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INTEGRATION OF CAMASSA-HOLM EQUATION WITH A SELF-CONSISTENT SOURCE OF INTEGRAL TYPE

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Abstract. The work is devoted to studying Camassa-Holm equation with a self-consistent source of integral type.

The source of the consistent equation corresponds to the continuous spectrum of a spectral problem related with the Camassa-Holm equation. As it is known, integrable systems admit operator Lax representation $L_t = [L, A]$, where L is a linear operator, while A is some skew-symmetric operator acting in a Hilbert space. A generalized Lax representation for the considered equation is of the form $L_t = [L, A] + C$, where C is the sum of differential operators with coefficients depending on solutions of spectral problems for the operator L. The construction of self-consistent source for the considered operator is based on the fact that exactly squares of eigenfunctions of the spectral problems are essential while solving integrable equations by the inverse scattering transform. Moreover, for the considered type of equations the evolution of the eigenfunctions in the generalized Lax representation has a singularity.

The application of the inverse scattering transform is based on the spectral problem related with the classical Camassa-Holm equation. We describe the evolution of scattering data of this spectral problem with a potential being a solution of the Camassa-Holm equation with a self-consistent source. While describing the evolution of the spectral data, we employ essentially Sokhotski-Plemelj formula. The results of the work on the evolution of the scattering data related with the discrete spectrum are based on the methods used in the previous works by the authors. The obtained results, formulated as a main theorem, allow us to apply the inverse scattering transform for solving the Cauchy problem for the considered equation. Our technique can be easily extended to higher analogues of the Camassa-Holm equation.

Keywords: Camassa-Holm equation, Jost solution, self-consistent source, evolution of scattering data, inverse scattering transform.

Mathematics Subject Classification: 39A23, 35Q51, 34K13, 34K29

1. Introduction

In 1967, American scientists C.S. Gardner, I.M. Green, M.D. Kruskal, R.M. Miura showed that the solution to the Korteweg-de Vries equation can be obtained for all "fast decaying" initial conditions [1], that is, for condition vanishing in a certain way as the coordinate tends to infinity. This method was called the inverse scattering transform (IST) since it employs essentially the problem on recovering the potential in the Sturm-Liouville operator on the entire axis by the scattering data (inverse scattering problem). Later in 1968 Lax [2] generalized essentially their ideas. Namely, he transformed the compatibility condition of linear problems into a convenient operator form representing it as the commutation condition of linear differential operators: $L_t = [L, A]$, where L is a linear operator and A is some skew-symmetric operator acting in a Hilbert space.

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In works by V.K. Melnikov [3], [4] some generalization of Lax equation was given in the form

$$L_t = [L, A] + C,$$

where C is the sum of differential operators with the coefficients depending on solutions of spectral problem for the operator L. Equations admitting such representation were called equations with a self-consistent source. We also note that in work by J. Leon and A. Latifi [5] a particular physical problem was provided, which was reduced to solving the KdV equation with a source. Nonlinear evolution equations with a self-consistent source arise also in the problems of hydrodynamics, physics of plasma, solid-state physics.

In 1993, by physical consideration, R. Camassa and D. Holm [6], derived the equation

$$u_t - u_{txx} + 3uu_x + 2\omega u_x = 2u_x u_{xx} + uu_{xxx},$$

which in dimensionless spatial-time coordinates (x, t) describes a unidirectional wave propagation in a shallow water over a plat bottom, $u(x, t)$ is a horizontal speed component describing a free surface, while a parameter $\omega > 0$ is related with a critical speed. In a modern literature, this equation is called Camassa-Holm equation.

Recently, the Camassa-Holm equation attracts a large interest as an example of an integrable system possessing more general wave equations in comparison with the KdV equation. An analysis made in [7], as well as in [8] and other works showed the existence of smooth solitons for all $\omega > 0$.

It was shown in works by A. Constantin, V. Gerdjikov, R. Ivanov [8], [9] that the inverse scattering transform is applicable for obtaining solutions to the Camassa-Holm equation.

In work of Chinese scientists Huang Ye-Hui, Yao Yu-Qin, Zeng Yun-Bo [10], the Camassa-Holm equation with a simplest self-consistent source was integrated by means of a direct method, a Darboux transform method.

