a Banach space. On a projective tensor product $\mathfrak{A}\widehat{\otimes}E$ there exists a structure of a unital left Banach $\mathfrak{A}$-module, which is uniquely defined by the formula

$$A \cdot (B \otimes X) = AB \otimes X, \quad A, B \in \mathfrak{A}, \quad X \in E.$$

A Banach module is called a *free unital Banach $\mathfrak{A}$-module* if it is topologically isomorphic to the module $\mathfrak{A}\widehat{\otimes}E$ for some Banach space E. For instance, the C^*-algebra $\mathfrak{A}$ itself is a free unital Banach $\mathfrak{A}$-module. Also a Banach direct l_1-sum of n copies of the module $\mathfrak{A}$ is a free Banach $\mathfrak{A}$-module since there is a topological isomoprphism of unital Banach $\mathfrak{A}$-modules

$$\bigoplus_1 \mathfrak{A} \cong \mathfrak{A}\widehat{\otimes}\mathbb{C}^n.$$

2. Embeddings of semigroup C^*-algebras induced by trivial extensions of semigroups

In this sections we enlarge the results obtained in work [22] on embeddings of C^*-algebras corresponding to semigroups forming normal extension of semigroups.

Theorem 2.1. *Let S be a semigroup with a left cancellation and (L, τ, σ) be a trivial extension of the semigroup S by means of a group G. Then there exists a unique isometric $*$-homomorphism $\varphi : C_r^*(S) \longrightarrow C_r^*(L)$ such that $\varphi(T_a) = T_{\tau(a)}$.*

Proof. Since (L, τ, σ) is a trivial extension, then up to an isomorphism of semigroups we have the identity

$$L = S \times G.$$

At the same time $\tau(a) = (a, e)$ and $\sigma(a, g) = g$ for all $a \in S$, $g \in G$.

There exists a canonical isomorphism of C^*-algebras [8, Lm. 2.16]:

$$\psi : C_r^*(S \times G) \longrightarrow C_r^*(S) \otimes_{\min} C_r^*(G) : T_{(a,g)} \mapsto T_a \otimes T_g.$$

We define a mapping

$$\theta : C_r^*(S) \longrightarrow C_r^*(S) \otimes_{\min} C_r^*(G) : A \mapsto A \otimes I.$$

It is obvious that θ is an injective $*$-homomorphism of C^*-algebras. Then the mapping

$$\varphi := \psi^{-1} \circ \theta : C_r^*(S) \longrightarrow C_r^*(S \times G)$$

is an injective $*$-homomorphism. It remains to confirm that $\varphi(T_a) = T_{\tau(a)}$. Indeed, since $\psi(T_{(a,e)}) = T_a \otimes I$, then $\varphi(T_a) = T_{(a,e)} = T_{\tau(a)}$. $\square$

We observe that Theorem 2.1 can be proved also without using Lemma 2.16 from [8]. This can be done in the same way how Theorem 3.1 was proved in [22].

Let us sketch the proof. We represent the Hilbert space $l^2(S \times G)$ as a direct sum of its subspaces

$$l^2(S \times G) = \bigoplus_{g \in G} H_g,$$

where the basis in the subspace H_g is the set $\{e_{(a,g)} \mid a \in S\}$. Each subspace H_g is invariant with respect to each operator $T_{(a,e)}$ and $T^*_{(a,e)}$, $a \in S$, and with respect to each operator monomial of form

$$\widehat{V}_{(\overline{a},\overline{e})} := \widehat{T}^{i_1}_{(a_1,e)} \widehat{T}^{i_2}_{(a_2,e)} \dots \widehat{T}^{i_k}_{(a_k,e)},$$

where $\overline{a} = (a_1, \dots, a_k) \in S^{\times k}$, $\overline{e} = (e, \dots, e) \in G^{\times k}$, $i_1, \dots, i_k \in \{-1, 1\}$, $k \in \mathbb{N}$.

