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ON DEGENERACY OF ORBITS OF NILPOTENT LIE ALGEBRAS

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Abstract. In the paper we discuss 7-dimensional orbits in $\mathbb{C}^4$ of two families of nilpotent 7-dimensional Lie algebras; this is motivated by the problem on describing holomorphically homogeneous real hypersurfaces. Similar to nilpotent 5-dimensional algebras of holomorphic vector fields in $\mathbb{C}^3$, the most part of algebras considered in the paper has no Levi non-degenerate orbits. In particular, we prove the absence of such orbits for a family of decomposable 7-dimensional nilpotent Lie algebra (31 algebra). At the same time, in the family of 12 non-decomposable 7-dimensional nilpotent Lie algebras, each containing at least three Abelian 4-dimensional ideals, four algebras has non-degenerate orbits. The hypersurfaces of two of these algebras are equivalent to quadrics, while non-spherical non-degenerate orbits of other two algebras are holomorphically non-equivalent generalization for the case of 4-dimensional complex space of a known Winkelmann surface in the space $\mathbb{C}^3$. All orbits of the algebras in the second family admit tubular realizations.

Keywords: homogeneous manifold, holomorphic function, vector field, Lie algebra, Abelian ideal.

Mathematics Subject Classification: 32M12, 32A10, 17B66, 14H10, 13A15

1. Introduction

At present, in the problem on description of locally holomorphically homogeneous real hypersurfaces of multi-dimensional complex spaces, two-dimensional [1] and three-dimensional cases [2] are completely studied. Basing on the classification of 3-dimensional real Lie algebras, E. Cartan showed that in $\mathbb{C}^2$ all homogeneous hypersufraces are the orbits of exactly such Lie algebras; a similar idea of using the classification of 5-dimensional Lie algebras allowed to complete the description of locally homogeneous hypersurfaces of 3-dimensional complex spaces.

At that, an important step in the case of 3-dimensional complex spaces was a statement on non-degeneracy of most 5-dimensional nilpotent Lie algebras: the orbits of only two of such algebras are non-degenerate quadrics

$$\operatorname{Im} z_3 = |z_1|^2 \pm |z_2|^2, \tag{1.1}$$

while the other nilpotent algebras can not have non-degenerate 5-dimensional orbits in $\mathbb{C}^3$ [3]. In particular, this statement is true for three decomposable nilpotent Lie algebras of dimension 5.

In view of this, it seemed natural to conjecture on the Levi degeneracy of non-spherical (not reducible to analogues of quadrics (1.1)) orbits of nilpotent Lie algebras in the space of arbitrary dimensions. However, in [4], one-parametric families of 7-dimensional nilpotent Lie algebras were studied and their non-spherical Levi non-degenerate orbits were found in $\mathbb{C}^4$.

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In the present paper we discuss 7-dimensional orbits in $\mathbb{C}^4$ of nilpotent 7-dimensional Lie algebras in two rather wide families. The first of them is the family of decomposable Lie algebras, while the other is the family of non-decomposable 7-dimensional nilpotent Lie algebras, each containing at least a 4-dimensional Abelian ideal.

For the first family, containing 31 Lie algebra, we prove that similarly to the case of 3-dimensional complex space, the statement on Levi degeneracy of 7-dimensional orbits of all such algebras remains true, see Theorem 2.1. In view of this, it is natural to conjecture on Levi degeneracy of all real hypersurfaces in the spaces $\mathbb{C}^n$ of arbitrary dimensions being the orbits of decomposable nilpotent $(2n-1)$-dimensional Lie algebras.

The second family we consider consists of 12 algebras and admits Levi non-degenerate orbits, see Theorem 6.2. Here 8 algebras follow a common line related with the degeneracy of the orbits of many nilpotent Lie algebras, see also [5] and [6]. At the same time, the orbits of two among 12 algebras are holomorphically equivalent to the quadrics

$$\operatorname{Im} z_4 = |z_1|^2 + |z_2|^2 \pm |z_3|^2. \tag{1.2}$$

An indefinite spherical surface corresponding to the minus sign at the term $|z_3|^2$ in equation (1.2) turns out to be the orbits of two *different* 7-dimensional Lie algebras being the subalgebras of a complete 24-dimensional algebra of the symmetries of this surface. Such phenomenon was mentioned in works [7], [4], [2] as rather natural for manifolds with rich algebras of symmetries, in particular, for spherical hypersurfaces.

Non-spherical integral hypersurfaces of other two algebras among 12 are, up to holomorphic equivalence, the surfaces described by the equations

$$\operatorname{Im} z_4 = z_1\bar{z}_2 + z_2\bar{z}_1 + |z_3|^2 \pm |z_1|^4. \tag{1.3}$$

These surfaces possess [8] the richest groups and algebras of symmetries among non-spherical non-degenerate homogeneous hypersurfaces in $\mathbb{C}^4$. For each of them the dimension of the holomorphic stabilizer, that is, of the local group of holomorphic transformations preserving the surface and a fixed point on it, is equal to 6. Hence, the dimension of the total Lie algebra of holomorphic vector fields on each of them is equal to $6+7=13$.

We call a pair of homogeneous non-spherical surfaces (1.3) *generalization of Winkelmann surfaces* ([9])

$$\operatorname{Im} z_3 = z_1\bar{z}_2 + z_2\bar{z}_1 + |z_1|^4 \tag{1.4}$$

in the space $\mathbb{C}^3$. On this pair, a maximum, equalling to 8, is attained by the dimension of Lie algebra of the holomorphic vector fields in the class of homogeneous non-degenerate non-spherical hypersurfaces in the space $\mathbb{C}^3$. A naturalness of the relation between surfaces (1.3) and (1.4) is justified by the similarity of these equations.

We note that by simple transformations, the equations of non-spherical non-degenerate homogeneous hypersurfaces in $\mathbb{C}^4$ obtained in [4] are reduced exactly to form (1.3).

We also recall that the degeneracy at the point 0 of a smooth surface in $\mathbb{C}^4$ containing the origin and given by the equation $\operatorname{Im} z_4 = F(z_1, z_2, z_3, \operatorname{Re} z_4)$, $dF(0)=0$, means the degeneracy of the matrix

$$H = \left(\partial^2 F/\partial z_k \partial \bar{z}_j\right)(0), \quad k, j \in \{1,2,3\}.$$

Hereafter we suppose that the studied homogeneous hypersurfaces being the orbits of 7-dimensional Lie algebras can be described exactly by such equations.

The technique used in this paper develops the ideas from [10] on representing abstract Lie algebras as the algebras of holomorphic vector fields in the multi-dimensional complex spaces. The consideration of only 7-dimensional orbits of such algebras in $\mathbb{C}^4$ determined by the system of seven basis equations

$$\operatorname{Re}\left(e_k(\Phi)_{|M}\right) \equiv 0, \qquad k = 1, \dots, 7, \tag{1.5}$$

means the completeness of the rank of these algebras. The condition of such completeness and the Levi non-degeneracy of the orbits of the discussed algebras turn out to be rigid filters allowing us to reduce the consideration of the large families of Lie algebras to studying just its particular representatives. At a final stage, the systems of partial differential equations (1.5) are integrated by means of standard methods.

The present paper is an extended version of a talk presented at the conference "Ufa Autumn Mathematical School" in November, 2020, see [11], [12].

We mention one more issue arose in view of the remarks of the referee on this work. The base for arguing in the paper is the classification of 7-dimensional (and also of 6-dimensional and 5-dimensional) nilpotent Lie algebras. For the 7-dimensional case such classification is provided, for instance, in a known paper by C. Seeley [13]. However, as M.P. Gong showed in [14], the paper by C. Seeley contains some errors and inaccuracies. Because of this the authors used needed classification lists from works [14] regarding them as more reliable.

2. Simplest cases of decomposable Lie algebras

The main result of the first part of the paper is the following statement.