In the present work we consider a system of equations

$$\begin{cases} u_t - u_{xxt} + 2\omega u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = \int\limits_{-\infty}^{\infty} (m_x' gf + 2(m+\omega)(g\,f)_x')dk, \\ g_{xx}(x,k,t) = \left(\frac{1}{4} + \lambda(m+\omega)\right) g(x,k,t), \\ f_{xx}(x,k,t) = \left(\frac{1}{4} + \lambda(m+\omega)\right) f(x,k,t), \qquad x, k \in \mathbb{R}, \qquad t \geqslant 0, \end{cases} \tag{1.1}$$

where

$$m(x,t) = u(x,t) - u_{xx}(x,t), \qquad \omega = const > 0,$$

$$m(x,t) + \omega > 0, \qquad \lambda(k) = -\frac{1}{\omega}\left(k^2 + \frac{1}{4}\right),$$

and $u = u(x, t)$ is a real function possessing a sufficient smoothness tending to its limits as $x \to \pm\infty$ fast enough so that

$$\int\limits_{-\infty}^{\infty} (1+|x|)\left(|u(x,t)| + \sum_{k=1}^{3}\left|\frac{\partial^k u(x,t)}{\partial x^k}\right|\right) dx < \infty, \qquad t \geqslant 0. \tag{1.2}$$

Problem (1.1)–(1.2) is considered with an initial condition

$$u(x,t)|_{t=0} = u_0(x),$$

where an initial function $u_0(x)$ possesses the following properties:

1) $u_0(x) - u''_0(x) + \omega > 0$, $x \in \mathbb{R}$,

2) $\int\limits_{-\infty}^{\infty} (1+|x|)\left(|u_0(x)|+|u''{}_0(x)|\right)dx<\infty$,

3) the equation $\psi_{xx}=\left(\frac{1}{4}+\lambda(m(x)+\omega)\right)\psi$ with $m(x)=u_0(x)-u_0''(x)$ possesses exactly N simple eigenvalues $\lambda_1(0),\lambda_2(0),\lambda_3(0),\dots,\lambda_N(0)$ in the interval $\left(-\frac{1}{4\omega};\,0\right)$.

The functions $g=g(x,k,t)$, $f=f(x,k,t)$ are continuous in the parameter k, possess the first order derivatives $g_k'=\frac{\partial g(x,k,t)}{\partial k}$, $f_k'=\frac{\partial f(x,k,t)}{\partial k}$, obey the inequalities

$$\int\limits_{-\infty}^{\infty}(|g(x,k,t)|^2+|f(x,k,t)|^2)dk<\infty,$$

$$\int\limits_{-\infty}^{\infty}\left(\left|\frac{\partial g(x,k,t)}{\partial x}\right|^2+\left|\frac{\partial f(x,k,t)}{\partial x}\right|^2\right)dk<\infty,$$

$$\int\limits_{-\infty}^{\infty}\left(\left|\frac{\partial g(x,k,t)}{\partial k}\right|^2+\left|\frac{\partial f(x,k,t)}{\partial k}\right|^2\right)dk<\infty,\qquad t\geqslant 0,\qquad x\in(-\infty;\infty),$$

and as $x\to\infty$, they have the following asymptotics:

$$\begin{aligned} f&\sim\alpha(k)e^{ikx}+\beta(k)e^{-ikx},\\ g&\sim\gamma(k)e^{ikx}+\delta(x)e^{-ikx}, \end{aligned}\tag{1.3}$$

where complex-valued functions $\alpha=\alpha(k,t)$, $\beta=\beta(k,t)$, $\delta=\delta(k,t)$, $\gamma=\gamma(k,t)$ are continuous in k and t, possess the first order derivatives and satisfy the following conditions for $t\geqslant 0$:

$$\begin{gathered} \int\limits_{-\infty}^{\infty}(|\alpha(k,t)|^2+|\beta(k,t)|^2+|\delta(k,t)|^2+|\gamma(k,t)|^2)dk<\infty,\\ \int\limits_{-\infty}^{\infty}\left(\left|\frac{\partial\alpha(k,t)}{\partial k}\right|^2+\left|\frac{\partial\beta(k,t)}{\partial k}\right|^2+\left|\frac{\partial\delta(k,t)}{\partial k}\right|^2+\left|\frac{\partial\gamma(k,t)}{\partial k}\right|^2\right)dk<\infty. \end{gathered}\tag{1.4}$$

We let

$$Q(k,t)=\beta(k,t)\gamma(k,t)+\alpha(-k,t)\delta(-k,t).$$

In this work we show how to construct a solution to Cauchy problem (1.1)–(1.4).