Then we define a mapping φ by the identities $\varphi(T_a) = T_{(a,e)}$, $\varphi(T^*_a) = T^*_{(a,e)}$ and we extend it to the operator monomials $\widehat{V}_{\overline{a}}$ of form (1.3):

$$\varphi(\widehat{V}_{\overline{a}}) = \widehat{V}_{(\overline{a},\overline{e})}$$

and on finite linear combinations A of form (1.4):

$$\varphi(A) = \sum_{i=1}^{m} \alpha_i \varphi(\widehat{V}_{\overline{a}_i}) = \sum_{i=1}^{m} \alpha_i \widehat{V}_{(\overline{a}_i,\overline{e})}.$$

The well-definiteness of such extension is proved by means of unitary operators

$$U_g : l^2(S) \longrightarrow H_g : e_a \mapsto e_{(a,g)},$$

for all $a \in S$, $g \in G$. The constructed mapping φ is a unital $*$-homomorphism from the algebra $\mathcal{P}(S)$ into the C^*-algebra $C^*_r(S \times G)$.

Finally, it can be shown that for each $A \in \mathcal{P}(S)$ the identity

$$\varphi(A) = \bigoplus_{g \in G} U_g A U^*_g$$

holds true. Therefore, φ is an isometric $*$-homomorphism. It remains to extend φ to an isometric $*$-homomorphism on the entire C^*-algebra $C^*_r(S)$.

We note that if the semigroup S contains the unit e, then the trivial extension L possesses a generating set. Indeed, as it can be easily checked, the generating set is

$$X = \{(e, g) \mid g \in G\}.$$

Then the existence of an isometric $*$-homomorphism $\varphi : C^*_r(S) \longrightarrow C^*_r(L)$ follows from [22, Thm. 3.1]. On the other hand, Theorem 2.1 provides an example of the extension L of the semigroup S, which does not possess a generating set but an embedding of C^*-algebras $C^*_r(S) \longrightarrow C^*_r(L)$ exists.

Example 2.1. *As the semigroup S we choose an additive semigroup of natural numbers $\mathbb{N}$. Let G be an arbitrary group with the unit e. Then the Cartesian product $\mathbb{N} \times G$ is a semigroup with respect to the multiplication*

$$(n, g)(m, h) = (n + m, gh), \tag{2.1}$$

where $n, m \in \mathbb{N}$, $g, h \in G$. An extension $(\mathbb{N} \times G, \tau, \sigma)$ of the semigroup $\mathbb{N}$, where $\tau(n) = (n, e)$ and $\sigma(n, g) = g$ possesses no generating set. Indeed, assume that such set X exists. Then the element $x_g = (x, g) \in X$ should be represented as $(x, g) = \tau(n)(x, g)$ for some $n \in \mathbb{N}$. But then $(x, g) = (n + x, g)$ and therefore $n = 0$. We obtain a contradiction. On the other hand, by Theorem 2.1, there exists an isometric $*$-homomorphism

$$\varphi : C^*_r(\mathbb{N}) \longrightarrow C^*_r(\mathbb{N} \times G),$$

such that $\varphi(T_n) = T_{\tau(n)}$, *where* $n \in \mathbb{N}$.

3. Trivial extensions of semigroups and modules over semigroup C^*-algebras

Throughout this section (L, τ, σ) is a trivial extension of the semigroup S by means of the group G. That is, up to an isomorphism of semigroups, we have the identity

$$L = S \times G,$$

and $\tau(a) = (a, e)$, $\sigma(a, g) = g$ for all $a \in S$, $g \in G$.

To prove the main result of the present section we employ a topological grading of the C^*-algebra $C_r^*(L)$ over the group G. The construction of such grading was made and justified in work [12] for an arbitrary reduced semigroup C^*-algebra under the existence of a surjective semigroup homomorphism from the corresponding semigroup onto the group G. The definitions of the graded and topologically graded C^*-algebra can be found in [23, Sects. 16.2, 19.2]. In what follows we give a short description of the construction allowing one to obtain a topological grading of the C^*-algebra $C_r^*(L)$ over the group G.

Since (L, τ, σ) is an extension of the semigroup S by means of the group G, we have a surjective semigroup homomorphism

$$\sigma : L \longrightarrow G.$$

For the semigroup L we consider a free semigroup $\mathcal{F}(L)$ of monomials (1.2) formed by the letters in the alphabet $\{T_x^{-1}, T_x^1 \mid x \in L\}$:

$$V_{\overline{x}} := T_{x_1}^{i_1} T_{x_2}^{i_2} \dots T_{x_k}^{i_k},$$

where $\overline{x} = (x_1, \dots, x_k) \in L^{\times k}$, $i_1, \dots, i_k \in \{-1, 1\}$, $k \in \mathbb{N}$.

