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ON DIVISIBLE QUANTUM DYNAMICAL MAPPINGS

R.N. GUMEROV, R.L. KHAZHIN

Abstract. In this paper we study quantum dynamical mappings called also quantum processes. The set of values of such mapping is a one-parameter family of completely positive trace-preserving linear operators defined on a finite-dimensional Hilbert space. In quantum information theory such operators are referred to as quantum channels. An important concept for quantum dynamical mappings is their divisibility. There are different types of this concept. The present paper deals with so-called completely positive divisible quantum processes. For two such processes, which are bijective and satisfy a commutativity condition, we construct a compound quantum process. It is shown that this compound quantum process is also completely positive divisible. Endowing a set of quantum channels with the norm topology, we consider continuous quantum processes and continuous completely positive evolutions. The latter are defined as two-parameter families of quantum channels satisfying additional properties. We prove that a continuous bijective completely positive divisible quantum process generates a continuous completely positive evolution. In order to illustrate the considered concepts and the results on them, we provide examples of quantum dynamical mappings with values in the set of qubit channels. In particular, a completely positive divisible compound quantum process is constructed for two bijective commuting quantum processes. Geometric and physical interpretations of this compound quantum process are given.

Keywords: Banach algebra, bijective process, completely positive divisible process, compound process, continuous completely positive evolution, positive divisible process, operator norm, quantum channel, quantum dynamical mapping, quantum process, topological group, trace norm.

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1. Introduction

The paper is devoted to one-parametric and two-parametric families consisting of completely positive trace-preserving operators. In quantum information theory such operators are called quantum channels. One-parametric families of quantum channels are called quantum dynamical mappings or processes. Quantum processes describe how the states of quantum system vary in time.

One of important characteristics of quantum processes is their divisibility. Various types of divisibility of quantum structures and closely related issues were studied in a series of papers, which served as a motivation for the present work. Infinitely divisible measuring and Markov mappings in quantum probability theory were studied in works [1] and [2]. In work [3], divisible and infinitesimally divisible quantum channels were considered as well as closely related continuous completely positive evolutions, the values of which were two-parametric families of

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quantum channels. Some of papers were devoted to positive and completely positively divisibilities, which are briefly called respectively a P-divisibility and a CP-divisibility. The relation between these types of divisibility of dynamical mapping and their tensor powers was presented in paper [4]. Bijective CP-divisible quantum processes were studied in works [5], [6]. The relations between so-called L-divisibility, P-divisibility and CP-divisibility were considered in [7]. The study of divisibility of quantum mapping is physically motivated and is of a sure mathematical interest, see [8]–[11] and the references therein.

In the present paper we consider bijective CP-divisible quantum processes. For two such processes satisfying an additional commutativity condition, we construct a compound quantum process. We prove that it is $CP(P)$-divisible. On the set of quantum channels we introduce a topology defined by means of trace (nuclear) form defined on the space of linear operator on a finite-dimensional Hilbert space. This allows us to consider continuous processes and evolutions. We prove that each bijective CP-divisible quantum process generates a continuous completely positive evolution. We provide examples of quantum dynamical mappings taking values in the set of one-qubit channels, which demonstrate the studied processes and their properties. Namely, we consider two bijective quantum processes, for which the compound process is CP-divisible. We provide a geometric and a physical interpretation of this compound quantum process. We also construct a quantum process consisting of unitary channels which is discontinuous at a fixed time.

The paper is organised as follows. It consists of the introduction and four sections. In the second section we collect definitions and facts needed for further presentation. In the third section we consider bijective CP-divisible quantum processes satisfying the commutativity condition and compound processes for them. The fourth section is devoted to continuous quantum processes and completely positive evolutions. In the fifth section we construct quantum processes which illustrate the considered notions and their properties.

2. Preliminaries

Throughout the paper $\mathcal{H}$ stands for a finite-dimensional complex Hilbert space. The complex vector space of all linear operators on $\mathcal{H}$ is denoted by $\mathfrak{L}(\mathcal{H})$.

A linear completely positive trace-preserving mapping $\Phi : \mathfrak{L}(\mathcal{H}) \longrightarrow \mathfrak{L}(\mathcal{H})$ is called a *quantum channel*. We recall that Φ is completely positive if for each $n \in \mathbb{N}$ the linear mapping Φ_n well-defined by the formula

$$\Phi_n : \mathfrak{L}(\mathcal{H} \otimes \mathbb{C}^n) \longrightarrow \mathfrak{L}(\mathcal{H} \otimes \mathbb{C}^n) : X \otimes Y \longmapsto \Phi(X) \otimes Y,$$

is positive, that is, for each $A \in \mathfrak{L}(\mathcal{H} \otimes \mathbb{C}^n)$ the identity $\Phi_n(A^*A) = B^*B$ for some $B \in \mathfrak{L}(\mathcal{H} \otimes \mathbb{C}^n)$. Here, as usually, $\mathbb{C}^n$ denotes the n-dimensional complex space with the standard scalar product, $*$ is the Hilbert adjoint. The symbol $\otimes$ is used for denoting both the Hilbert tensor product of spaces and the tensor product of operators. A convex set of all quantum channels from $\mathfrak{L}(\mathcal{H})$ in $\mathfrak{L}(\mathcal{H})$ is denoted by $\mathcal{O}_c(\mathcal{H})$. This is a semigroup, in which as the multiplication, the composition of the operators serves; it is denoted by the symbol $\circ$. The unit in the semigroup $\mathcal{O}_c(\mathcal{H})$ is the identical channel $\mathcal{I} : \mathfrak{L}(\mathcal{H}) \longrightarrow \mathfrak{L}(\mathcal{H})$. Further information on quantum channels can be found in book [12].