2. Scattering problem

We consider an equation

$$\psi_{xx}(x,k)=\left(\frac{1}{4}+\lambda(m(x)+\omega)\right)\psi(x,k),\tag{2.1}$$

where $m(x)=u(x)-u_{xx}(x)$, $\omega=const>0$, $m(x)+\omega>0$, with a function $u(x)$ satisfying the condition

$$\int\limits_{-\infty}^{\infty}(1+|x|)\left(|u(x)|+|u''(x)|\right)dx<\infty.\tag{2.2}$$

Under condition (2.2), there exists a Jost solution for equation (2.1) with the following asymptotics:

$$\begin{aligned} \psi_1&=e^{-ikx}+o(1), &\quad \psi_2&=e^{ikx}+o(1), &\quad x&\to+\infty,\\ \varphi_1&=e^{-ikx}+o(1), &\quad \varphi_2&=e^{ikx}+o(1), &\quad x&\to-\infty. \end{aligned}$$

For real k, the pairs (φ_1, φ_2) and (ψ_1, ψ_2) are the pairs of linearly independent solutions of equation (2.1) and this is why

$$\varphi_1(x,k) = a(k)\psi_1(x,k) + b(k)\psi_2(x,k).$$

The function $a(k)$ is continued analytically in the upper half-plane and possesses there finitely many simple zeroes $k = ik_n$, $k_n > 0$, and $\lambda_n = -\frac{1}{\omega}(-k_n^2 + \frac{1}{4})$, $n = 1, 2, \ldots, N$, is an eigenvalue of equation (2.1) so that $\varphi_1(x, ik_n) = b_n\psi_2(x, ik_n)$, $n = 1, 2, \ldots, N$.

The set $\{r(k) = \frac{a(k)}{b(k)},\ k \in \mathbb{R},\ k_n,\ b_n,\ \ n = 1, 2, \ldots, N\}$ is called scattering data for equation (2.1). A direct scattering problem is to determine the scattering data by the function $u(x)$. An inverse scattering problem is to recover the function $m(x)$, and hence, $u(x)$ in equation (2.1) by the scattering data. It was shown in work [9] that the function $u(x)$ is uniquely recovered by the scattering data.

3. Evolution of spectral characteristics corresponding to continuous spectrum

Let $u = u(x,t)$ be a solution to problem (1.1)–(1.4). For the sake of convenience, in this section we omit the dependence on t if it is inessential. We consider a system of equations:

$$\psi_{xx} = \left(\frac{1}{4} + \lambda(m(x) + \omega)\right)\psi, \tag{3.1}$$

$$\frac{\partial F}{\partial x} = (m(x) + \omega)\psi(x,k)g(x,\eta), \qquad \eta \in \mathbb{R}, \tag{3.2}$$

where $\lambda = -\frac{1}{\omega}\left(k^2 + \frac{1}{4}\right)$, and $g(x,\eta)$ and $f(x,\eta)$ are solutions of the equation

$$y''_{xx} = \left(\frac{1}{4} + \xi(m + \omega)\right)y, \qquad \xi = -\frac{1}{\omega}\left(\eta^2 + \frac{1}{4}\right).$$

We construct the following functions

$$\vartheta = \psi'_t - \left(\frac{1}{2\lambda} - u\right)\psi'_x - \frac{u_x}{2}\psi - \gamma_1\psi - \lambda\int_{-\infty}^{\infty} f(x,\eta)F(x,k,\eta)d\eta, \tag{3.3}$$

$$\begin{aligned} G(x,k,\eta) =& g(x,\eta)\frac{\partial\psi(x,k)}{\partial x} - \frac{\partial g(x,\eta)}{\partial x}\psi(x,k) - (\lambda - \xi)F(x,k,\eta) \\ =& g(x,\eta)\frac{\partial\psi(x,k)}{\partial x} - \frac{\partial g(x,\eta)}{\partial x}\psi(x,k) - \frac{1}{\omega}\left(\eta^2 - k^2\right)F(x,k,\eta). \end{aligned}$$

For arbitrary k we have the following system:

$$\begin{cases} \vartheta_{xx} - \left(\frac{1}{4} + \lambda(m(x) + \omega)\right)\vartheta = -\lambda\displaystyle\int_{-\infty}^{\infty}(m(x) + \omega)f(x,\eta)G(x,k,\eta)d\eta, \\ \dfrac{\partial G(x,k,\eta)}{\partial x} \equiv 0, \qquad x \in \mathbb{R}, \qquad \eta \in (-\infty; \infty). \end{cases} \tag{3.4}$$