Theorem 2.1. *A real hypersurface in $\mathbb{C}^4$ being an orbit of a nilpotent decomposable 7-dimensional Lie algebra is necessary Levi degenerate.*

We split the proof of this theorem into several cases related with possible structures of decomposable algebras and the dimensions of the Abelian ideals they contain. The complete proof consists of particular cases discussed in Sections 2-5 of the paper.

Let us consider possible structures of decomposable 7-dimensional Lie algebras.

The list of such algebras can be easily formed in view of the representation of the number 7 as a sum of several smaller natural numbers equalling to the dimensions of non-decomposable algebras-summands. Formally speaking, there are 14 decompositions; for instance, there are three ways of representing the number 7 as a sum of two numbers:

$$7 = 6 + 1 = 5 + 2 = 4 + 3.$$

Apart of this, there are 4 representations of this number as a sum of three terms, 3 representations of this number as a sum of three terms, 3 representations of four terms, 2 representations of five terms and two single representations of six and seven terms.

We note that the direct sum $g = g_1 \oplus \ldots \oplus g_n$ of Lie algebras is nilpotent if and only if each term in the sum is nilpotent. Since there exist no two-dimensional nilpotent algebras, we should remove from the formal representations of the seven as a sum of small terms all decompositions containing at least one term «2». Then we get a specified list of 7 options:

$$\begin{aligned} 7 &= (6+1) = (4+3) = (5+1+1) = (3+3+1) = (4+1+1+1) \\ &= (3+1+1+1+1) = (1+1+1+1+1+1+1). \end{aligned} \tag{2.1}$$

One more point is taking into consideration the number of nilpotent (non-decomposable) Lie algebras of «smaller» dimensions. According to [14], there exist 20 such algebras of the dimension six and extra 6 nilpotent (non-decomposable) Lie algebras have the dimension five. Each of the dimensions 1, 3, 4 has exactly one nilpotent non-decomposable representative. In view of this, the list of decompositions (2.1) corresponds to

$$20 + 1 + 6 + 1 + 1 + 1 + 1 = 31$$

different nilpotent decomposable 7-dimensional Lie algebras.

For many of the mentioned 31 decomposable Lie algebras the absence of real Levi non-degenerate hypersurfaces, being the orbits of holomorphic realizations of these algebras, can be established rather easily. The base of such conclusion is the following statement proved in [5].

Theorem 2.2. *If a 7-dimensional Lie algebra g_7 possesses a 5-dimensional Abelian subalgebra I_5 and in the complement $g_7 \setminus I_5$ there exists an element commuting with a 4-dimensional subalgebra $h_4 \subset I_5$, then all integral hypersurfaces of the holomorphic realization of the algebra g_7 in the space $\mathbb{C}^4$ are Levi degenerate.*

Remark 2.1. *An obvious corollary of this theorem is the statement on Levi degeneration of all orbits in $\mathbb{C}^4$ of each holomorphic realization of 7-dimensional algebra g_7 having a 6-dimensional Abelian subalgebra.*

We first apply Theorem 2.2 and its corollary to decomposable algebras containing in their decompositions at most 4-dimensional terms.

Hereinafter nilpotent non-decomposable terms of smaller dimensions $k \in \{1,3,4,5,6\}$ forming a discussed 7-dimensional algebra are denoted by $\mathfrak{g}_k$.

Proposition 2.1. *7-dimensional orbits in $\mathbb{C}^4$ of all realizations of the five algebras*

$$\mathfrak{g}_4 \oplus \mathfrak{g}_3, \quad 2\mathfrak{g}_3 \oplus \mathfrak{g}_1, \quad \mathfrak{g}_4 \oplus 3\mathfrak{g}_1, \quad \mathfrak{g}_3 \oplus 4\mathfrak{g}_1, \quad 7\mathfrak{g}_1 \tag{2.2}$$

can be only Levi degenerate.

Proof. We begin the proof of Proposition 2.1 from the end of list (2.2). The Abelian algebra $7\mathfrak{g}_1$ contains a 6-dimensional Abelian ideal and this is why by the corollary of Theorem 2.2 it can possess only Levi degenerate 7-dimensional orbits.

The three-dimensional Heisenberg algebra $\mathfrak{g}_3$ with the only relation

$$[e_1, e_2] = e_3 \tag{2.3}$$

contains a pair of commuting vectors e_2, e_3. A linear hull h_2 of these vectors is a 2-dimensional Abelian subalgebra (and even an Abelian ideal) in $\mathfrak{g}_3$. A sum of h_2 with four one-dimensional Abelian algebras forms a 6-dimensional Abelian subalgebra in a 7-dimensional Lie algebra $\mathfrak{g}_3 \oplus 4\mathfrak{g}_1$.

Similarly, the only non-trivial 4-dimensional nilpotent Lie algebra with the relations

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = e_4$$

has a large by dimension 3-dimensional Abelian ideal $h_3 =< e_2, e_3, e_4 >$. Then a 7-dimensional algebra $\mathfrak{g}_4 \oplus 3\mathfrak{g}_1$ also possesses a 6-dimensional Abelian ideal.

The first two Lie algebras in list (2.2) have only 5-dimensional Abelian ideal. In the case of the algebra $\mathfrak{g}_4 \oplus \mathfrak{g}_3$ of two terms this is $h_5 = h_3 + h_2$, and in this second case, $h_5 = h_2^{(1)} + h_2^{(2)} + \mathfrak{g}_1$, where $h_2^{(1)}$, $h_2^{(2)}$ are two-dimensional Abelian ideals of two 3-dimensional terms belonging to the algebra $g_7 = 2\mathfrak{g}_3 \oplus \mathfrak{g}_1$.

In addition to this ideal, there exists an element commuting with four independent vectors in the ideal h_5.

In the first case, as such element, we can take e_5 in $\mathfrak{g}_3$ (assuming that this subalgebra of the total algebra g_7 is described by the relation $[e_5, e_6] = e_7$) not belonging to the two-dimensional ideal h_2. This element commutes with the basis e_2, e_3, e_4 in the ideal h_3 and also with the element e_7 in $\mathfrak{g}_3$.

In the second case, we denote by $\mathfrak{g}_3^{(1)}$ and $\mathfrak{g}_3^{(2)}$ two 3-dimensional terms belonging to g_7. Here the element e_1 is from the first 3-dimensional term (with relation (2.3)) complement to $h_2^{(1)}$ and it commutes with two basis elements of the two-dimensional ideal $h_2^{(2)}$ of the second 3-dimensional algebra $\mathfrak{g}_3^{(2)}$, with the one-dimensional term $\mathfrak{g}_1$ and also with the element e_3.

Then Theorem 2.2 and its corollary complete the proof on the absence of non-degenerate orbits in $\mathbb{C}^4$ for all five Lie algebras in (2.2). □

We proceed to studying decomposable Lie algebras containing according to (2.1) a 5-dimensional or a 6-dimensional term.

3. Algebras with 5-dimensional and 6-dimensional non-decomposable terms

Proposition 3.1. *7-dimensional orbits in $\mathbb{C}^4$ for all realizations of the Lie algebras of the form $\mathfrak{g}_5 \oplus 2\mathfrak{g}_1$ with nilpotent non-decomposable term of dimension 5 can be only Levi degenerate.*

Proof. In accordance with [14] (see also [15]), we write the table of commutation relations for all six nilpotent non-decomposable Lie algebras of dimension 5; hereinafter by s_{jk} we denote the commutator $[e_j, e_k]$.

Table 3.1: Non-decomposable 5-dimensional nilpotent Lie algebras [14]

Algebras	s_{12}	s_{13}	s_{14}	s_{15}	s_{23}	s_{24}	s_{25}	s_{34}	s_{35}	s_{45}
$N_{5,1}$	e_3	e_4	e_5		e_5					
$N_{5,2,1}$	e_3	e_4	e_5							
$N_{5,2,2}$	e_4		e_5		e_5					
$N_{5,2,3}$	e_3	e_4			e_5					
$N_{5,3,1}$	e_5							e_5		
$N_{5,3,2}$	e_4	e_5								

By this table we see easily that the algebras $N_{5,2,1} \oplus 2\mathfrak{g}_1$ and $N_{5,3,2} \oplus 2\mathfrak{g}_1$ possess a 6-dimensional Abelian ideal $I_6 =< e_2, e_3, e_4, e_5 > \oplus 2\mathfrak{g}_1$.