We shall also use capital Greek letters for denoting operator-valued functions of form

$$\Phi : [0; T] \longrightarrow \mathcal{O}_c(\mathcal{H}) : t \longmapsto \Phi(t), \quad \text{where} \quad \Phi(0) = \mathcal{I},$$

and associated one-parameteric families of quantum channels

$$\Phi = \{\Phi(t) \in \mathcal{O}_c(\mathcal{H}) \mid 0 \leqslant t \leqslant T,\ \Phi(0) = \mathcal{I}\},$$

which are called *quantum dynamical mappings* or *quantum processes*. Each of these Φ will be shortly called a *process*. Hereinafter in the text, T is an arbitrary fixed positive number.

A quantum process Φ is called *divisible* if for all $0 \leqslant t_1 \leqslant t_2 \leqslant T$ there exists a linear operator $\Phi(t_2, t_1) : \mathfrak{L}(\mathcal{H}) \longrightarrow \mathfrak{L}(\mathcal{H})$ such that the identity holds

$$\Phi(t_2) = \Phi(t_2, t_1) \circ \Phi(t_1). \tag{2.1}$$

At the same time, if each operator $\Phi(t_2, t_1)$ is a quantum channel, then the process Φ is called *completely positive divisible* or *CP-divisible*. If each operator $\Phi(t_2, t_1)$ preserves the positivity and the trace, then the process Φ is called *positive divisible* or *P-divisible*.

At the next step we define quantum processes, which are apriori divisible. We first recall that the operator inverse to a completely positive operator is not necessarily completely positive.

A quantum process Φ is called *bijective* if for each $t \in [0; T]$ there exists a linear operator $\Phi(t)^{-1} : \mathfrak{L}(\mathcal{H}) \longrightarrow \mathfrak{L}(\mathcal{H})$ being inverse for the operator $\Phi(t)$, that is, the identity

$$\Phi(t) \circ \Phi(t)^{-1} = \Phi(t)^{-1} \circ \Phi(t) = \mathcal{I} \tag{2.2}$$

holds.

It is obvious that each bijective quantum process is divisible. Indeed, for such process in identity (2.1), as the linear operator $\Phi(t_2, t_1)$, the composition of the operators $\Phi(t_2) \circ \Phi(t_1)^{-1}$ serves, where $0 \leqslant t_1 \leqslant t_2 \leqslant T$.

On the complex vector space $\mathfrak{L}(\mathcal{H})$ we consider a *trace (nuclear) norm* defined by the formula

$$\|A\|_{tr} := \operatorname{tr} \sqrt{A^* A}, \qquad A \in \mathfrak{L}(\mathcal{H}), \tag{2.3}$$

where the symbol tr denotes the trace of the arithmetic square root of the positive operator $A^*A \in \mathfrak{L}(\mathcal{H})$. In its turn, this norm induces the operator norm $\| \cdot \|$ on the complex vector space $\mathfrak{B}(\mathfrak{L}(\mathcal{H}))$ of all linear operators acting from the normed space $(\mathfrak{L}(\mathcal{H}), \| \cdot \|_{tr})$ into itself. This norm can be defined by the formula

$$\|\Lambda\| := \max \{ \|\Lambda(A)\|_{tr} \mid A \in \mathfrak{L}(\mathcal{H}), \|A\|_{tr} \leqslant 1 \}, \tag{2.4}$$

where $\Lambda \in \mathfrak{B}(\mathfrak{L}(\mathcal{H}))$. For each positive trace-preserving operator $\Phi \in \mathfrak{B}(\mathfrak{L}(\mathcal{H}))$ the identity $\|\Phi\| = 1$ holds [13, Cor. 3.40].

As each operator norm, the induced trace norm is *(sub)multiplicative*, that is, for all $\Lambda_1, \Lambda_2 \in \mathfrak{B}(\mathfrak{L}(\mathcal{H}))$ the identity holds:

$$\|\Lambda_2 \circ \Lambda_1\| \leqslant \|\Lambda_1\| \cdot \|\Lambda_2\|. \tag{2.5}$$

The normed space $(\mathfrak{B}(\mathfrak{L}(\mathcal{H})), \| \cdot \|)$ with the composition as the multiplication and with the identity mapping $\mathcal{I}$ as the unit becomes a unital Banach algebra [14, Ch. 5].

In the literature on quantum information theory, for instance, [13], the norm $\| \cdot \|$ is also called an *induced trace norm*. As it is known, in contrast to a so-called completely bounded norm (see, for instance, [15]) it defines the metrics, which is not supported by the physical justification for introducing it on the set of quantum channels. However, for studying topological properties of a considered mapping the norm $\| \cdot \|$ is sufficient. This is explained by the fact that all norms on a given finite-dimensional vector space are equivalent and define the same topology.

We recall that the group of all invertible elements $\operatorname{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H})))$ of the algebra $(\mathfrak{B}(\mathfrak{L}(\mathcal{H})), \| \cdot \|)$ is a topological group [16, Thm. 1.2.43]. At the same time, the consistence of algebraic and topological structures in $\operatorname{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H})))$ allows us to speak about the continuity of the mapping

$$\Pi : \operatorname{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H}))) \times \operatorname{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H}))) \to \operatorname{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H}))) : (\Lambda_1, \Lambda_2) \mapsto \Lambda_1 \circ \Lambda_2^{-1} \tag{2.6}$$

from the Cartesian square of the group with the topology of the product into the group [17, Sect. 17].