On the base of (3.2) and by means of Jost solutions to equations (3.1) we introduce the notations:

$$\begin{aligned} F_- &= \int_{-\infty}^{x}(m(z) + \omega)\varphi_1(z,k)g(z,\eta)dz, \\ F_+ &= -\int_{x}^{\infty}(m(z) + \omega)\psi_2(z,k)g(z,\eta)dz. \end{aligned} \tag{3.5}$$

For each $\eta \in (-\infty, \infty)$, the functions F_- and F_+ are analytic as $\operatorname{Im} k > 0$, therefore, by equations (1.1), (3.1) and the asymptotics of the Jost solutions we obtain the identities:

$$\begin{aligned} F_-(x,k,\eta) &= \frac{\omega}{\eta^2 - k^2}\left(g(x,\eta)\frac{\partial \varphi_1(x,k)}{\partial} - \frac{\partial g(x,\eta)}{\partial}\varphi_1(x,k)\right), \\ F_+(x,k,\eta) &= \frac{\omega}{\eta^2 - k^2}\left(g(x,\eta)\frac{\partial \psi_2(x,k)}{\partial} - \frac{\partial g(x,\eta)}{\partial}\psi_2(x,k)\right), \end{aligned} \tag{3.6}$$

and the right hand side in expansion (3.6) is valid for the values $k^2 \neq \eta^2$. Similar to identity (3.5), for identity (3.3) as $\operatorname{Im} k > 0$ we let

$$\begin{aligned} \vartheta_1 &= \varphi_{1t} - \left(\frac{1}{2\lambda} - u\right)\varphi_{1x} - \frac{u_x}{2}\varphi_1 - \gamma_1\varphi_1 - \lambda\int_{-\infty}^{\infty} f(x,\eta)F_-(x,k,\eta)d\eta, \\ \vartheta_2 &= \psi_{2t} - \left(\frac{1}{2\lambda} - u\right)\psi_{2x} - \frac{u_x}{2}\psi_2 - \gamma_1\psi_2 - \lambda\int_{-\infty}^{\infty} f(x,\eta)F_+(x,k,\eta)d\eta; \end{aligned} \tag{3.7}$$

these functions are also analytic in the upper half-plane with respect to the parameter k, moreover, for real non-zero values k the functions $F_-(x,k,\eta)$, $F_+(x,k,\eta)$ have singularities as $\eta = k$, $\eta = -k$.

Substituting expansions (3.6) into identities (3.7), as $\operatorname{Im} k \to +0$, we calculate the functions ϑ_1 and ϑ_2. Then for $\operatorname{Im} k = 0$ the following identities hold:

$$\begin{aligned} \vartheta_1 =& \varphi_{1t} - \left(\frac{1}{2\lambda} - u\right)\varphi_{1x} - \frac{u_x}{2}\varphi_1 - \gamma_1\varphi_1 - \lambda\int_{-\infty}^{\infty} f(x,\eta)F_-(x,k,\eta)d\eta \\ &+ \Phi_1^-(k)f(x,k) + \Phi_2^-(k)f(x,-k), \end{aligned} \tag{3.8}$$

$$\begin{aligned} \vartheta_2 =& \psi_{2t} - \left(\frac{1}{2\lambda} - u\right)\psi_{2x} - \frac{u_x}{2}\psi_2 - \gamma_1\psi_2 - \lambda\int_{-\infty}^{\infty} f(x,\eta)F_+(x,k,\eta)dk \\ &+ \Phi_1^+(k)f(x,k) + \Phi_2^+(k)f(x,-k), \end{aligned} \tag{3.9}$$

where the integral is treated in the sense of the principle value, while the functions Φ_1^-, Φ_2^-, Φ_1^+ and Φ_2^+ are determined by the following identities:

$$\begin{aligned} \Phi_1^-(k) &= -\frac{\pi i\omega\lambda}{2k}\left(g(x,k)\frac{\partial \varphi_1(x,k)}{\partial x} - \frac{\partial g(x,k)}{\partial x}\varphi_1(x,k)\right), \\ \Phi_2^-(k) &= -\frac{\pi i\omega\lambda}{2k}\left(g(x,-k)\frac{\partial \varphi_1(x,k)}{\partial x} - \frac{\partial g(x,-k)}{\partial x}\varphi_1(x,k)\right), \\ \Phi_1^+(k) &= -\frac{\pi i\omega\lambda}{2k}\left(g(x,k)\frac{\partial \psi_2(x,k)}{\partial x} - \frac{\partial g(x,k)}{\partial x}\psi_2(x,k)\right), \\ \Phi_2^+(k) &= -\frac{\pi i\omega\lambda}{2k}\left(g(x,-k)\frac{\partial \psi_2(x,k)}{\partial x} - \frac{\partial g(x,-k)}{\partial x}\psi_2(x,k)\right). \end{aligned} \tag{3.10}$$