For other four algebras $\mathfrak{g}_5$ in this table the sum $\mathfrak{g}_5 \oplus 2\mathfrak{g}_1$ has a 5-dimensional Abelian ideal of form $I_3 \oplus 2\mathfrak{g}_1$. Here for the algebra $N_{5,1}$ the Abelian ideal I_3 is a linear hull $< e_3, e_4, e_5 >$, for $N_{5,2,2}$ this is $< e_2, e_4, e_5 >$, while for two remaining algebras $N_{5,2,3}$ and $N_{5,3,1}$ we let $I_3 =< e_3, e_4, e_5 >$.

It remains to observe that in the complement $\mathfrak{g}_5 \setminus I_3$ to each of these 5-dimensional ideals there is an element commuting with two elements in I_3:

e_2 commutes with $< e_4, e_5 > \subset I_3$ in the case of the algebras $N_{5,1}$, $N_{5,2,3}$ and $N_{5,3,1}$,

e_3 commutes with $< e_4, e_5 > \subset I_3$ in the case of the algebra $N_{5,2,2}$.

Then each 7-dimensional algebra $\mathfrak{g}_7 = \mathfrak{g}_5 \oplus 2\mathfrak{g}_1$ with the first term from Table 3.1 has a 5-dimensional Abelian ideal I_5 and an element in the complement to this ideal commuting with four independent elements in I_5. By Theorem 2.2, all 7-dimensional orbits in $\mathbb{C}^4$ of the Lie algebras discussed in this proposition can be only degenerate. □

We begin the study of 20 decomposable algebras containing a 6-dimensional non-decomposable term with the following technical statement.

Proposition 3.2. *Thirteen of 20 algebras of the form $\mathfrak{g}_6 \oplus \mathfrak{g}_1$ have a 5-dimensional ideal and an element in the complement commuting with a 4-dimensional subalgebra of this ideal.*

Five of seven remaining algebras have a 5-dimensional Abelian ideal and an element in its complement commuting only with a 3-dimensional subalgebra of this ideal.

Two of seven algebras have only a 4-dimensional Abelian ideal and two elements in its complement commuting with 2-dimensional subalgebras of such ideal.

Remark 3.1. *For the mentioned 13 Lie algebras in Proposition 3.2 the statement of Theorem 2.1 holds thanks to Theorem 2.2.*

Remark 3.2. *We specify that two of these 13 algebras have 6-dimensional Abelian ideal and hence, they also satisfy formal conditions on a 5-dimensional ideal in Proposition 3.2.*

Proof. The proof of Proposition 3.2 requires an accurate consideration, for example, with a computer assistance, of commutation relations similar to ones given in Table 3.1 and provided, for instance, in [14]. We briefly discuss just 3 algebras of the mentioned 13 and we discuss in details 7 most interesting algebras of form $\mathfrak{g}_6 \oplus \mathfrak{g}_1$ not satisfying the assumptions of Theorem 2.2.

So, two 6-dimensional algebras:

$N_{6,2,1}$ with four commutation relations $[e_1, e_i] = e_{i+1}, 2 \leqslant i \leqslant 5$ and $N_{6,3,4}$ with three relations $[e_1, e_2] = e_3$; $[e_2, e_3] = e_5$; $[e_2, e_4] = e_6$ contain Abelian ideals $I_5^{(1)} =< e_2, e_3, e_4, e_5, e_6 >$ and $I_5^{(2)} =< e_1, e_3, e_4, e_5, e_6 >$, respectively.

The one-dimensional term $\mathfrak{g}_1$ included into the direct sum with each of these algebras obviously increases by one the dimension of the Abelian ideal of the obtained 7-dimensional algebras.

As an example of a general situation for 13 algebras we consider three non-trivial commutation relations defining $N_{6,3,6}$, which is the last in the list [14] of 20 non-decomposable 6-dimensional nilpotent algebras:

$$[e_1, e_2] = e_4; \qquad [e_1, e_3] = e_5; \qquad [e_2, e_3] = e_6.$$

This algebra has only a 4-dimensional Abelian ideal $I_4 =< e_3, e_4, e_5, e_6 >$, and in its direct sum with the one-dimensional algebra $\mathfrak{g}_1$ we obtain a 5-dimensional ideal $I_5 = I_4 \oplus \mathfrak{g}_1$. At the same time, the element e_2 in the complement to I_5 commutes with three fields e_4, e_5, e_6 in I_4 as well as with the basis field of the one-dimensional term $\mathfrak{g}_1$.

Now let us describe as a table the commutation relations for seven most interesting algebras (of twenty) not satisfying the assumptions of Theorem 2.2. We also observe that the basis element e_6 belongs to the center of each of discussed 20 Lie algebras. This is the reason why in Table 3.2 we write out only 10 instead of formal 15 commutation relations for each of the involved algebras: trivial relations with e_6 are omitted.

Table 3.2: 6-dimensional Lie algebras with «low-dimensional» Abelian ideals [14]

Algebras	s_{12}	s_{13}	s_1	s_{15}	s_{23}	s_{24}	s_{25}	s_{34}	s_{35}	s_{45}
$N_{6,1,1}$	e_3	e_4	e_5	e_6	e_5	e_6				
$N_{6,1,2}$	e_3	e_4	e_5		e_5		e_6	$-e_6$		
$N_{6,1,4}$	e_3	e_4	e_6		e_6		e_6			
$N_{6,2,2}$	e_3	e_4	e_5				e_6	$-e_6$		
$N_{6,2,3}$	e_4		e_5	e_6	e_5			$-e_6$		
$N_{6,2,5}$	e_3	e_4		e_6	e_5	e_6				
$N_{6,3,1}$	e_4	e_5					e_6		e_6	

This table implies the following specifications of Proposition 3.2 on Abelian ideals I_k of the discussed 7-dimensional algebras and the elements in the complements to these ideals commuting with the subalgebras of such ideals ($\mathfrak{g}_1 =< e_7 >$):

$N_{6,1,1} \oplus \mathfrak{g}_1$: $I_5 =< e_3, e_4, e_5, e_6, e_7 >$, e_2 commutes with $< e_5, e_6, e_7 >$,
$N_{6,1,4} \oplus \mathfrak{g}_1$: $I_5 =< e_3, e_4, e_5, e_6, e_7 >$, e_1 commutes with $< e_5, e_6, e_7 >$,
$N_{6,2,3} \oplus \mathfrak{g}_1$: $I_5 =< e_2, e_4, e_5, e_6, e_7 >$, e_3 commutes with $< e_5, e_6, e_7 >$,
$N_{6,2,5} \oplus \mathfrak{g}_1$: $I_5 =< e_3, e_4, e_5, e_6, e_7 >$, e_1 commutes with $< e_4, e_6, e_7 >$,
$N_{6,3,1} \oplus \mathfrak{g}_1$: $I_5 =< e_1, e_4, e_5, e_6, e_7 >$, e_2 commutes with $< e_4, e_6, e_7 >$;
$N_{6,1,2} \oplus \mathfrak{g}_1$ and $N_{6,2,2} \oplus \mathfrak{g}_1$: $I_4 =< e_4, e_5, e_6, e_7 >$, e_2 commutes with $< e_4, e_6, e_7 >$.

The proof is complete. □

Our next considerations are related exactly with selected ideals of the seven discussed algebras.