We define a mapping $\mathrm{ind} : \mathcal{F}(L) \longrightarrow G$ by the formula

$$\mathrm{ind}\,(V_{\overline{x}}) = \sigma(x_1)^{i_1} \sigma(x_2)^{i_2} \dots \sigma(x_k)^{i_k}.$$

It is easy to see that the mapping ind is an involutve surjective homomorphism of the semigroups. The value $\mathrm{ind}\,(V_{\overline{x}})$ is called a *σ-index of a monomial* $V_{\overline{x}}$.

It was shown in [12, Lm. 1] that if two monomials define the same operator monomial, then their σ-indices coincide, that is, if $\widehat{V}_{\overline{x}} = \widehat{V}_{\overline{y}}$, then $\mathrm{ind}\,(V_{\overline{x}}) = \mathrm{ind}\,(V_{\overline{y}})$. This is the quantity $\mathrm{ind}\,(V_{\overline{x}}) \in G$ can be also called *σ-index of the operator monomial* $\widehat{V}_{\overline{x}}$. It is easy to confirm that the set of monomials with σ-index e is an involutive semigroup in the group of all monomials $\mathcal{F}(L)$.

Let $\mathfrak{A}_e$ stand for the C^*-subalgebra in the C^*-algebra $C_r^*(L)$ generated by the set of all operator monomials with σ-index e. Let $\mathfrak{A}_g$ be a Banach subspace in the C^*-algebra $C_r^*(L)$ being a closure of the linear span of the set of all operator monomials with σ-index g, $g \in G$.

The family of subspaces $\{\mathfrak{A}_g \mid g \in G\}$ forms a topological G-grading of the reduced semigroup C^*-algebra $C_r^*(L)$ [12, Thm. 2].

Now we prove one technical lemma.

Lemma 3.1. *Let S be a semigroup with a left cancellation and G be a group with a unit e. Then in the C^*-algebra $C_r^*(L)$ for all $a_1, \dots, a_k \in S$ and $g_1, \dots, g_k \in G$ the following identity holds for the operator monomials:*

$$\widehat{T}_{(a_1,g_1)}^{i_1} \dots \widehat{T}_{(a_{k-1},g_{k-1})}^{i_{k-1}} \widehat{T}_{(a_k,g_k)}^{i_k} = \widehat{T}_{(a_1,e)}^{i_1} \dots \widehat{T}_{(a_{k-1},e)}^{i_{k-1}} \widehat{T}_{(a_k,g^{i_k})}^{i_k}, \tag{3.1}$$

where $g = g_1^{i_1} g_2^{i_2} \dots g_k^{i_k}$, $i_1, \dots, i_k \in \{-1, 1\}$, $k \in \mathbb{N}$, $k \geq 2$.

Proof. We consider an arbitrary monomial

$$\widehat{T}^{i_1}_{(a_1,g_1)} \dots \widehat{T}^{i_{k-1}}_{(a_{k-1},g_{k-1})}\widehat{T}^{i_k}_{(a_k,g_k)}.$$

We shall prove the lemma by induction in the length k of the monomial.

Let $k = 2$. Then identity (3.1) becomes:

$$\widehat{T}^{i}_{(a,p)}\widehat{T}^{j}_{(b,q)} = \widehat{T}^{i}_{(a,e)}\widehat{T}^{j}_{(b,(p^i q^j)^j)}, \tag{3.2}$$

where $a, b \in S$, $p, q \in G$, $i, j \in \{-1, 1\}$. Let us prove; in order to do this, we consider four cases.