Since the functions $g(x,k)$, $g(x,-k)$, $\varphi_1(x,k)$, $\psi_2(x,k)$ are solutions to equation (3.1), the functions Φ_1^-, Φ_2^-, Φ_1^+ and Φ_2^+ are independent of x.

Letting

$$\begin{aligned} G_-(x,k,\eta) &= g(x,\eta)\frac{\partial \varphi_1(x,k)}{\partial x} - \frac{\partial g(x,\eta)}{\partial x}\varphi_1(x,k) - \frac{1}{\omega}\left(\eta^2 - k^2\right)F_-(x,k,\eta), \\ G_+(x,k,\eta) &= g(x,\eta)\frac{\partial \psi_2(x,k)}{\partial x} - \frac{\partial g(x,\eta)}{\partial x}\psi_2(x,k) - \frac{1}{\omega}\left(\eta^2 - k^2\right)F_+(x,k,\eta), \end{aligned} \tag{3.11}$$

according to expansion (3.6), in the upper closed half-plane $\operatorname{Im} k \geqslant 0$ we have:

$$G_{-} = G_{+} \equiv 0, \qquad \eta \in (-\infty, \infty)\,. \tag{3.12}$$

By identities (3.4), (3.11), (3.12) and the definition of the functions $\vartheta_1(x,k)$, $\vartheta_2(x,k)$ for each $k \in (-\infty, \infty)$ we obtain:

$$\vartheta_{1xx} - \left(\frac{1}{4} + \lambda\,(m+\omega)\right)\vartheta_1 = \vartheta_{2xx} - \left(\frac{1}{4} + \lambda\,(m+\omega)\right)\vartheta_2 = 0. \tag{3.13}$$

Identities (3.12)-(3.13) are a simplified form of system (3.3). For each $k \in (-\infty, \infty)$, according to asymptotic expansions (1.3) we have:

$$\begin{aligned} f(x,k) &= \alpha(k)\psi_2(x,k) + \beta(k)\psi_1(x,k),\\ g(x,k) &= \gamma(k)\psi_2(x,k) + \delta(k)\psi_1(x,k). \end{aligned} \tag{3.14}$$

On the other hand, the expansions hold:

$$\begin{aligned} f(x,k) &= p(k)\varphi_2(x,k) + q(k)\varphi_1(x,k),\\ g(x,k) &= l(k)\varphi_2(x,k) + s(k)\varphi_1(x,k), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} p(k) &= \alpha(k)a(k) - \beta(k)b(k), \qquad & q(k) &= -\alpha(k)b(-k) - \beta(k)a(-k),\\ l(k) &= \gamma(k)a(k) - \delta(k)b(k), \qquad & s(k) &= -\gamma(k)b(-k) + \delta(k)a(-k). \end{aligned}$$

Therefore, by identities (3.10) and expansions (3.14), (3.15) we obtain $\Phi_1^-(k) = -\pi\omega\lambda l(k)$. Similarly we find:

$$\begin{aligned} \Phi_1^-(k) &= -\pi\omega\lambda\, l(k), \qquad & \Phi_1^+(k) &= \pi\omega\lambda\,\delta(k),\\ \Phi_2^-(k) &= -\pi\omega\lambda\, s(-k), \qquad & \Phi_2^+(k) &= \pi\omega\lambda\,\gamma(-k). \end{aligned} \tag{3.16}$$

According (3.13), for real $k \neq 0$ the function $\vartheta_1(x,k)$ is expressed as a linear combination of solutions $\varphi_1(x,k)$, $\varphi_2(x,k)$, and $\vartheta_2(x,k)$ is expressed via the solutions $\psi_1(x,k)$, $\psi_2(x,k)$. By (3.8), (3.9) and asymptotic expansions for the Jost function we let

$$\begin{aligned} \vartheta_1(x,k) &= K^-(k)\varphi_1(x,k) + K_0^-(k)\varphi_2(x,k),\\ \vartheta_2(x,k) &= \left(-\frac{ik}{\lambda} + K^+(k)\right)\psi_2(x,k) + K_0^+(k)\psi_1(x,k),\\ \vartheta_2(x,-k) &= K^+(-k)\psi_1(x,k) + K_0^+(-k)\psi_2(x,k), \end{aligned} \tag{3.17}$$

where the functions $K^-(k)$, $K^+(k)$, $K_0^-(k)$, $K_0^-(k)$ are independent of x.