3. Compound quantum processes

We consider a pair of arbitrary quantum dynamical mappings

$$\Phi, \Omega : [0; T] \longrightarrow \mathcal{O}_c(\mathcal{H}).$$

Since the set of quantum channels $\mathcal{O}_c(\mathcal{H})$ is closed with respect to the composition of the operators, then the following definition is well-defined. A quantum dynamical mapping

$$\Sigma(\Omega, \Phi) : [0; T] \longrightarrow \mathcal{O}_c(\mathcal{H}) : t \longmapsto \Omega(t) \circ \Phi(t),$$

is called a *compound quantum processes* for Φ and Ω.

As usually, for the processes Φ and Ω by $[\Phi(s), \Omega(t)]$ we denote the commutator of the channels $\Phi(s)$ and $\Omega(t)$, where $s, t \in [0; T]$. In what follows we consider the processes satisfying the condition

$$[\Omega(t_1), \Phi(t_2)] = 0 \quad \text{for all} \quad t_1, t_2 \in [0; T] \quad \text{such that} \quad t_1 \leqslant t_2. \tag{3.1}$$

We shall say that the processes Φ and Ω *commute* if the commutator in (3.1) vanishes for all $t_1, t_2 \in [0; T]$.

We observe that for the commuting processes Φ and Ω the compound quantum processes $\Sigma(\Omega, \Phi)$ and $\Sigma(\Phi, \Omega)$ coincide. It is clear that the compound process for the bijective quantum processes is bijective.

Proposition 3.1. *Let Φ and Ω be bijective (positively) completely positive divisible processes satisfying condition (3.1). Then the compound quantum process $\Sigma(\Omega, \Phi)$ is also bijective (positively) completely positive divisible.*

Proof. We shall prove the proposition for the case of completely positive divisible processes Φ and Ω. The case of positive divisible processes can be proved in the same way.

For the brevity of writing we introduce the notation $\Sigma := \Sigma(\Omega, \Phi)$. It is obvious that the compound quantum process Σ is bijective. We need to show that $\Sigma(t_2) \circ \Sigma(t_1)^{-1}$ is a quantum channel for all $0 \leqslant t_1 \leqslant t_2 \leqslant T$.

In order to do this, we fix arbitrary $t_1 \leqslant t_2$ and using the definition of compound channel and the properties of inverse operators, we write the identity

$$\Sigma(t_2) \circ \Sigma(t_1)^{-1} = (\Omega(t_2) \circ \Phi(t_2)) \circ (\Phi(t_1)^{-1} \circ \Omega(t_1)^{-1}). \tag{3.2}$$

Since the channels Φ and Ω satisfy the condition (3.1), the identity

$$\Omega(t_1) \circ \Phi(t_1) = \Phi(t_1) \circ \Omega(t_1)$$

holds, which implies the relation

$$\Phi(t_1)^{-1} \circ \Omega(t_1)^{-1} = \Omega(t_1)^{-1} \circ \Phi(t_1)^{-1}. \tag{3.3}$$

Substitution of (3.3) into (3.2) and the associativity of the semigroup operation in $\mathcal{O}_c(\mathcal{H})$ give the identity

$$\Sigma(t_2) \circ \Sigma(t_1)^{-1} = \Omega(t_2) \circ (\Phi(t_2) \circ \Omega(t_1)^{-1}) \circ \Phi(t_1)^{-1}. \tag{3.4}$$

Since the channels Φ and Ω satisfy condition (3.1), the identity

$$\Omega(t_1) \circ \Phi(t_2) = \Phi(t_2) \circ \Omega(t_1) \tag{3.5}$$

is true. Multiplying both sides of identity (3.5) from the left and the right by $\Omega(t_1)^{-1}$, we obtain:

$$\Phi(t_2) \circ \Omega(t_1)^{-1} = \Omega(t_1)^{-1} \circ \Phi(t_2). \tag{3.6}$$

Finally, substituting (3.6) into (3.4), we obtain the representation:

$$\Sigma(t_2) \circ \Sigma(t_1)^{-1} = (\Omega(t_2) \circ \Omega(t_1)^{-1}) \circ (\Phi(t_2) \circ \Phi(t_1)^{-1}). \tag{3.7}$$

Since the quantum channels Φ and Ω are completely positive divisible, the operators $\Phi(t_2) \circ \Phi(t_1)^{-1}$ and $\Omega(t_2) \circ \Omega(t_1)^{-1}$ are quantum channels. In this way, identity (3.7) shows that the linear operator $\Sigma(t_2)\circ\Sigma(t_1)^{-1}$ is represented as the composition of two quantum channels. This is why it is also a quantum channel and this completes the proof. □

4. Continuous bijective processes and completely positive evolutions

We equip the segment $[0;T]$ and the square $[0;T]\times[0;T]$ by natural topologies, while the set $\mathcal{O}_c(\mathcal{H})$ is equipped by the topology of the subspace in the Banach space $(\mathfrak{B}(\mathfrak{L}(\mathcal{H})), \|\cdot\|)$. The space $\mathcal{O}_c(\mathcal{H})$ is compact [13, Prop. 2.28]. In this section we consider continuous quantum dynamical mappings of form

$$\Phi : [0;T] \longrightarrow \mathcal{O}_c(\mathcal{H})$$

and the continuous evolutions generated by them.

We recall [3, Sect. V] that a *continuous completely positive evolution* is a continuous mapping

$$\Psi : [0,T]\times[0,T] \longrightarrow \mathcal{O}_c(\mathcal{H}) : (t_2,t_1) \longmapsto \Psi(t_2,t_1)$$

satisfying the following two conditions:

$$\Psi(t_3,t_2)\circ\Psi(t_2,t_1) = \Psi(t_3,t_1) \qquad \text{for all} \quad 0\leqslant t_1\leqslant t_2\leqslant t_3\leqslant T; \tag{4.1}$$

$$\lim_{\delta\to 0}\|\Psi(t+\delta,t)-\mathcal{I}\| = 0 \qquad \text{for all} \quad t\in[0,T). \tag{4.2}$$

By the continuity of the multiplication in the Banach algebra of operators $\mathfrak{B}(\mathfrak{L}(\mathcal{H}))$, the compound process for two continuous processes is continuous. For the completeness of the presentation, we provide a detailed justification of this fact in the proof of the following statement.