1) Let $i = j = 1$. Then

$$T_{(a,p)}T_{(b,q)} = T_{(ab,pq)} = T_{(a,e)}T_{(b,pq)}.$$

2) Let $i = j = -1$. Then

$$T^*_{(a,p)}T^*_{(b,q)} = T^*_{(ba,qp)} = (T_{(b,qp)}T_{(a,e)})^* = T^*_{(a,e)}T^*_{(b,(p^{-1}q^{-1})^{-1})}.$$

3) Let $i = -1$, $j = 1$. We calculate $T^*_{(a,p)}T_{(b,q)}$ on an arbitrary basis vector $e_{(c,h)} \in l^2(S \times G)$. If $T^*_{(a,p)}T_{(b,q)}e_{(c,h)} \neq 0$, then

$$T^*_{(a,p)}T_{(b,q)}e_{(c,h)} = e_{(d,p^{-1}qh)}$$

for some $d \in S$ such that $ad = bc$. Indeed,

$$\langle T^*_{(a,p)}T_{(b,q)}e_{(c,h)}, e_{(d,p^{-1}qh)}\rangle = \langle e_{(bc,qh)}, T_{(a,p)}e_{(d,p^{-1}qh)}\rangle = \langle e_{(bc,qh)}, e_{(ad,qh)}\rangle \neq 0$$

if and only if $ad = bc$. On the other hand, in this case,

$$T^*_{(a,e)}T_{(b,p^{-1}q)}e_{(c,h)} = e_{(d,p^{-1}qh)}.$$

If there exists no such d, then

$$T^*_{(a,p)}T_{(b,q)}e_{(c,h)} = T^*_{(a,e)}T_{(b,p^{-1}q)}e_{(c,h)} = 0.$$

Thus, we have the identity of operators:

$$T^*_{(a,p)}T_{(b,q)} = T^*_{(a,e)}T_{(b,p^{-1}q)}.$$

4) Let $i = 1$, $j = -1$. We again calculate $T_{(a,p)}T^*_{(b,q)}$ on an arbitrary basis vector $e_{(c,h)} \in l^2(S \times G)$. If $T_{(a,p)}T^*_{(b,q)}e_{(c,h)} \neq 0$, then

$$T_{(a,p)}T^*_{(b,q)}e_{(c,h)} = e_{(ad,pq^{-1}h)}$$

for some $d \in S$ such that $c = bd$. On the other hand, in this case we have:

$$T_{(a,e)}T^*_{(b,qp^{-1})}e_{(c,h)} = e_{(ad,(qp^{-1})^{-1}h)} = e_{(ad,pq^{-1}h)}.$$

If there exists no such d, then

$$T_{(a,p)}T^*_{(b,q)}e_{(c,h)} = T_{(a,e)}T^*_{(b,qp^{-1})}e_{(c,h)} = 0.$$

Thus, we have an identity:

$$T_{(a,p)}T^*_{(b,q)} = T_{(a,e)}T^*_{(b,qp^{-1})} = T_{(a,e)}T^*_{(b,(pq^{-1})^{-1})}.$$

The considered four cases prove completely identity (3.2).

Now we consider a monomial of an arbitrary length k. By the induction assumption we have the identity

$$\widehat{T}^{i_1}_{(a_1,g_1)} \dots \widehat{T}^{i_{k-1}}_{(a_{k-1},g_{k-1})}\widehat{T}^{i_k}_{(a_k,g_k)} = \widehat{T}^{i_1}_{(a_1,e)} \dots \widehat{T}^{i_{k-2}}_{(a_{k-2},e)}\widehat{T}^{i_{k-1}}_{(a_{k-1},h^{i_{k-1}})}\widehat{T}^{i_k}_{(a_k,g_k)}, \tag{3.3}$$

where $h = g_1^{i_1}g_2^{i_2}\dots g_{k-1}^{i_{k-1}}$. We apply formula (3.2) to the product $\widehat{T}^{i_{k-1}}_{(a_{k-1},h^{i_{k-1}})}\widehat{T}^{i_k}_{(a_k,g_k)}$. This gives the identity

$$\widehat{T}^{i_{k-1}}_{(a_{k-1},h^{i_{k-1}})}\widehat{T}^{i_k}_{(a_k,g_k)} = \widehat{T}^{i_{k-1}}_{(a_{k-1},e)}\widehat{T}^{i_k}_{(a_k,(hg_k^{i_k})^{i_k})} = \widehat{T}^{i_{k-1}}_{(a_{k-1},e)}\widehat{T}^{i_k}_{(a_k,g^{i_k})}, \tag{3.4}$$

where $g = g_1^{i_1} \dots g_{k-1}^{i_{k-1}} g_k^{i_k}$. Taking into consideration identities (3.3) and (3.4), we obtain desired identity (3.1). □

In the next theorem we see what is the C^*-subalgebra $\mathfrak{A}_e$ of the semigroup C^*-algebra $C_r^*(L)$.