Now we are going to determine these functions introducing the notations

$$C^-(k) = -i\omega\lambda \int_{-\infty}^{\infty} \left(\frac{p(\eta)s(\eta)}{\eta+k} - \frac{q(\eta)l(\eta)}{\eta-k}\right) d\eta,$$

$$C(k) = \lambda\omega\pi\left(p(k)l(k) + q(-k)s(-k)\right),$$

we get

$$\Phi_-(x,k) \sim C^-(k)e^{-ikx} + C(k)e^{ikx} \qquad \text{as} \qquad x \to -\infty. \tag{3.18}$$

Similarly we obtain

$$\Phi_+(x,k) \sim C_0(k)e^{-ikx} + C^+(k)e^{ikx} \qquad \text{as} \qquad x \to \infty,$$

where

$$C^{+}(k) = -i\omega\lambda \int_{-\infty}^{\infty} \left(\frac{\alpha(\eta)\delta(\eta)}{\eta - k} - \frac{\beta(\eta)\delta(\eta)}{\eta + k} \right) d\eta,$$

$$C_0(k) = -\lambda\omega\pi \left(\alpha(-k)\gamma(-k) + \beta(k)\delta(k) \right). \tag{3.19}$$

We introduce the function

$$\vartheta(x,k) = \vartheta_1(x,k) - a(k)\vartheta_2(x,-k) - b(k)\vartheta_2(x,k). \tag{3.20}$$

By identities (3.8) and (3.9), expansions (3.14), (3.15), identity (3.16) and according to definition (3.6) and the expansion of the fundamental system of solutions we get the following identity

$$\vartheta(x,k) = \left(\frac{\partial a(k)}{\partial t} - 2\lambda\omega\pi a(k)Q(k) \right) \psi_1(x,k) + \left(\frac{\partial b(k)}{\partial t} + 2a(k)C_0(-k) \right) \psi_2(x,k). \tag{3.21}$$

On the other hand, passing to the limit $x \to -\infty$ in (3.8), employing identities (3.15), (3.16), asymptotic expansions of the Jost solutions to equation (3.1), (3.18),

$$K^{-}(k) = C^{-}(k) - \pi\omega\lambda \left(l(k)q(k) + s(-k)p(-k) \right), \tag{3.22}$$

$$K_0^{-}(k) = C(k) - \pi\omega\lambda \, \left(l(k)q(k) + s(-k)p(-k) \right), \tag{3.23}$$

we find:

$$\vartheta_1(x,k) = K^{-}(k)\varphi_1(x,k) + K_0^{-}(k)\varphi_2(x,k).$$

Passing to the limit as $x \to +\infty$ and introducing the notation $\gamma_1 = \frac{ik}{2\lambda}$,

$$K^{+}(k) = C^{+}(k) + \pi\omega\lambda \left(\delta(k)\alpha(k) + \gamma(-k)\beta(-k) \right),$$

$$K_0^{+}(k) = C_0(k) + \pi\omega\lambda \, \left(\delta(k)\beta(k) + \alpha(-k)\gamma(-k) \right),$$

we obtain:

$$\vartheta_2(x,k) = K_0^{+}(k)\psi_1(x,k) + \left(\frac{ik}{\lambda} + K^{+}(k) \right) \psi_2(x,k).$$

Similarly, making the change $k = -k$ in (3.11), we consider the function $\vartheta_2(x,-k)$. Then by the notations (3.22), (3.23) for $\gamma_1 = \frac{ik}{2\lambda}$ we get:

$$\vartheta_2(x,-k) = K_0^{-}(-k)\psi_2(x,k) + K^{+}(-k)\psi_1(x,k).$$

Employing respectively (3.18) and (3.19), we have:

$$K_0^{+}(k) = K_0^{-}(k) \equiv 0.$$

Therefore, identities (3.17) can be rewritten as

$$\begin{aligned} \vartheta_1(x,k) &= K^{-}(k)\varphi_1(x,k), \\ \vartheta_2(x,k) &= \left(-\frac{ik}{\lambda} + K^{+}(k) \right) \psi_2(x,k), \\ \vartheta_2(x,-k) &= K^{+}(-k)\psi_1(x,k). \end{aligned} \tag{3.24}$$