Proposition 4.1. *Let Φ and Ω be continuous bijective (positively) completely positive divisible quantum processes, which satisfy condition (3.1). Then the compound quantum process $\Sigma(\Omega,\Phi)$ is also continuous bijective (positively) completely positive divisible.*

Proof. In view of Proposition 3.1 it remains to confirm that the compound process is continuous. In order to do this, we consider the mappings

$$[0;T] \xrightarrow{\ \Omega\Delta\Phi\ } \mathcal{O}_c(\mathcal{H})\times\mathcal{O}_c(\mathcal{H}) \xrightarrow{\ \ m\ \ } \mathcal{O}_c(\mathcal{H}).$$

Here $\Omega\Delta\Phi$ is the diagonal of the mappings Ω and Φ with the values in the Cartesian square of the space of quantum channels $\mathcal{O}_c(\mathcal{H})$ with the topology of the product, that is,

$$\Omega\Delta\Phi(t) = (\Omega(t),\Phi(t))$$

for each $t\in[0;T]$. The symbol m denotes the composition of operators in $\mathcal{O}_c(\mathcal{H})$.

Both these mappings are continuous. Indeed, the diagonal $\Omega\Delta\Phi$ is continuous since by the assumptions both mappings

$$p_1\circ(\Omega\Delta\Phi) = \Omega \quad \text{and} \quad p_2\circ(\Omega\Delta\Phi) = \Phi$$

are continuous; here $p_i : \mathcal{O}_c(\mathcal{H})\times\mathcal{O}_c(\mathcal{H}) \longrightarrow \mathcal{O}_c(\mathcal{H})$ is the projection on ith coordinate, $i=1,2$, [18, Thm. 12.10]. The mapping m is continuous by the multiplicative inequality (2.5).

This is why the compound quantum process

$$\Sigma(\Omega,\Phi) = m\circ(\Omega\Delta\Phi)$$

is continuous as a composition of continuous mappings. □

Now we are going to prove the criterion of continuity for bijective quantum processes. In its formulation and below in the text there are limits depending on the parameter $t\in[0;T]$. We note that for the boundary values of the parameter, that is, as $t=0$ or $t=T$, we mean respectively the right or the left limit.

Proposition 4.2. *The bijective quantum process*

$$\Phi : [0;T] \longrightarrow \mathcal{O}_c(\mathcal{H})$$

is continuous if and only if for each $t \in [0;T]$ *the identity holds:*

$$\lim_{\delta\to 0} \Phi(t) \circ \Phi(t+\delta)^{-1} = \mathcal{I}. \tag{4.3}$$

Proof. Below, in both implications to be proved, t is an arbitrary fixed number in the segment $[0;T]$ and as δ we choose real numbers such that the condition $t+\delta \in [0;T]$ holds.

Necessity. We introduce the following mappings: a continuous constant mapping

$$\Phi_1 : [-t, T-t] \longrightarrow \mathrm{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H}))) : \delta \longmapsto \Phi(t);$$

and a mapping

$$\Phi_2 : [-t, T-t] \longrightarrow \mathrm{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H}))) : \delta \longmapsto \Phi(t+\delta),$$

which is continuous as a composition of continuous mappings. The diagonal of these mappings

$$\Phi_1 \triangle \Phi_2 : [-t, T-t] \longrightarrow \mathrm{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H}))) \times \mathrm{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H}))) : \delta \longmapsto (\Phi(t), \Phi(t+\delta))$$

is continuous by the continuity of the mappings

$$p_1 \circ (\Phi_1 \triangle \Phi_2) = \Phi_1 \quad \text{and} \quad p_2 \circ (\Phi_1 \triangle \Phi_2) = \Phi_2,$$

where $p_i : \mathrm{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H}))) \times \mathrm{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H}))) \longrightarrow \mathrm{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H})))$ is the projection on ith coordinate, $i = 1, 2$ [18, Thm. 12.10].

The composition $\Pi \circ (\Phi_1 \triangle \Phi_2)$ of the continuous mapping Π, see (2.6), with the diagonal mapping $\Phi_1 \triangle \Phi_2$ is continuous. This fact and identity (2.2) imply relation (4.3).

Sufficiency. Let us prove the continuity of the mapping Φ at a point t, namely, the validity of the identity

$$\Phi(t) = \lim_{\delta\to 0} \Phi(t+\delta).$$

It is implied, in view of condition (4.3), for instance, from the continuity of the composition made of the continuous diagonal mapping

$$[\Pi \circ (\Phi_1 \triangle \Phi_2)] \triangle \Phi_2) : [-t, T-t] \longrightarrow \mathrm{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H}))) \times \mathrm{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H}))) :$$
$$\delta \longmapsto (\Phi(t) \circ \Phi(t+\delta)^{-1}, \Phi(t+\delta))$$

and of the continuous multiplication in the topological group $\mathrm{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H})))$. □

Remark 4.1. *It is also clear that the continuity of the mapping* Φ *is equivalent to the validity of the condition*

$$\lim_{\delta\to 0} \Phi(t+\delta) \circ \Phi(t)^{-1} = \mathcal{I} \tag{4.4}$$

for each $t \in [0;T]$.