4. Auxiliary statements

Lemma 4.1. ([4]). *Let a real hypersurface $M \subset \mathbb{C}^4$ be Levi non-degenerate near some its point Q and be an orbit of a 7-dimensional Lie algebra g of holomorphic vector fields in this space. Let I_4 be a 4-dimensional Abelian subalgebra in g with a fixed basis e_4, e_5, e_6, e_7. By a* *holomorphic change of coordinates in the space $\mathbb{C}^4$ defined in the neighbourhood of the point Q this basis can be reduced to one of the following three forms:*

$$1)\ \begin{matrix}(1, & 0, & 0, & 0),\\ (0, & 1, & 0, & 0),\\ (0, & 0, & 1, & 0),\\ (0, & 0, & 0, & 1),\end{matrix} \qquad 2)\ \begin{matrix}(0, & b_4(z_1), & c_4(z_1), & d_4(z_1)),\\ (0, & 1, & 0, & 0),\\ (0, & 0, & 1, & 0),\\ (0, & 0, & 0, & 1),\end{matrix} \qquad 3)\ \begin{matrix}(0, & 1, & 0, & 0),\\ (0, & 0, & c_5(z_1), & d_5(z_1)),\\ (0, & 0, & 1, & 0),\\ (0, & 0, & 0, & 1).\end{matrix}$$

Lemma 4.2. *Let $M \subset \mathbb{C}^4$ be a Levi non-degenerate hypersurface, on which there exists a 7-dimensional algebra of holomorphic vector fields with a 5-dimensional Abelian subalgebra I_5. Then the first of three cases in Lemma 4.1 is possible for none of quadruple of independent fields in I_5.*

Proof. We consider some basis e_1, e_2, e_3, e_4, e_5 in a 5-dimensional Abelian algebra I_5 assuming that the quadruple of the fields in the basis is flattened and having the following form in some coordinates:

$$e_1 = (1,0,0,0), \qquad e_2 = (0,1,0,0), \qquad e_3 = (0,0,1,0), \qquad e_4 = (0,0,0,1).$$

The fifth basis field e_5 in the algebra I_5 commuting with this quadruple should have only constant components since each field in the quadruple means differentiating with respect to one of the complex variables in the space $\mathbb{C}^4$. Considering instead of e_5 its linear combination with the fields in the quadruple, we can suppose that it reads as

$$e_5 = (iA_5, iB_5, iC_5, iD_5),$$

where A_5, B_5, C_5, D_5 are real constants.

But the vector field $e_5^* = -ie_5 = (A_5, B_5, C_5, D_5)$ is also tangential to M as a linear combination of the fields in the basis quadruple and this means that M is Levi degenerate. The obtained contradiction and the non-degeneracy of M complete the proof. □

Lemma 4.3. *Let a 7-dimensional algebra g_7 of holomorphic vector fields on a non-degenerate hypersurface $M \subset \mathbb{C}^4$ has a 5-dimensional Abelian subalgebra I_5 containing a 3-dimensional subalgebra h_3, with which some element in the complement $g_7 \setminus I_5$ commutes. Then under the simplification of the quadruple of independent fields containing some basis h_3 and an arbitrary fourth field in $I_5 \setminus h_3$, the flattening only of two fields in h_3 and of the third field in $I_5 \setminus h_3$ is impossible.*

Proof. As a basis in the 5-dimensional Abelian subalgebra I_5, we regard the fields e_3, e_4, e_5, e_6, e_7 and $h_3 =< e_5, e_6, e_7 >$. Simplifying the quadruple of the fields e_4, e_5, e_6, e_7 by the scheme of Lemma 4.1, we need to show that in the discussed situation, under an arbitrary choice of the basis fields e_5, e_6, e_7 the third case of this lemma is impossible.

Supposing that it is possible, we have three flattened fields e_4, e_6, e_7, two of which belong to 3-dimensional subalgebra, while the third does not. The field

$$e_3 = (a_3(z), b_3(z), c_3(z), d_3(z)),$$

also belonging to the Abelian subalgebra I_5, commutes with the flattened triple of the fields. This is why its components can depend *only* on the variable z_1. And the first component $a_3(z_1)$ is to be identically zero, since otherwise by using the technique of work [10] the field e_3 can be flattened keeping the flattening for the fields e_4, e_6, e_7, while this is impossible by Lemma 4.2.

Then the first components $a_k(z_1)$ of entire basis quintuple e_3, e_4, e_5, e_6, e_7 are zero. We use the existence of the field

$$e_1 = (a_1(z), b_1(z), c_1(z), d_1(z)) \in g_7 \setminus I_5$$

commuting with the fields $e_5 = (0, 0, c_5(z_1), d_5(z_1))$, $e_6 = \partial/\partial z_3$, $e_7 = \partial/\partial z_4$. By the conditions $[e_1, e_6] = [e_1, e_7] = 0$ the components of e_1 depend at most on the variables z_1, z_2. Then

$$0 = [e_1, e_5] = a_1(z_1, z_2) \cdot (0, 0, c_5'(z_1), d_5'(z_1)).$$

Therefore, $a_1(z_1, z_2)c_5'(z_1) \equiv 0$, $a_1(z_1, z_2)d_5'(z_1) \equiv 0$. Here either $a_1(z_1, z_2) \equiv 0$ or both coefficients $c_5(z_1)$, $d_5(z_1)$ are independent of the variable z_1, that is, are constants.

But for a Levi non-degenerate surface M the coefficient $a_1(z)$ can not be zero since 6 zeroes in the column of the first components of the basis fields indicate the degeneration of M. And a linear hull of independent over $\mathbb{R}$ triple of vector fields

$$e_5 = (0, 0, C_5, D_5), \qquad e_6 = (0, 0, 1, 0), \qquad e_7 = (0, 0, 0, 1)$$

with constant coefficients always contains two non-trivial fields of form Z, iZ. The presence of such pair of the fields tangential to the hypersurface M also indicates its Levi degeneracy.

The obtained contradictions complete the proof. □

Corollary 4.1. *Under the assumptions of Lemma 4.3, it is possible to flatten each basis of a three-dimensional subalgebra h_3 commuting with an element in the complement $g_7 \setminus I_5$.*

Simply speaking, the search of non-degenerate homogeneous orbits for Lie algebras with such properties can be made restricting oneself by Case 2 in Lemma 4.1.

5. Completing of proof of Theorem 2.1

Theorem 2.1 will be completely proved after the consideration of the orbits of the set of seven «exceptional» Lie algebras in Table 3.2. Below we discuss two groups of algebras, into which this set is naturally partitioned.

5.1. Decomposable algebras with 5-dimensional Abelian ideals.

Proposition 5.1. *Holomorphic realizations in the space $\mathbb{C}^4$ of five algebras*

$$N_{6,1,1} \oplus \mathfrak{g}_1, \qquad N_{6,1,4} \oplus \mathfrak{g}_1, \qquad N_{6,2,3} \oplus \mathfrak{g}_1, \qquad N_{6,2,5} \oplus \mathfrak{g}_1, \qquad N_{6,3,1} \oplus \mathfrak{g}_1, \tag{5.1}$$

containing a 5-dimensional Abelian ideal, do not *have Levi non-degenerate 7-dimensional orbits.*

Proof. For each of five mentioned 7-dimensional Lie algebras we employ the inclusions $h_3 \subset I_5 \subset g_7$ given in Section 4 and corresponding discussions from the previous section; we also recall that h_3 commutes with some element $g_7 \setminus I_5$.

At the same time, I_5, h_3 are formed, generally speaking, in different ways for each of the five algebras. For instance, for the algebra $N_{6,1,1} \oplus \mathfrak{g}_1$ we have $I_5 =< e_3, e_4, e_5, e_6, e_7 >$, while the element e_2 commutes with a 3-dimensional algebra $h_3 =< e_5, e_6, e_7 >$.

We prove Proposition 5.1 exactly for this algebra; its validity for other algebras in list (5.1) can be established in the same way.

Thus, according to Lemma 4.3, by means of a holomorphic change of the coordinates we flatten the quadruple of the fields e_2, e_5, e_6, e_7. We fix the obtained form of the basis:

$$\begin{aligned}
e_1 &= (a_1(z), b_1(z), c_1(z), d_1(z)),\\
e_2 &= (1, 0, 0, 0),\\
e_3 &= (0, b_3(z_1), c_3(z_1), d_3(z_1)),\\
e_4 &= (0, b_4(z_1), c_4(z_1), d_4(z_1)),\\
e_5 &= (0, 1, 0, 0),\\
e_6 &= (0, 0, 1, 0),\\
e_7 &= (0, 0, 0, 1)
\end{aligned}$$

of the algebra $N_{6,1,1} \oplus \mathfrak{g}_1$ after such flattening and continue consideration of the commutation relations in the realization of this algebra.