Similarly, employing the expansion of the fundamental system of solutions to equation (3.1), by (3.20) we conclude that

$$\vartheta(x,k) = a(k) \left(K^{-}(k) - K^{+}(k) \right) \psi_1(x,k) + b(k) \left(\frac{ik}{\lambda} + K^{-}(k) - K^{+}(k) \right) \psi_2(x,k). \tag{3.25}$$

Combining identities (3.21) and (3.25) and comparing the coefficients at $\psi_1(x,k)$ and $\psi_2(x,k)$, we get the evolution equations for $a(k)$ and $b(k)$:

$$\frac{\partial a(k)}{\partial t} - 2\lambda\pi\omega a(k)Q(k) = a(k)\left(K^-(k) - K^+(-k)\right), \tag{3.26}$$

$$\frac{\partial b(k)}{\partial t} - 2a(k)C_0(-k) = b(k)\left(\frac{ik}{\lambda} + K^-(k) - K^+(k)\right). \tag{3.27}$$

Multiplying equation (3.27) by $a(k)$ and deducting it from (3.26) multiplied by $b(k)$, we obtain:

$$\begin{aligned}\frac{\partial r(k,t)}{\partial t} &= \left(-\frac{4ik\omega}{4k^2+1} + (4k^2+1)\frac{\pi}{2}Q(k,t) - K^+(k,t) + K^+(-k,t)\right) r(k,t)\\ &- 2C_0(-k,t), \qquad \operatorname{Im} k = 0.\end{aligned} \tag{3.28}$$

It can be shown that the functions $K^+(k)$, $K^-(k)$ can be analytically continued with respect to k in the upper half-plane $\operatorname{Im} k > 0$. It is obvious that as $\operatorname{Im} k > 0$, the identities hold:

$$K^+(k) = -i\omega\lambda\int_{-\infty}^{\infty}\left(\frac{\alpha(\eta)\delta(\eta)}{\eta-k} - \frac{\beta(\eta)\delta(\eta)}{\eta+k}\right)d\eta,$$

$$K^-(k) = -i\omega\lambda\int_{-\infty}^{\infty}\left(\frac{p(\eta)s(\eta)}{\eta+k} - \frac{q(\eta)l(\eta)}{\eta-k}\right)d\eta.$$

4. Evolution of spectral characteristics associated with discrete spectrum

We introduce a notation

$$G_n(x) = \vartheta_1(x,ik_n) - b_n\vartheta_2(x,ik_n), \qquad n = 1,2,\ldots,N, \tag{4.1}$$

and in the case of the continuous spectrum we obtain:

$$\vartheta_1(x,ik_n) = \varphi_{1nt} - \left(\frac{1}{2\lambda_n} - u\right)\varphi_{1nx} - \frac{u_x}{2}\varphi_{1n} - \gamma_1\varphi_{1n} - \lambda_n\int_{-\infty}^{\infty} f(x,\eta)F_-(x,ik_n,\eta)d\eta, \tag{4.2}$$

$$\vartheta_2(x,ik_n) = \psi_{2nt} - \left(\frac{1}{2\lambda_n} - u\right)\psi_{2nx} - \frac{u_x}{2}\psi_{2n} - \gamma_1\psi_{2n} - \lambda_n\int_{-\infty}^{\infty} f(x,\eta)F_+(x,ik_n,\eta)dk, \tag{4.3}$$

where

$$\lambda_n = \lambda(ik_n) = -\frac{1}{\omega}\left(-k_n^2 + \frac{1}{4}\right), \qquad n = 1,2,\ldots,N,$$

$$F_-(x,ik_n,\eta) = \int_{-\infty}^{x}(m(z)+\omega)\varphi_{1n}(z)g(z,\eta)dz, \tag{4.4}$$

$$F_+(x,ik_n,\eta) = -\int_{x}^{\infty}(m(z)+\omega)\psi_{2n}(z)g(z,\eta)dz. \tag{4.5}$$

Substituting expansions (4.2), (4.3) into (4.1), according to the identity $\varphi_{1n}(x) = b_n\psi_{2n}(x)$ and in view of formulae (4.4) and (4.5) and Lemma 3 from work [11], we obtain:

$$G_n(x) = \frac{\partial b_n}{\partial t}\psi_{2n} - \lambda_n b_n\int_{-\infty}^{\infty} f(x,\eta)\frac{1}{\lambda_n - \xi}W\{g,\psi_{2n}\}\big|_{-\infty}^{\infty}d\eta = \frac{\partial b_n}{\partial t}\psi_{2n}. \tag{4.6}$$