Now we are going to construct a continuous completely positive evolution by a continuous bijective completely positive process. In order to do this, we shall need an auxiliary statement. In order to formulate this statement, we consider two triangles, one being above and the other below the diagonal of the square $[0;T] \times [0;T]$:

$$A = \{(t_1; t_2) \in [0;T] \times [0;T] \mid t_1 \leqslant t_2\},$$
$$B = \{(t_1; t_2) \in [0;T] \times [0;T] \mid t_1 \geqslant t_2\}.$$

It is obvious that A and B are closed subsets in the space $[0;T] \times [0;T]$ with the topology of the product. They intersect at the diagonal of the square:

$$A \cap B = \{(t;t) \mid t \in [0;T]\}.$$

In the formulation of the auxiliary statement, A and B are considered as the spaces with the topologies induced from the space $[0;T] \times [0;T]$. This statement is a particular case of the gluing lemma from topology, see, for instnace, [18, Thm. 8.7].

Lemma 4.1. *Let* $\Gamma : A \longrightarrow \mathcal{O}_c(\mathcal{H})$ *and* $\Upsilon : B \longrightarrow \mathcal{O}_c(\mathcal{H})$ *be continuous mappings. If* $\Gamma(t,t) = \Upsilon(t,t)$ *for each* $t \in [0;T]$*, then the formula*

$$\Psi(t_1,t_2) := \begin{cases} \Gamma(t_1,t_2) & \text{as } (t_1,t_2) \in A\,; \\ \Upsilon(t_1,t_2) & \text{as } (t_1,t_2) \in B \end{cases}$$

defines a continuous mapping $\Psi : [0;T] \times [0;T] \longrightarrow \mathcal{O}_c(\mathcal{H})$.

In the next theorem we describe how one can construct a continuous completely positive evolution by a continuous bijective completely positive process.

Theorem 4.1. *Assume that we are given a continuous bijective completely positive divisible quantum process* $\Phi : [0;T] \longrightarrow \mathcal{O}_c(\mathcal{H})$*. Then the formula*

$$\Psi(t_1,t_2) = \begin{cases} \Phi(t_2) \circ \Phi(t_1)^{-1}, & \text{if } t_1 \leqslant t_2\,; \\ \Phi(t_1) \circ \Phi(t_2)^{-1}, & \text{if } t_1 \geqslant t_2 \end{cases} \tag{4.5}$$

defines a continuous completely positive evolution

$$\Psi : [0;T] \times [0;T] \longrightarrow \mathcal{O}_c(\mathcal{H}).$$

Proof. Since the bijective quantum process Φ is completely positive divisible, then $\Phi(t) \circ \Phi(s)^{-1} \in \mathcal{O}_c(\mathcal{H})$ as $0 \leqslant s \leqslant t \leqslant T$.

Let A and B be topological spaces defined above in the formulation of Lemma 4.1. We consider two mappings:

$$\Gamma : A \longrightarrow \mathcal{O}_c(\mathcal{H}) : (t_1,t_2) \longmapsto \Phi(t_2) \circ \Phi(t_1)^{-1};$$
$$\Upsilon : B \longrightarrow \mathcal{O}_c(\mathcal{H}) : (t_1,t_2) \longmapsto \Phi(t_1) \circ \Phi(t_2)^{-1}.$$

Our aim is to show that they satisfy the assumptions of Lemma 4.1.

First, for each $t \in [0;T]$ the identity $\Gamma(t,t) = \Upsilon(t,t) = \mathcal{I}$ holds. This implies that formula (4.5) well-defines a mapping on the square $[0;T] \times [0;T]$ with the values in the space of quantum channels $\mathcal{O}_c(\mathcal{H})$.

Second, the mappings Γ and Υ are continuous. To prove this fact, we consider the following continuous mappings. Let

$$R : A \longrightarrow [0,T] \times [0,T] : (t_1,t_2) \longmapsto (t_2,t_1);$$
$$\widetilde{\Phi} : [0,T] \longrightarrow \mathrm{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H}))) : t \longmapsto \Phi(t).$$

Then we consider the Cartesian square of the mapping $\widetilde{\Phi}$:

$$\widetilde{\Phi} \times \widetilde{\Phi} : [0,T] \times [0,T] \longrightarrow \mathrm{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H}))) \times \mathrm{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H}))) : (t_1,t_2) \longmapsto (\Phi(t_1), \Phi(t_2)).$$

The continuity of the introduced mappings is implied by the definition of the continuity and simple rules of constructing continuous functions, see, for instance, [18, Thm. 8.6, 12.10]. Now we consider the composition of continuous mappings:

$$\widetilde{\Gamma} = \Pi \circ (\widetilde{\Phi} \times \widetilde{\Phi}) \circ R : A \longrightarrow \mathrm{Inv}(\mathfrak{B}(\mathfrak{L}(\mathcal{H}))),$$

where Π is mapping (2.6). Since the mapping $\widetilde{\Gamma}$ is continuous, $\Gamma(t_1,t_2) = \widetilde{\Gamma}(t_1,t_2)$ for each pair $(t_1,t_2) \in A$, then employing [18, Thm.8.6, Item 4)], we get the continuity of Γ. In the same way, without using the operator R, we prove the continuity of the mapping Υ.

Thus, the mappings Γ and Υ satisfy the assumptions of Lemma 4.1. Therefore, the mapping Ψ is continuous.

Property (4.1) for the mapping Ψ can be checked straightforwardly by using formulae (4.5) and (2.2).

The continuity of the mapping Ψ and an arbitrary norm, as well as the identity $\Psi(t,t) = \mathcal{I}$, valid for each $t \in [0,T]$, imply condition (4.2) for Ψ.