Among 21 relations for the pairs of basis fields, not considered ones are the conditions for the six commutators $[e_1, e_k]$, $(k = 2, \dots, 7)$, and two commutators $[e_2, e_3]$, $[e_2, e_4]$. By means of the relations $[e_2, e_3] = e_5$, $[e_2, e_4] = e_6$ we easily specify the forms of the fields:

$$e_3 = (0, z_1 + B_3, C_3, D_3), \qquad e_4 = (0, B_4, z_1 + C_4, D_4), \tag{5.2}$$

where B_k, C_k, D_k are some complex constants.

Extra two relations $[e_1, e_6] = [e_1, e_7] = 0$ mean that the components of the field e_1 are independent of the variables z_3, z_4. The relation $[e_1, e_5] = e_6$, allows us to specify the dependence of this field on the variable z_2 and it yields that

$$e_1 = (a_1(z_1), b_1(z_1), -z_2 + c_1(z_1), d_1(z_1)) \tag{5.3}$$

with some holomorphic functions a_1, b_1, c_1, d_1.

A field of such form should satisfy the following three relations

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = e_4, \qquad [e_1, e_4] = e_5.$$

However, the latter contains a contradiction since by (5.2) and (5.3) we have:

$$[e_1, e_4] = a_1(z_1) \cdot (0, 0, 1, 0) - B_4 \cdot (0, 0, -1, 0) \neq (0, 1, 0, 0) = e_5.$$

Thus, the assumption on the existence of at least one non-degenerate orbit of the Lie algebra $N_{6,1,1} \oplus \mathfrak{g}_1$ gives rise to the contradiction.

The proof of the proposition for this algebra is complete. As it has been mentioned above, other algebras mentioned in the formulation of the proposition can be considered in the same way. □

5.2. Decomposable algebras with 4-dimensional Abelian algebras.

Proposition 5.2. *Holomorphic realizations in the space $\mathbb{C}^4$ of the algebras $N_{6,1,2} \oplus \mathfrak{g}_1$ and $N_{6,2,2} \oplus \mathfrak{g}_1$ with 4-dimensional maximal Abelian ideals have no Levi non-degenerate orbits.*

Proof. According to Table 3.2, two of these algebras differ just by one commutator $[e_2, e_3]$. The arguing below does not involve this commutator and is common for both discussed algebras.

Supposing that there exists a realization of one of the discussed 7-dimensional Lie algebras g_7 with non-degenerate orbits, we consider three cases of Lemma 4.1 on simplification of the fixed basis e_4, e_5, e_6, e_7 in the ideal I_4 of this algebra related exactly with its non-degenerate orbit.

In *the first case* for the flattened quadruple of the basis fields in the ideal I_4 we consider the commutation relation of each of the remaining basis fields in the algebra g_7 with this quadruple. For the field e_1 we have:

$$[e_1, e_4] = e_5, \qquad [e_1, e_5] = 0, \qquad [e_1, e_6] = 0, \qquad [e_1, e_7] = 0.$$

By these relations we obtain a simplified form of the field:

$$e_1 = (A_1, -z_1 + B_1, C_1, D_1)$$

with some complex coefficients A_1, B_1, C_1, D_1. Similar considerations related with the field e_3 reduce it to the form

$$e_3 = (A_3, B_3, z_1 + C_3, D_3)$$

with constants A_3, B_3, C_3, D_3. Calculating the commutator of such field and employing the relation $[e_1, e_3] = e_4$, we obtain a contradiction:

$$A_1(0, 0, 1, 0) - A_3(0, -1, 0, 0) = (1, 0, 0, 0).$$

In *the second case* we suppose that the quadruple of the fields e_4, e_5, e_6, e_7 read as

$$\begin{aligned} e_4 &= (0, \quad b_4(z_1), \quad c_4(z_1), \quad d_4(z_1)),\\ e_5 &= (0, \quad 1, \quad 0, \quad 0),\\ e_6 &= (0, \quad 0, \quad 1, \quad 0),\\ e_7 &= (0, \quad 0, \quad 0, \quad 1). \end{aligned}$$

Here we consider the commutators of the fields e_1, e_3 with three flattened fields in the ideal I_4. Since all six commutators vanish, we conclude that the components of the fields e_1 and e_3 depend only on the variable z_1. We note that in view of the simplified form of the basis in the ideal I_4, in this case the first components of the fields $a_1(z_1)$, $a_3(z_1)$ can not vanish simultaneously.

In view of this, we consider two subcases. In the first subcase we assume that $a_1(z_1) \neq 0$ and then similar to the arguing of two previous section we can flatten the basis field e_1 to the form $e_1 = (1, 0, 0, 0)$ keeping the triple of fields e_5, e_6, e_7 flattened. The form of the fields e_3 and e_4 is also preserved but with changed, generally speaking, functional coefficients $a_k(z_1)$, $b_k(z_1)$, $c_k(z_1)$, $d_k(z_1)$, $k = 3, 4$. Then by the pair of commutation relations $[e_1, e_4] = e_5$ and $[e_1, e_3] = e_4$ we obtain a specified form of the fields

$$e_4 = (0, z_1 + B_4, C_4, D_4), \qquad e_3 = (A_3, \frac{1}{2}(z_1 + B_4)^2 + B_3, C_4 z_1 + C_3, D_4 z_1 + D_3).$$

However, calculating the commutator $[e_3, e_4] = -e_6$, we arrive at a contradiction since the left hand side of the latter identity $A_3 \cdot (0, 1, 0, 0)$ does not coincide with the right hand side.

In the second subcase we let $a_1(z_1) \equiv 0$ but then $a_3(z_1) \neq 0$. Similarly to the previous subcase, we flatten the field e_3 by a holomorphic change to $e_3 = (1, 0, 0, 0)$. Calculating in this case the commutator $[e_3, e_4] = -e_6$, we obtain a specified form of the field:

$$e_4 = (0, B_4, -z_1 + C_4, D_4)$$

with some constants B_4, C_4, D_4. Then by the relation $[e_1, e_3] = e_4$ we obtain a similar specification with additional constants A_1, B_1, C_1, D_1:

$$e_1 = (A_1, -B_4 z_1 + B_1, \frac{1}{2}(z_1 - C_4), -D_4 z_1 + D_1).$$

It remains to find the commutator of the fields e_1, e_4 taking into consideration the found facts. We have:

$$[e_1, e_4] = A_1 \cdot (0, 0, -1, 0),$$

and this contradicts to the commutation relation $[e_1, e_4] = e_5 = (0, 1, 0, 0)$.

Finally, in *the third case* we have a quadruple of basis fields in the ideal I_4 of the form

$$\begin{aligned} e_4 &= (0, 1, \quad 0, \quad 0),\\ e_5 &= (0, 0, \quad c_5(z_1), \quad d_5(z_1)),\\ e_6 &= (0, 0, \quad 1, \quad 0),\\ e_7 &= (0, 0, \quad 0, \quad 1). \end{aligned}$$

We discuss here extra two fields in the complement to this ideal, namely, e_1 and e_3. The commutator of each of these fields with the fields e_6 and e_7 vanishes. This is why the components of the fields e_1 and e_3 depend at most on the variables z_1, z_2. Bearing this in mind and the commutation relations

$$\begin{aligned} [e_1, e_5] &= a_1(z_1, z_2) \cdot (0, 0, c_5'(z_1), d_5'(z_1)) = 0,\\ [e_3, e_5] &= a_3(z_1, z_2) \cdot (0, 0, c_5'(z_1), d_5'(z_1)) = 0, \end{aligned}$$

we obtain one of the two situations:

either 1) $a_1(z_1, z_2) \equiv 0, \quad a_3(z_1, z_2) \equiv 0$,

or 2) $c_5(z_1) = C_5 = const, \quad d_5(z_1) = D_5 = const.$

In the first situation on a non-degenerate hypersurface $M \subset \mathbb{C}^4$ we have six linearly independent holomorphic vector fields e_k, $(k = 1, 3, 4, 5, 6, 7)$ with identically vanishing first components. As it was mentioned above, this is impossible.