On the other hand, according to identities (3.24) with $k = ik_n$, identities (4.1) can be rewritten as

$$G_n(x) = K^-(ik_n)\varphi_{1n} - b_n\left(\frac{k_n}{\lambda_n} + K^+(ik_n)\right)\psi_{2n} = b_n\left(K^-(ik_n) - \frac{k_n}{\lambda_n} - K^+(ik_n)\right)\psi_{2n}, \quad (4.7)$$

where $\lambda_n = \lambda(ik_n)$, $n = 1, 2, \ldots, N$. Therefore, comparing identities (4.6) and (4.7), we obtain evolution equations for b_n:

$$\frac{db_n(t)}{dt} = \left(\frac{4\omega k_n}{1-4k_n^2} + K^-(ik_n,t) - K^+(ik_n,t)\right) b_n(t), \qquad n = 1, 2, \ldots, N. \quad (4.8)$$

By Lemma 3 in work [11] we easily obtain:

$$\frac{dk_n(t)}{dt} = 0, \qquad n = 1, 2, \ldots, N. \quad (4.9)$$

Thus, identities (3.28), (4.8) and (4.9) can be summarized in the following theorem.

Theorem 4.1. *If the functions $u(x,t)$, $g(x,t,k)$, $f(x,t,k)$ solve problem (1.1)–(1.4), then the scattering data for equation (2.1) with the function $u(x,t)$ vary in t as follows:*

$$\begin{aligned}
\frac{dr(k,t)}{dt} =& \left(-\frac{4ik\omega}{4k^2+1} + (4k^2+1)\frac{\pi}{2}Q(k,t) - K^+(k,t) + K^+(-k,t)\right) r(k,t)\\
&- 2C_0(-k,t), \quad \operatorname{Im} k = 0,\\
\frac{db_n(t)}{dt} =& \left(\frac{4\omega k_n}{1-4k_n^2} + K^-(ik_n,t) - K^+(ik_n,t)\right) b_n(t),\\
\frac{dk_n(t)}{dt} =& 0, \qquad n = 1, 2, \ldots, N,
\end{aligned}$$

where

$$\begin{aligned}
K^+(k,t) =& i\left(k^2+\frac{1}{4}\right)\int_{-\infty}^{\infty}\left(\frac{\alpha(\eta,t)\delta(\eta,t)}{\eta-k} - \frac{\beta(\eta,t)\gamma(\eta,t)}{\eta+k}\right)d\eta\\
&- \left(k^2+\frac{1}{4}\right)\pi\left(\alpha(k,t)\delta(k,t) + \gamma(-k,t)\beta(-k,t)\right),\\
K^-(k,t) =& i\left(k^2+\frac{1}{4}\right)\int_{-\infty}^{\infty}\left(\frac{p(\eta,t)s(\eta,t)}{\eta+k} - \frac{q(\eta,t)l(\eta,t)}{\eta-k}\right)d\eta\\
&+ \left(k^2+\frac{1}{4}\right)\pi\left(l(k,t)q(k,t) + s(-k,t)p(-k,t)\right),
\end{aligned}$$

$$p(k,t) = \alpha(k,t)a(k,t) - \beta(k,t)b(k,t), \qquad q(k,t) = -\alpha(k,t)b(-k,t) + \beta(k,t)a(-k,t),$$
$$l(k,t) = \gamma(k,t)a(k,t) - \delta(k,t)b(k,t), \qquad s(k,t) = -\gamma(k,t)b(-k,t) + \delta(k,t)a(-k,t),$$
$$Q(k,t) = \beta(k,t)\gamma(k,t) + \alpha(-k,t)\delta(-k,t),$$
$$C_0(-k,t) = \left(k^2+\frac{1}{4}\right)\pi\left(\alpha(k,t)\gamma(k,t) + \beta(-k,t)\delta(-k,t)\right).$$

The obtained identities determine completely the evolution of the scattering data and this allows us to apply the inverse scattering data for solving problem (1.1)–(1.4).

We also mention work [11], where a problem on integrating the Camassa-Holm equation with a self-consistent source was considered in the case of moving eigenvalues. In the case of the sine-Gordon equation and the Toda chain such problems were considered in works [12], [13].

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