Theorem 3.1. *Let S be a semigroup with a left cancellation and (L, τ, σ) be a trivial extension of this semigroup by means of the group G. Let $\mathfrak{A}_e$ be the C^*-subalgebra in the C^*-algebra $C_r^*(L)$ generated by the operator monomials of σ-index e, where e is the unit of the group G. Then an isometric isomorphism of C^*-algebras holds:*

$$C_r^*(S) \cong \mathfrak{A}_e.$$

Proof. An operator monomial the C^*-algebra $C_r^*(L)$ reads as

$$\widehat{V}_{(\overline{a},\overline{g})} := \widehat{T}^{i_1}_{(a_1,g_1)} \widehat{T}^{i_2}_{(a_2,g_2)} \dots \widehat{T}^{i_k}_{(a_k,g_k)}, \tag{3.5}$$

where $\overline{a} = (a_1, \dots, a_k) \in S^{\times k}$, $\overline{g} = (g_1, \dots, g_k) \in G^{\times k}$, $i_1, \dots, i_k \in \{-1, 1\}$, $k \in \mathbb{N}$. At the same time, the σ-index of operator monomial (3.5) is equal to

$$\operatorname{ind}(V_{(\overline{a},\overline{g})}) = g_1^{i_1} g_2^{i_2} \dots g_k^{i_k}. \tag{3.6}$$

By Theorem 2.1 we obtain that there exists an isometric $*$-homomorphism

$$\varphi : C_r^*(S) \longrightarrow C_r^*(L) : T_a \mapsto T_{(a,e)}.$$

Then for each operator monomial $\widehat{V_{\overline{a}}} \in C_r^*(S)$ of form (1.3) the identity holds: $\varphi(\widehat{V_{\overline{a}}}) = \widehat{V}_{(\overline{a},\overline{e})}$, where $\overline{a} = (a_1, \dots, a_k) \in S^{\times k}$, $\overline{e} = (e, \dots, e) \in G^{\times k}$, $k \in \mathbb{N}$. Since $\operatorname{ind}(V_{(\overline{a},\overline{e})}) = e$, then $\varphi(\mathcal{P}(S)) \subset \mathfrak{A}_e$ and therefore, we can consider a corestriction of φ on $\mathfrak{A}_e$:

$$\varphi^0 : C_r^*(S) \longrightarrow \mathfrak{A}_e, \tag{3.7}$$

which is an injective $*$-homomorhpism. Let us show that φ^0 is surjective.

It follows from Lemma 3.1 and formula (3.6) that if $\operatorname{ind}(V_{(\overline{a},\overline{g})}) = e$, then we have the identity

$$\widehat{V}_{(\overline{a},\overline{g})} = \widehat{T}^{i_1}_{(a_1,g_1)} \widehat{T}^{i_2}_{(a_2,g_2)} \dots \widehat{T}^{i_k}_{(a_k,g_k)} = \widehat{T}^{i_1}_{(a_1,e)} \widehat{T}^{i_2}_{(a_2,e)} \dots \widehat{T}^{i_k}_{(a_k,e)} = \widehat{V}_{(\overline{a},\overline{e})}.$$

This means that a dense subalgebra in the C^*-algebra $\mathfrak{A}_e$ coincides with the set of all possible finite linear combinations

$$\sum_{i=1}^{m} \alpha_i \widehat{V}_{(\overline{a}_i,\overline{e})}$$

of the operators of form $\widehat{V}_{(\overline{a},\overline{e})}$, where $\overline{a} = (a_1, \dots, a_k) \in S^{\times k}$, $\overline{e} = (e, \dots, e) \in G^{\times k}$, $k \in \mathbb{N}$. This implies the surjectivity of the homomorphism φ^0. Thus, φ^0 is an isometric isomorphism of the C^*-algebras $C_r^*(S)$ and $\mathfrak{A}_e$. □