Thus, the mapping Ψ is a continuous completely positive evolution. The proof is complete. □

Remark 4.2. *The validity of condition (4.2) for the mapping Ψ in Theorem 4.1 follows from properties (4.3) and (4.4).*

5. Quantum dynamical mappings with values in set of one-qubit quantum channels

Throughout this section, by a complex Hilbert space $\mathcal{H}$ we mean the two-dimensional complex space $\mathbb{C}^2$ with the standard scalar product and the canonical orthogonormal basis. By $\mathscr{M}_2(\mathbb{C})$ we denote the space of complex-valued 2×2 matrices. We shall often identify an operator from $\mathfrak{L}(\mathcal{H})$ with its matrix in $\mathscr{M}_2(\mathbb{C})$ using the same notations. In its turn, in the space $\mathfrak{L}(\mathcal{H})$ the Pauli basis is fixed $\{I, \sigma_x, \sigma_y, \sigma_z\}$ consisting of the identity operator and the Pauli operators:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We also identify the operators in $\mathfrak{B}(\mathfrak{L}(\mathcal{H}))$ with the corresponding 4×4 matrices in the Pauli basis.

5.1. Commuting bijective completely positive divisible quantum processes.

We fix a real parameter k and consider a linear mapping $\Phi_k : \mathfrak{L}(\mathcal{H}) \longrightarrow \mathfrak{L}(\mathcal{H})$ defined by its action on the basis vectors:

$$\Phi_k(I) = I, \quad \Phi_k(\sigma_x) = k\cdot\sigma_x, \quad \Phi_k(\sigma_y) = k\cdot\sigma_y, \quad \Phi_k(\sigma_z) = \sigma_z, \tag{5.1}$$

that is, in the Pauli basis the matrix of the operator Φ_k reads as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The structure of quantum channels of form $\Phi : \mathscr{M}_2(\mathbb{C}) \longrightarrow \mathscr{M}_2(\mathbb{C})$ was studied in details in [19]. In particular, there were obtained necessary and sufficient conditions ensuring that a linear mapping in $\mathfrak{B}(\mathfrak{L}(\mathcal{H}))$ is completely positive [19, see (12), (13)]. These results, see also [20, App. B], implies the following condition:

The mapping Φ_k is a quantum channel if and only if the identity holds:

$$|k| \leqslant 1. \tag{5.2}$$

Let $\mathfrak{S}(\mathcal{H}) := \{\rho\in\mathfrak{L}(\mathcal{H}) \mid \rho\geqslant 0;\, tr\rho = 1\}$ be the set of all density operators. We choose an arbitrary $\rho\in\mathfrak{S}(\mathcal{H})$ and expand it over the Pauli basis:

$$\rho = \frac{1}{2}(I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z), \tag{5.3}$$

where r_x, r_y, r_z are real parameters such that the identity $r_x^2 + r_y^2 + r_z^2 \leqslant 1$ holds. The vector formed by these parameters is called a *Bloch vector* of a given $\rho\in\mathfrak{S}(\mathcal{H})$ and we say that the set of states is the unit ball centered at the origin in the space $\mathbb{R}^3$.

Acting by the quantum channel Φ_k on ρ, we obtain the expression

$$\Phi_k(\rho) = \frac{1}{2}(I + k\cdot r_x\sigma_x + k\cdot r_y\sigma_y + r_z\sigma_z),$$

that is, the Bloch vector is transformed by the rule $(r_x, r_y, r_z) \to (k\cdot r_x,\, k\cdot r_y,\, r_z)$. Thus, the unit ball in $\mathbb{R}^3$ shrinks in k times along the axis z.

Let we are given a monotonically decaying function $k : [0, T] \longrightarrow (0, 1]$ such that the condition $k(0) = 1$ holds. By the function k we define the process

$$\Phi_{dec} := \{\Phi_{k(t)} \in O_c(H) \mid 0 \leqslant t \leqslant T,\ \Phi_{k(0)} = \mathcal{I}\}$$

formed by the decoherence channels $\Phi_{k(t)}$ defined by formulae (5.1).

Proposition 5.1. *The quantum process Φ_{dec} is bijective completely positive divisible.*

Proof. This process is bijective since the function k is positive and hence, for each $t \in [0; T]$ there exists an inverse linear operator

$$\Phi^{-1}_{k(t)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{k(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{k(t)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us show that the process Φ_{dec} is completely positive divisible. In order to do this, we consider a pair of numbers $t_1, t_2 \in [0; T]$ such that $t_1 \leqslant t_2$. By condition (5.2), the composition

$$\Phi_{k(t_2)} \circ \Phi^{-1}_{k(t_1)} = \Phi_{\frac{k(t_2)}{k(t_1)}}$$

is a quantum channel if and only if the inequality $\frac{k(t_2)}{k(t_1)} \leqslant 1$ holds. But it is true since the function $k(t)$ decreases monotonically and $t_1 \leqslant t_2$. The proof is complete. □

In order to construct the second process, we consider a family of unitary operators:

$$\Omega = \left\{ U(t) = \exp\left(\frac{it\sigma_z}{2}\right) \in \mathfrak{L}(\mathcal{H}) \mid \quad 0 \leqslant t \leqslant T \right\}. \tag{5.4}$$

By this family we define the process

$$\Phi_{\Omega} := \left\{\Phi_{U(t)} \in O_c(H) \mid 0 \leqslant t \leqslant T,\ \Phi_{U(0)} = \mathcal{I}\right\}$$

consisting of unitary channels acting by the rule

$$\Phi_{U(t)}(X) = U(t) \circ X \circ (U(t))^*, \quad X \in \mathfrak{L}(\mathcal{H}).$$

The following proposition is obvious.