The second situation is also impossible on a non-degenerate hypersurface since in the linear hull of independent over $\mathbb{R}$ triple of the vector fields

$$e_5 = (0, 0, C_5, D_5), \qquad e_6 = (0, 0, 1, 0), \qquad e_7 = (0, 0, 0, 1)$$

with constant coefficients there are two non-trivial fields Z, iZ. The presence of such pair of the fields tangential to the hypersurface M indicates its Levi degeneracy.

The contradictions arising in each of the considered cases and subcases complete the proof. □

This proposition is a completing part in the proof of Theorem 2.1 and the first part of the paper.

Proceeding to the second part, we note that the main point of the above arguing is the presence of Abelian subalgebras (ideals) of dimension at least 4 of considered 7-dimensional Lie algebras. At the same time, at least some of the considered algebras contain more than one Abelian subalgebra of the maximal possible dimension.

For instance, the algebra $N_{6,1,2} \oplus \mathfrak{g}_1$ (see Table 3.2), apart of the written 4-dimensional ideal $I_4 =< e_4, e_5, e_6, e_7 >$, possesses extra three similar Abelian ideals:

$$I_4' =< e_1, e_5, e_6, e_7 >, \qquad I_4'' =< e_2, e_4, e_6, e_7 >, \qquad I_4''' =< e_3, e_5, e_6, e_7 > .$$

Such property of 7-dimensional algebras is also informative in the problem on describing their orbits. We shall employ this in the concluding sections of the paper.

6. Non-decomposable algebras with three 4-dimensional ideals

Below we discuss 7-dimensional Lie algebras having only 4-dimensional Abelian ideals but the number of such ideals in each considered algebra is supposed to be at least three. Among 149 non-decomposable nilpotent Lie algebras, 4 algebras (17, 157, 147A, 37D_1 in the indexing of work [14]) have more than three Abelian ideals and 8 algebras

$$247D, \quad 247E, \quad 247Q, \quad 247R, \quad 147D, \quad 137D, \quad 1357A, \quad 1357D \tag{6.1}$$

possess exactly three such ideals.

It turns out that opposite to the first part of the paper, 4 of these 12 Lie algebras have Levi non-degenerate orbits in the space $\mathbb{C}^4$. An intermediate but important result on the description of such surfaces is the following statement.

Theorem 6.1. *Let a real non-degenerate hypersurface $M \subset \mathbb{C}^4$ be an orbit of 7-dimensional algebra of holomorphic vector fields. If this algebra does not contain 5-dimensional Abelian subalgebras but contains three 4-dimensional Abelian subalgebras, then M is holomorphically equivalent to a tubular hypersurface, the equation of which depends only on the imaginary parts of four complex variables.*

A complete description of non-degenerate orbits of considered 12 algebras is provided by the next theorem.

Theorem 6.2. *1) Up to a holomorphic equivalence, Levi non-degenerate orbits in $\mathbb{C}^4$ of 7-dimensional Heisenberg algebra 17 containing eight different 4-dimensional Abelian ideals are the spherical surfaces (quadrics):*

$$\operatorname{Im} z_4 = |z_1|^2 + |z_2|^2 \pm |z_3|^2; \tag{6.2}$$

2) three algebras 157, $147A$, $37D_1$, *each having four different Abelian ideals, do not admit Levi non-degenerate* 7*-dimensional orbits in* $\mathbb{C}^4$*;*

3) five algebras $247D$, $247E$, $247Q$, $247R$, $147D$ *containing three different Abelian ideals also do not admit non-degenerate* 7*-dimensional orbits in the space* $\mathbb{C}^4$*;*

4) up to a holomorphic equivalence, non-degenerate orbits of two algebras $137D$, $1357A$ *are only non-spherical surfaces*

$$\operatorname{Im} z_4 = z_1\bar{z}_3 + z_3\bar{z}_1 + |z_2|^2 \pm |z_1|^4; \tag{6.3}$$

5) all non-degenerate orbits of the algebra $1357D$ *are holomorphically equivalent to an indefinite quadrics*

$$\operatorname{Im} z_4 = |z_1|^2 + |z_2|^2 - |z_3|^2. \tag{6.4}$$

The remaining part of the paper is devoted to the proof of Theorem 6.2. We split all 12 discussed algebras into separated blocks; the properties of the algebras in the blocks we are interesting in are formulated and proved in Propositions 6.1, 6.2, 7.1, 7.2, 8.1, 8.2, 9.1, 9.2. Theorem 6.1 will be obtained as a corollary of one of these propositions.

Before we proceed to these statements, we first write commutation relations for all 12 discussed algebras. For the first quadruple of the algebras 17, 157, $147A$, $37D_1$, in some bases they read as:

$$\begin{aligned}
&17\ : \quad [e_1,e_2]=[e_3,e_4]=[e_5,e_6]=e_7\\
&157\ : \quad [e_1,e_2]=e_3,\ [e_1,e_3]=e_7,\ [e_2,e_4]=e_7,\ [e_5,e_6]=e_7;\\
&147A\ : \ [e_1,e_2]=e_4,\ [e_1,e_3]=e_5,\ [e_1,e_6]=e_7,\ [e_2,e_5]=e_7,\ [e_3,e_4]=e_7;\\
&37D1\ : \ [e_1,e_2]=e_5,\ [e_1,e_3]=e_6,\ [e_1,e_4]=e_7,\ [e_2,e_3]=-e_7,\\
&\qquad\quad [e_2,e_4]=e_6,\ [e_3,e_4]=-e_5.
\end{aligned} \tag{6.5}$$

The commutation relations for eight algebras (6.1) are

$$\begin{aligned}
&247D\ : \ [e_1,e_k]=e_{k+2},\ k=2,3;\ [e_1,e_4]=e_6,\ [e_2,e_5]=e_7,\ [e_3,e_4]=e_7;\\
&247E\ : \ [e_1,e_k]=e_{k+2},\ k=2,3,4;\ [e_1,e_5]=e_6,\ [e_2,e_5]=e_7,\ [e_3,e_4]=e_7;\\
&247Q\ : \ [e_1,e_k]=e_{k+2},\ k=2,3,4;\ [e_2,e_3]=e_6,\ [e_2,e_5]=e_7;\ [e_3,e_4]=e_7;\\
&247R\ : \ [e_1,e_k]=e_{k+2},\ k=2,3,4;\ [e_1,e_5]=e_6,\ [e_2,e_3]=e_6,\\
&\qquad\quad [e_2,e_5]=e_7;\ [e_3,e_4]=e_7;
\end{aligned} \tag{6.6}$$

$$\begin{aligned}
&147D\ : \ [e_1,e_2]=e_4,\ [e_1,e_3]=-e_6,\ [e_1,e_5]=e_7,\ [e_1,e_6]=e_7,\ [e_2,e_3]=e_5,\\
&\qquad\quad [e_2,e_6]=e_7,\ [e_3,e_4]=-2e_7;\\
&137D\ : \ [e_1,e_2]=e_5,\ [e_1,e_4]=e_6,\ [e_1,e_6]=e_7,\ [e_2,e_3]=e_6,\\
&\qquad\quad [e_2,e_4]=e_7,\ [e_3,e_5]=-e_7;\\
&1357A\ : \ [e_1,e_2]=e_4,\ [e_1,e_4]=e_5,\ [e_1,e_5]=e_7,\ [e_2,e_3]=e_5,\\
&\qquad\quad [e_2,e_6]=e_7,\ [e_3,e_4]=-e_7;\\
&1357D\ : \ [e_1,e_2]=e_3,\ [e_1,e_6]=e_7,\ [e_2,e_k]=e_{k+2},\ k=3,4;\\
&\qquad\quad [e_2,e_5]=e_7,\ [e_3,e_4]=e_7.
\end{aligned} \tag{6.7}$$

We shall write out interesting for us 4-dimensional ideals of the considered algebras as needed. We just note that it is natural to employ computer programs [16] for finding out such Abelian subalgebras and ideals or ones of other dimensions in a long list in [14] of 149 algebras as well as for determining the number of the ideals in each such algebra. For the discussed here 12 algebras, all statement on the ideals can be easily checked straightforwardly. For instance, in the algebra $247D$ with the above written commutation relations there are following Abelian ideals:

$$I_4' =< e_3, e_5, e_6, e_7 >, \qquad I_4'' =< e_4, e_5, e_6, e_7 >, \qquad I_4''' =< e_4, e_2, e_6, e_7 > . \qquad (6.8)$$

We observe one feature of the situation with three Abelian ideals for the considered 12 algebras useful for further discussions. It is convenient to formulate this feature in terms of non-ordered sets of symbols in 7-valued alphabet $\{1, 2, 3, 4, 5, 6, 7\}$ with a fixed number of the elements in the set.