On the underlaying space of the C^*-algebra $C_r^*(L)$ we define a structure of the Banach $C_r^*(S)$-module by defining the operation of the left external multiplication as follows:

$$A \cdot B = \varphi^0(A)B, \tag{3.8}$$

where $A \in C_r^*(S)$, $B \in C_r^*(L)$ and $\varphi^0 : C_r^*(S) \longrightarrow \mathfrak{A}_e$ is isometric isomorphism (3.7) from Theorem 3.1. In what follows we shall show that if L is a trivial extension of the semigroup S by means of a finite group, then this module is free. But first we need one auxiliary statement.

We fix an arbitrary element $a \in S$. In the C^*-algebra $C_r^*(L)$, for each $g \in G$ we consider the operators of form $V_g := T^*_{(a,e)} T_{(a,g)}$. Let us show that V_g are unitary operators. Indeed, taking into consideration Lemma 3.1, we obtain the identities:

$$V_g V_g^* = T^*_{(a,e)} T_{(a,g)} T^*_{(a,g)} T_{(a,e)} = T^*_{(a,e)} T_{(a,e)} T^*_{(a,e)} T_{(a,gg^{-1})} = I,$$
$$V_g^* V_g = T^*_{(a,g)} T_{(a,e)} T^*_{(a,e)} T_{(a,g)} = T^*_{(a,e)} T_{(a,e)} T^*_{(a,e)} T_{(a,g^{-1}g)} = I.$$

Lemma 3.2. *For each $g \in G$ the identity*

$$\mathfrak{A}_g = C_r^*(S) \cdot V_g$$

holds, that is, the space $\mathfrak{A}_g$ is a cyclic Banach $C_r^*(S)$-module and the element V_g is a cyclic element of the module $\mathfrak{A}_g$.

Proof. In view of identity (3.8) and the fact that $\varphi^0 : C_r^*(S) \longrightarrow \mathfrak{A}_e$ is an isometric isomorphism, in order to prove the lemma, it is sufficient to prove the identity

$$\mathfrak{A}_g = \{AV_g \mid A \in \mathfrak{A}_e\}.$$

Since $\operatorname{ind}(V_g) = g$ and $\|V_g\| = 1$, the proof of the needed identity reproduce literally the proof of Lemma 5 in [12]. □

Theorem 3.2. *Let S be a semigroup with a left cancellation, G be a finite group and (L, τ, σ) be a trivial extension of the semigroup S by means of the group G. Then there exists a topological isomorphism of the Banach $C_r^*(S)$-modules*

$$C_r^*(L) \cong \bigoplus_1 C_r^*(S),$$

where the number of the terms in the direct l_1-sum is equal to the order of the group G. In other words, the C^*-algebra $C_r^*(L)$ is a free Banach $C_r^*(S)$-module.

Proof. We first observe that since the group G is finite, then as it was shown in [12, Thm. 4], the underlaying space of C^*-algebra $C_r^*(L)$ is represented as the direct sum of its subspaces:

$$C_r^*(L) = \bigoplus_{g \in G} \mathfrak{A}_g.$$

This means that each element $A \in C_r^*(L)$ is uniquely represented as a finite sum

$$A = \sum_{g \in G} A_g,$$

where $A_g \in \mathfrak{A}_g$.

To prove the theorem, it is sufficient to show the existence of an isomorphism of $C_r^*(S)$-modules

$$\bigoplus_{g \in G} \mathfrak{A}_g \cong \bigoplus_1 C_r^*(S).$$

It follows from Lemma 3.2 that the $C_r^*(S)$-module $\mathfrak{A}_g$ is topologically isomorphic to the quotient module $C_r^*(S)/\operatorname{Ann}\{V_g\}$ [25, Prop. VI.2.3], where

$$\operatorname{Ann}\{V_g\} := \{A \in C_r^*(S) \mid A \cdot V_g = 0\}$$

is the annulator of the element V_g. Since V_g is a unitary element, it is easy to confirm that $\operatorname{Ann}\{V_g\} = 0$. Therefore, we have a topological isomorphism of Banach $C_r^*(S)$-modules