Proposition 5.2. *The quantum process Φ_{Ω} is bijective completely positive divisible.*

By direct calculations we find the formulae for the transformations of the vectors in the Pauli basis under the action of a channel $\Phi_{U(t)}$ from the family Φ_{Ω}:

$$\Phi_{U(t)}(I) = I, \qquad \Phi_{U(t)}(\sigma_x) = \cos(t) \cdot \sigma_x - \sin(t) \cdot \sigma_y,$$
$$\Phi_{U(t)}(\sigma_y) = \sin(t) \cdot \sigma_x + cos(t) \cdot \sigma_y, \qquad \Phi_{U(t)}(\sigma_z) = \sigma_z.$$

In the matrix form they read as

$$\Phi_{U(t)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(t) & sin(t) & 0 \\ 0 & -sin(t) & cos(t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Writing the density operator $\rho \in \mathfrak{S}(\mathcal{H})$, see (5.3), as the column vector

$$\rho = \frac{1}{2} \begin{pmatrix} 1 \\ r_x \\ r_y \\ r_z \end{pmatrix} \tag{5.5}$$

and applying to it the linear transformation $\Phi_{U(t)}$, we obtain the vector

$$\Phi_{U(t)}(\rho) = \frac{1}{2}\begin{pmatrix} 1 \\ \cos(t)\cdot r_x + \sin(t)\cdot r_y \\ -\sin(t)\cdot r_x + \cos(t)\cdot r_y \\ r_z \end{pmatrix}.$$

This transformation corresponds to the rotation of the Bloch vector (r_x, r_y, r_z) by the angle t around the axis z.

It is straightforward to confirm that for all $t_1, t_2 \in [0;T]$ the identity holds:

$$\Phi_{U(t_1)} \circ \Phi_{k(t_2)} = \Phi_{k(t_2)} \circ \Phi_{U(t_1)},$$

that is, the following proposition is true.

Proposition 5.3. *The quantum processes Φ_{dec} and Φ_Ω commute.*

Remark 5.1. *It should be noted that the channels $\Phi_{k(t)}$ forming the process Φ_{dec} are covariant with respect to the maximal Abelian group of unitary operators $\{\exp(it\sigma_z) \mid t \in \mathbb{R}\}$. This fact guarantees the commuting of the processes Φ_{dec} and Φ_Ω. The definition of covariant channels and related notions can be found in* [21]–[23].

We consider a compound process

$$\Sigma(\Phi_{dec}, \Phi_\Omega) := \{\Phi_{k(t)} \circ \Phi_{U(t)} \in O_c(H) \mid 0 \leqslant t \leqslant T,\ \Phi(0) = I\}.$$

Propositions 5.1, 5.2, 5.3 and 3.1 imply the following theorem.

Theorem 5.1. *The quantum process $\Sigma(\Phi_{dec}, \Phi_\Omega)$ is bijective completely positive divisible.*

We complete this subsection by geometric and physical interpretations of compound quantum process $\Sigma(\Phi_{dec}, \Phi_\Omega)$. We note that each channel in this process is written in the matrix form as follows:

$$\Phi_{k(t)} \circ \Phi_{U(t)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & k(t)\cdot cos(t) & k(t)\cdot sin(t) & 0 \\ 0 & -k(t)\cdot sin(t) & k(t)\cdot cos(t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Under the action of the linear transformation $\Phi_{k(t)} \circ \Phi_{U(t)}$ on an arbitrary state represented by the Bloch vector (5.5) we obtain the column vector

$$\Phi_{k(t)} \circ \Phi_{U(t)}(\rho) = \frac{1}{2}\begin{pmatrix} 1 \\ k(t)(cos(t)\cdot r_x - sin(t)\cdot r_y) \\ k(t)(sin(t)\cdot r_x + cos(t)\cdot r_y) \\ r_z \end{pmatrix}.$$

Since the function $k : [0,T] \to (0,1]$ decreases monotonically, as the parameter t grows, the Bloch vector (r_x, r_y, r_z) goes along a spiral approaching the axis Oz.

5.2. Continuous bijective completely positive divisible quantum dynamical mapping. Here, demonstrating an application of Proposition 4.2, we prove the continuity of the process Φ_Ω in the previous subsection.

We shall make use of the matrix form of writing. We consider a one-parametric family of unitary matrices (5.4):

$$U(t) = \begin{pmatrix} e^{\frac{it}{2}} & 0 \\ 0 & e^{-\frac{it}{2}} \end{pmatrix}, \quad t \in [0;T].$$

We have a bijective completely positive divisible quantum process Φ_Ω consisting of unitary quantum channels

$$\Phi_{U(t)} : \mathscr{M}_2(\mathbb{C}) \longrightarrow \mathscr{M}_2(\mathbb{C}) : X \longmapsto U(t)X(U(t))^*, \quad t \in [0;T].$$

Let us show the validity of the following statement.

Proposition 5.4. *The quantum process Φ_Ω is continuous.*

Proof. By Proposition 4.2, it is sufficient to show the validity of condition (4.3) for an arbitrary $t \in [0;T]$. In order to do this, we fix $t, t+\delta \in [0;T]$. It is easy to see that formulae given below are true; they induce trace norm (2.4) and trace norm (2.3), while the maximum is taken over all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{M}_2(\mathbb{C}), \quad \text{for which} \quad \left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_{tr} \leqslant 1.$$

Indeed, we have the estimate

$$\begin{aligned}
\left\| \Phi_{U(t)} \circ \Phi^{-1}_{U(t+\delta)} - \mathcal{I} \right\| &= \max \left\{ \left\| (\Phi_{U(t)} \circ \Phi^{-1}_{U(t+\delta)} - \mathcal{I}) \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \right\|_{tr} \right\} \\
&= \max \left\| \begin{pmatrix} 0 & b(e^{-i\delta} - 1) \\ c(e^{i\delta} - 1) & 0 \end{pmatrix} \right\|_{tr} \\
&= \max\, tr \begin{pmatrix} |c(e^{i\delta} - 1)| & 0 \\ 0 & |b(e^{-i\delta} - 1)| \end{pmatrix} \\
&= |e^{i\delta} - 1| \max(|c| + |b|) \leqslant |e^{i\delta} - 1| \max(|c| + |b| + |a| + |d|).
\end{aligned}$$