In order to do this, we encode a 4-dimensional linear ideal being a linear hull of the elements $\{e_j, e_k, e_l, e_m\}$ in a 7-dimensional algebra by means a non-ordered set (j, k, l, m) of length 4. Then we can measure a *distance* (of Hamming type) between such codings as the number of symbols differing one coding from the other.

For instance, ideals (6.8) in the algeba $247D$ are associated with the sets

$$J_1 = (3, 5, 6, 7), \qquad J_2 = (4, 5, 6, 7), \qquad J_3 = (4, 2, 6, 7). \qquad (6.9)$$

At the same time, the distances from the set J_2 both to J_1 and J_3 is equal 1, while J_1 and J_3 are separated by two units. We shall say in this case that an *isosceles triangle with sides (1,1,2)* in the set of the discussed codings of lengths 4 corresponds to the ideals I_4', I_4'', I_4''' in the algebra $247D$.

We also note that the algebra $37D_1$ possesses 4 Abelian ideals with codings

$$J_1 = (1, 5, 6, 7), \qquad J_2 = (2, 5, 6, 7), \qquad J_3 = (3, 5, 6, 7), \qquad J_4 = (4, 5, 6, 7).$$

Each two of them can be separated by the unit distance in the set of codings of length 4 and this differs the algebra $37D_1$ from the algebra $247D$. It turns out that among 12 discussed algebras only the algebra $37D_1$ turns to be exceptional in the sense of the structure of 4-dimensional ideals.

Proposition 6.1. *Among* 4*-dimensional Abelian ideals for each of* 12 *algebras* (6.1), (6.5), *except the algebra* $37D_1$*, there exist three ideals forming* $(1, 1, 2)$*-triangle in the set of codings of length* 4.

Proof. The proof of this statement can be made by a simple enumerations; the provided discussions of the ideals in the algebras $247D$ and $37D_1$ serve as illustrations. □

We consider separately the algebra $37D_1$ selected in Proposition 6.1 and after that we proceed to considering other 11 algebras in set (6.1), (6.5).

Proposition 6.2. *A realization of algebra* $37D_1$ *as an algebra of holomorphic vector fields on a Levi non-degenerate hypersurface* $\mathbb{C}^4$ *is impossible.*

Proof. Supposing the existence of Levi non-degenerate orbit of such algebra and using Lemma 4.1, we discuss separately three cases, which arise while trying to simplify the basis e_4, e_5, e_6, e_7 of the Abelian ideal I_4'' in this algebra.

In the first case we assume that

$$e_4 = (1, 0, 0, 0), \qquad e_5 = (0, 1, 0, 0), \qquad e_6 = (0, 0, 1, 0), \qquad e_7 = (0, 0, 0, 1). \qquad (6.10)$$

Commutation relations (6.5) for each of three remaining basis fields of a 7-dimensional algebra $37D_1$ with flattened fields e_4, e_5, e_6, e_7 allow simplify essentially the form of the fields e_1, e_2, e_3. For instnace, by the only non-trivial relation $[e_1, e_4] = e_7$ of such type involving the field e_1 we get:

$$e_1 = (A_1, B_1, C_1, -z_1 + D_1),$$

where A_1, B_1, C_1, D_1 are some complex constants.

In the same way we can get simplified representations for the fields e_2, e_3:

$$e_2 = (A_2, B_2, -z_1 + C_2, D_2), \qquad e_3 = (A_3, z_2 + B_3, C_3, D_3).$$

Remark 6.1. *If a Lie algebra of vector fields in* $\mathbb{C}^4$ *contains two fields of form*

$$e_1 = (a_1(z), B_1, c_1(z), d_1(z)),$$
$$e_2 = (a_2(z), B_2, c_2(z), d_2(z)),$$

where B_1, B_2 *are some constants, then the second component of the commutator* $[e_1, e_2]$ *is zero.*

Taking into consideration this remark and the relation

$$[e_1, e_2] = e_5 = (0, 1, 0, 0),$$

which holds in the algebra $37D_1$, we arrive at a contradiction.

In the second case of Lemma 4.1, only three fields e_5, e_6, e_7 in the same ideal are flattened to condition (6.10), and $e_4 = (0, b_4(z_1), c_4(z_1), d_4(z_1))$. We also employ the presence of one more 4-dimensional ideal $I'_4 =< e_3, e_5, e_6, e_7 >$ in the discussed algebra. The components of the field e_3 commuting with other three basis fields I'_4 can depend only on the variable z_1, that is, the entire field can be represented as

$$e_3 = (a_1(z_1), b_1(z_1), c_1(z_1), d_1(z_1)).$$

If at the same time $a_1(z_1) \equiv 0$, then $[e_3, e_4] = 0$ and this contradicts to the relation $[e_3, e_4] = -e_5$ in this algebra. Therefore, $a_1(z_1) \neq 0$ (possibly, at a shifted point of the surface) and then entire field e_3 can be flattened to the form $e_3 = (1, 0, 0, 0)$ using the technique of work [10] (see also [7]).

Considering now the commutators of the fields e_1, e_2 with the flattened basis of the ideal I'_4, similarly to the Case 1 we can simplify these fields to the form

$$e_1 = (A_1, B_1, z_1 + C_1, D_1), \qquad e_2 = (A_2, B_2, C_2, z_1 + D_2).$$

By the remark used above, the second component of the commutator $[e_1, e_2]$ is zero and this contradicts to the relation $[e_1, e_2] = e_5$.

Finally, in the third case, it is important for us that $e_5 = (0, 0, c_5(z_1), d_5(z_1))$ in the coordinates obtained under the holomorphic transformation and only the pair of the fields e_6, e_7 is of the same form as in (6.10). Since these two fields belong to the center $Z =< e_5, e_6, e_7 >$ of the algebra $37D_1$, the components of all fields in this algebra can depend only on the variables z_1, z_2.

Calculating in this case the commutators of the fields e_1, e_2, e_3 with the field e_5, we obtain:

$$[e_1, e_5] = a_1(z_1, z_2)(0, 0, c'_5(z_1), d'_5(z_1)) = 0,$$
$$[e_2, e_5] = a_2(z_1, z_2)(0, 0, c'_5(z_1), d'_5(z_1)) = 0,$$
$$[e_3, e_5] = a_3(z_1, z_2)(0, 0, c'_5(z_1), d'_5(z_1)) = 0.$$

Formally speaking, three such identities are possible only in two cases:

a) $c_5(z_1) = Const, \qquad d_5(z_1) = Const$;

b) $a_1(z) = a_2(z) = a_3(z) = 0$.

But for the basis vector fields of 7-dimensional algebra tangential to a non-degenerate hypersurface $M \subset \mathbb{C}^4$, none of these cases is possible, see the proofs of Lemma 4.3 and Proposition 5.1.

Thus, under the assumption on the existence of a non-degenerate orbit of the algebra $37D_1$ all three possible situations of the simplification of the ideal $I =< e_4, e_5, e_6, e_7 >$ of this algebra lead to contradictions. □

In order to describe the orbits of remaining 11 Lie algebras in the set (6.1), (6.5) and to prove Theorems 6.1 and 6.2, we shall employ the property of mutual location of their 4-dimensional Abelian ideals stated in Proposition 6.1.