$$\psi_g : C_r^*(S) \longrightarrow \mathfrak{A}_g : A \mapsto A \cdot V_g.$$

We define a linear mapping:

$$\alpha : \bigoplus_1 C_r^*(S) \longrightarrow \bigoplus_{g \in G} \mathfrak{A}_g$$

by the formula

$$\alpha(B) = \sum_{g \in G} \psi_g(B_g),$$

where $B = (B_g)_{g\in G} \in \bigoplus_1 C_r^*(S)$. It is easy to confirm that α is surjective. The injectivity of α is implied by the linear independence of the family of the subspaces $\{\mathfrak{A}_g\}_{g\in G}$. The continuity of α follows from the chain of inequalities

$$\|\alpha(B)\| \leqslant \sum_{g\in G} \|\psi_g(B_g)\| \leqslant \max_{g\in G} \|\psi_g\| \cdot \sum_{g\in G} \|B_g\| = \max_{g\in G} \|\psi_g\| \cdot \|B\|_1.$$

Since α is a bijective bounded linear operator, then by the Banach theorem on the inverse operator it has a bounded inverse operator

$$\alpha^{-1} : \bigoplus_{g\in G} \mathfrak{A}_g \longrightarrow \bigoplus_1 C_r^*(S).$$

It is obvious that α and α^{-1} are morphisms of left $C_r^*(S)$-modules. Therefore, the mapping α is a topological isomorphism of Banach $C_r^*(S)$-modules. Thus, the C^*-algebra $C_r^*(L)$ is a free Banach $C_r^*(S)$-module. □

In work [13], there were described conditions under which the underlaying space of an arbitrary topologically graded semigroup C^*-algebra $C_r^*(L)$ possesses a structure of a free Banach module over its subalgebra $\mathfrak{A}_e$. Namely, let $X := \{x_g \mid g \in G\}$ be a set of elements in L such that the condition $X \cap \sigma^{-1}(g) = \{x_g\}$ holds. Let G be a finite group and in the semigroup L there exists a set X such that each element is invertible in L. Then the C^*-algebra $C_r^*(L)$ is a free Banach $\mathfrak{A}_e$-module [13, Thm. 2].

If under the assumptions of Theorem 3.2 the semigroup S contains the unit e, then in the semigroup L there exists a set X such that each its element is invertible:

$$X = \{(e, g) \mid g \in G\}.$$

Then Theorem 3.2 is a corollary of Theorem 3.1 and [13, Thm. 2]. On the other hand, if the semigroup S contains no unity, then Theorem 3.2 provides an example showing that the statement inverse to [13, Thm. 2] is wrong.

Example 3.1. *Let G be a finite group. As in Example 2.1, we consider an additive semigroup of natural numbers $\mathbb{N}$ and the Cartesian product $\mathbb{N}\times G$ with a multiplication defined by formula (2.1). Then the C^*-algebra $C_r^*(\mathbb{N} \times G)$ is a free Banach module over the Toeplitz algebra $\mathcal{T} = C_r^*(\mathbb{N})$ and we have an isomorphism of Banach $\mathcal{T}$-modules:*

$$C_r^*(\mathbb{N} \times G) \cong \bigoplus_1 \mathcal{T}.$$

At the same time, the semigroup $\mathbb{N} \times G$ contains no other subgroups.

BIBLIOGRAPHY

1. L.A. Coburn. *The C^*-algebra generated by an isometry* // Bull. Amer. Math. Soc. **73**:5, 722–726 (1967).
2. L.A. Coburn. *The C^*-algebra generated by an isometry II* // Trans. Amer. Math. Soc. **137**, 211–217 (1969).
3. R.G. Douglas. *On the C^*-algebra of a one-parameter semigroup of isometries* // Acta Math. **128**:3, 143–151 (1972).
4. G.J. Murphy. *Ordered groups and Toeplitz algebras* // J. Oper. Theory. **18**:2, 303–326 (1987).
5. G.J. Murphy. *Toeplitz operators and algebras* // Math. Z. **208**:3, 355–362 (1991).
6. A. Nica. *C^*-algebras generated by isometries and Wiener – Hopf operators* // J. Oper. Theory. **27**:1, 17–52 (1992).
7. M. Laca, I. Raeburn. *Semigroup crossed products and the Toeplitz algebras of nonabelian groups* // J. Funct. Anal. **139**:2, 415–440 (1996).