Then on the space $\mathscr{M}_2(\mathbb{C})$ we consider l_1-norm defined by the formula

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_1 = |a| + |b| + |c| + |d|.$$

Since all norms on $\mathscr{M}_2(\mathbb{C})$ are equivalent, there exists a real number $C > 0$ such that

$$\| \cdot \|_1 \leqslant C \cdot \| \cdot \|_{tr}.$$

In view of this inequality and the above obtained estimate, we have the inequality

$$\left\| \Phi_{U(t)} \circ \Phi^{-1}_{U(t+\delta)} - \mathcal{I} \right\| \leqslant C \cdot |e^{i\delta} - 1|.$$

Passing here to the limit as $\delta \longrightarrow 0$, we obtained needed identity (4.3). □

Remark 5.2. *In the case of a continuous function k it can be shown that, for instance,* [18, Thm. 12.10]*, the quantum process Φ_{dec} from Subsection 5.1 is continuous.*

5.3. Discontinuous bijective completely positive divisible quantum dynamical mapping. We fix arbitrary positive numbers t_0, δ_1, δ_2 such that the conditions hold: $0 < t_0 < T$ and $\delta_1 - \delta_2 \neq 2\pi n$, where $n \in \mathbb{Z}$.

We define a one-parametric family of unitary matrices by the formula

$$U(t) = \begin{cases} e^{it} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } 0 \leqslant t \leqslant t_0; \\ \begin{pmatrix} e^{i(t+\delta_1)} & 0 \\ 0 & e^{i(t+\delta_2)} \end{pmatrix} & \text{if } t_0 < t \leqslant T. \end{cases}$$

For each of the matrices $U(t)$ we consider a bijective completely positively trace-preserving mapping

$$\Phi_{U(t)} : \mathscr{M}_2(\mathbb{C}) \longrightarrow \mathscr{M}_2(\mathbb{C}) : X \longmapsto U(t) X (U(t))^*.$$

We note that for each $t \in [t; t_0]$ the channel $\Phi_{U(t)}$ is the identical mapping on $\mathscr{M}_2(\mathbb{C})$.

As a result, we have a bijective completely positive divisible quantum process

$$\Phi := \left\{ \Phi_{U(t)} \in O_c(\mathcal{H}) \mid 0 \leqslant t \leqslant T \right\}$$

consisting of unitary channels.

Proposition 5.5. *The quantum process* $\Phi : [0;T] \longrightarrow O_c(\mathcal{H})$ *is discontinuous at the point* t_0.

Proof. Let us establish the validity of the following relation:

$$\lim_{\tau\to 0+} \Phi_{U(t_0+\tau)} \neq \Phi_{U(t_0)}.$$

Let $\tau \in (0; T - t_0]$. To estimate from below the operator norm $\left\|\Phi_{U(t_0+\tau)} - \Phi_{U(t_0)}\right\|$ induced by the trace norm $\|\cdot\|_{tr}$ on the space of the matrices $\mathscr{M}_2(\mathbb{C})$, we introduce a density matrix:

$$S := \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

It is easy to see that the identity holds:

$$[\Phi_{U(t_0+\tau)} - \Phi_{U(t_0)}](S) = \frac{1}{2}\begin{pmatrix} 0 & e^{i(\delta_1-\delta_2)} - 1) \\ e^{i(\delta_2-\delta_1)} - 1) & 0 \end{pmatrix}.$$

Therefore, the absolute value of this matrix is written as

$$\left|[\Phi_{U(t_0+\tau)} - \Phi_{U(t_0)}](S)\right| = \frac{1}{2}\begin{pmatrix} |e^{i(\delta_1-\delta_2)} - 1| & 0 \\ 0 & |e^{i(\delta_1-\delta_2)} - 1| \end{pmatrix}.$$

Hereinafter, for a matrix $A \in \mathscr{M}_2(\mathbb{C})$ we employ a notation $|A| = \sqrt{A^*A}$.

This is why for each $\tau \in (0; T - t_0]$ the following lower bound for the norm of the difference of quantum channels holds:

$$\begin{aligned}\left\|\Phi_{U(t_0+\tau)} - \Phi_{U(t_0)}\right\| &= \max\left\{\left\|[\Phi_{U(t_0+\tau)} - \Phi_{U(t_0)}](X)\right\|_{tr} \mid X \in \mathscr{M}_2(\mathbb{C}), \|X\|_{tr} \leqslant 1\right\} \\ &\geqslant \left\|[\Phi_{U(t_0+\tau)} - \Phi_{U(t_0)}](S)\right\|_{tr} = tr\left|[\Phi_{U(t_0+\tau)} - \Phi_{U(t_0)}](S)\right| \\ &= |e^{i(\delta_1-\delta_2)} - 1| \neq 0.\end{aligned}$$

Thus, we have shown that the bijective completely positive divisible quantum process Φ is discontinuous at the point t_0. □

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Renat Nelsonovich Gumerov,
N.I. Lobachevsky Institute of Mathematics and Mechanics,
Kazan Federal University,
Kremlevskaya 35,
420008, Kazan, Russia
E-mail: `Renat.Gumerov@kpfu.ru`

Ruslan Linarovich Khazhin,
N.I. Lobachevsky Institute of Mathematics and Mechanics,
Kazan Federal University,
Kremlevskaya 35,
420008, Kazan, Russia
E-mail: `ruslan.khazhin@mail.ru`