8. X. Li *Semigroup C^*-algebras and amenability of semigroups* // J. Funct. Anal. **262**:10, 4302–4340 (2012).
9. S.A. Grigoryan, T.A. Grigoryan, E.V. Lipacheva, A. S. Sitdikov. *C^*-Algebra generated by the path semigroup* // Lobachevskii J. Math. **37**:6, 740–748 (2016).
10. S.A. Grigoryan, E.V. Lipacheva, A.S. Sitdikov. *Nets of graded C^*-algebras over partially ordered sets* // Alg. Anal. **30**:6, 1–19 (2018). [St. Petersburg Math. J. **30**:6, 901–915 (2019).]
11. R.N. Gumerov, E.V. Lipacheva. *Topological grading of semigroup C^*-algebras* // Herald of the Bauman Moscow State Technical University, Series Natural Sciences. **90**:3, 44–55 (2020).
12. E.V. Lipacheva. *On graded semigroup C^*-algebras and Hilbert modules* // Trudy MIAN. **313**, 131–142 (2021). [Proc. Steklov Inst. Math. **313**, 120–130 (2021).]
13. E.V. Lipacheva. *A Semigroup C^*-algebra which is a Free Banach Module* // Lobachevskii J. Math. **42**:10, 2386–2391 (2021).
14. B.V. Novikov. *Semigroup cohomologies: a survey* // Fund. Prikl. Matem. **7**:1, 1–18 (2001). (in Russian).
15. A.H. Clifford. *Extensions of semigroups* // Trans. Amer. Math. Soc. **68**:2, 165–173 (1950).
16. L. Rédei. *Die Verallgemeinerung der Schreierschen Erweiterungstheorie* // Acta Sci. Math. Szeged. **14**, 252–273 (1952).
17. L.M. Gluskin, I.L. Perepelitsyn. *Normal extensions of semi-groups* // Izv. VUZov. Matem. 12, 46–54 (1972). (in Russian).
18. L.M. Gluskin. *Normal extensions of commutative semigroups* // Izv. VUZov. Matem. **29**:9, 14–22 (1985). [Russian Math. (Iz. VUZ). **29**:9, 16–27 (1985).]
19. I.S. Berdnikov, R.N. Gumerov, E.V. Lipacheva. *On the Stone-Čech compactification functor and the normal extensions of monoids* // Lobachevskii J. Math. **42**:10, 2295–2305 (2021).
20. S.A. Grigoryan, R.N. Gumerov, E.V. Lipacheva. *On extensions of semigroups and their applications to Toeplitz algebras* // Lobachevskii J. Math. **40**:12, 2052–2061 (2019).
21. R.N. Gumerov. *Normal extensions of semigroups and embeddings of semigroup C^*-algebras* // Trudy MFTI. **12**:1, 74–82 (2020). (in Russian).
22. E.V. Lipacheva. *Extensions of semigroups and morphisms of semigroup C^*-algebras* // Sibir. Matem. Zh. **62**:1, 82–96 (2021). [Siberian Math. J. **62**:1, 66–76 (2021).]
23. R. Exel. *Partial dynamical systems, Fell bundles and applications.* Amer. Math. Soc., Providence, RI. (2017).
24. E.S. Lyapin. *Semigroups.* Fizmatgiz, Moscow (1960). [Amer. Math. Soc., Providence, R.I. (1974).]
25. A.Ya Helemskii. *Banach and polynormed algebras. General theory, representations, homologies.* Nauka, Moscow (1989). [*Banach and locally convex algebras.* Clarendon Press, Oxford (1993).]

Ekaterina Vladimirovna Lipacheva,
Kazan State Power Engineering University,
Krasnoselskaya str. 51,
420066, Kazan, Russia

Lobachevskii Institute of Mathematics and Mechanics,
Kazan Federal University,
Kremlevskaya str. 35,
420008, Kazan, Russia
E-mail: elipacheva@gmail.com