7. Reduction to tubes and degeneracies of orbits for 8 algebras

Proposition 7.1. *Let a 7-dimensional algebra of holomorphic vector fields on a real non-degenerate hypersurface $M \subset \mathbb{C}^4$ contains no 5-dimensional Abelian subalgebras but has three Abelian 4-dimensional subalgebras forming $(1,1,2)$-triangle in the set of their codings of length 4. Then it is possible to flatten a quadruple of basis field in one of the Abelian subalgebras by a holomorphic change of variables.*

Proof. Let g be a 7-dimensional algebra of holomorphic vector fields in $\mathbb{C}^4$ satisfying the assumptions of Proposition 7.1. For the sake of convenience, we redenote the basis fields of the algebra g and its 4-dimensional Abelian subalgebras I', I'', I''' in accordance with formula (6.9), that is,

$$I' =< e_3, e_5, e_6, e_7 >, \quad I'' =< e_4, e_5, e_6, e_7 >, \quad I''' =< e_2, e_4, e_6, e_7 > .$$

Remark 7.1. *Since the Lie algebra g contains no 5-dimensional Abelian subalgebras, then none of the two commutators $[e_3, e_4]$, $[e_2, e_5]$ can vanish.*

Then we apply Lemma 4.1 to the basis of the ideal I''. If it is possible to flatten completely this basis (Case 1), then the proposition is true.

Let the set of fields (e_4, e_5, e_6, e_7) be in the second case of Lemma 4.1. Then we consider the field e_3 and the Abelian ideal I'. Since all three its fields e_5, e_6, e_7 are flattened, the components of the commuting with it field $e_3 = (a_3, b_3, c_3, d_3)$ can depend only on the variable z_1.

If at the same time $a_3(z_1) \neq 0$, then the entire field

$$e_3 = (a_3(z_1), b_3(z_1), c_3(z_1), d_3(z_1)) \tag{7.1}$$

can be reduced, as in the proof of Lemma 4.3, by a holomorphic change of variables to the form $e_3 = \partial/\partial z_1$ with simultaneous keeping of other basis fields e_5, e_6, e_7 of subalgebra I'. This also proves the proposition.

The situation $a_3(z_1) \equiv 0$ in this case is impossible. Indeed, for two fields e_3 and e_4 depending only on the variable z_1 and having identically vanishing components a_3, a_4, their commutator $[e_3, e_4]$ should also vanish. But this contradicts to the remark made in the beginning of the proof.

Finally, in the third case of Lemma 4.1, the fields e_4, e_5, e_6, e_7 read as

$$\begin{aligned} e_4 &= (0, 1, \quad 0, \quad 0), \\ e_5 &= (0, 0, \quad c_5(z_1), \quad d_5(z_1)), \\ e_6 &= (0, 0, \quad 1, \quad 0), \\ e_7 &= (0, 0, \quad 0, \quad 1). \end{aligned}$$

Recalling the third Abelian subalgebra I''', we obtain a simplified form of the field e_2, the components of which, similar to (7.1), turn out to be depending at most on z_1. At the same time, similar to the arguing in Case 2, the component $a_2(z_1)$ can not vanish identically since then $[e_2, e_5] = 0$.

The field $e_2 = (a_2(z_1), b_2(z_1), c_2(z_1), d_2(z_1))$ with a non-zero component $a_2(z_1)$ can be flattened to the form $e_2 = (1, 0, 0, 0)$ keeping the fields e_4, e_6, e_7 flattened. The proposition turns out to be true also in this case. □

Remark 7.2. *Together with Proposition 7.1, Theorem 6.1 is also proved since the flattening of a quadruple of independent fields on the hypersurface M means the tubularity property for it.*

This property plays an important role in general descriptions of holomorphically homogeneous hypersurfaces in the complex spaces of arbitrary dimensions similarly to the studied cases with the homogeneous property in $\mathbb{C}^2$ and $\mathbb{C}^3$. However, for an informative illustration of Theorem 6.1

and for completing the proof of Theorem 6.2 we also need to reduce the number of the discussed Lie algebras.

Proposition 7.2. *Seven Lie algebras* 157, 147A, 247D, 247E, 247Q, 247R, 147D, *the* 4*-dimensional ideal of which form* $(1,1,2)$*-triangles in set of codings, have no non-degenerate orbits in* $\mathbb{C}^4$.

Proof. Proposition 7.1 allows us to reduce the consideration of each algebra, three Abelian ideals of which has structure $(1,1,2)$-triangle to three rather simple checkings. As an example, we consider them for the algebra 247D.

Supposing that this algebra has a Levi non-degenerate orbit, we consider separately three cases of flattening basis fields for each of three Abelian ideals in the algebra. We first flatten the fields

$$\begin{aligned} e_4 &= (1,0,0,0),\\ e_5 &= (0,1,0,0),\\ e_6 &= (0,0,1,0),\\ e_7 &= (0,0,0,1), \end{aligned}$$

corresponding to the set J_2 in (6.9).

Commutation relations (6.6) for each of three remaining basis fields of the 7-dimensional algebra g with flattened fields e_4, e_5, e_6, e_7 allow us to simplify essentially the form of the fields e_1, e_2, e_3. For instance, by the only nontrivial relation $[e_1, e_4] = e_6$ of such type involving the field e_1 we get

$$e_1 = (A_1, B_1, -z_1 + C_1, D_1),$$

where A_1, B_1, C_1, D_1 are some complex constants.

Similarly, for the fields e_2, e_3 we obtain simplified representations of the form

$$e_2 = (A_2, B_2, C_2, -z_2 + D_2), \qquad e_3 = (A_3, B_3, C_3, -z_1 + D_3).$$

For such fields e_1, e_2 the first component of its commutator vanishes and this contradicts to the relation $[e_1, e_2] = e_4 = (1,0,0,0)$ in the algebra 247D.

Similar contradictions are obtained also under the flattening of the bases in the ideals I' and I''' in the algebra 247D. For the ideal I', the flattening of its basis fields e_3, e_5, e_6, e_7 and consideration of these fields with the remaining triple e_1, e_2, e_4 of the basis fields in the algebra 247D give rise to the formulae

$$e_1 = (A_1, -z_1 + B_1, C_1, D_1), \qquad e_4 = (A_4, B_4, C_4, z_1 + D_4). \tag{7.2}$$

Applying the above used remark to the commutator $[e_1, e_4] = e_6 = (0,0,1,0)$ and to the third components of the fields (7.2), we obtain a contradiction in this case, too.

Finally, after flattening of the basis fields,

$$\begin{aligned} e_2 &= (1,0,0,0),\\ e_4 &= (0,1,0,0),\\ e_6 &= (0,0,1,0),\\ e_7 &= (0,0,0,1), \end{aligned}$$

in the ideal I''' we obtain a similar representation for

$$\begin{aligned} e_1 &= (A_1, -z_1 + B_1, -z_2 + C_1, D_1),\\ e_3 &= (A_3, B_3, C_3, -z_2 + D_3),\\ e_5 &= (A_5, B_5, C_5, z_1 + D_5). \end{aligned} \tag{7.3}$$

Considering then the commutation relations $[e_1, e_5] = 0$ and $[e_3, e_5] = 0$, we obtain:

$$A_1(0,0,0,1) - (A_5(0,-1,0,0) + B_5(0,0,-1,0)) = 0,$$
$$A_3(0,0,0,1) - B_5(0,0,0,-1) = 0.$$

The vanishing of two separate components of two commutators $[e_1, e_5]$ and $[e_3, e_5]$ we obtain easily the identities $A_1 = A_3 = A_5 = 0$, meaning that six basis fields of the algebra $247D$ possess in this case identically zero first components. This situation is incompatible with the Levi non-degeneracy of 7-dimensional surface, on which the algebra of tangential fields $247D$ is defined.

In the same way we establish that the assumption on the existence of non-degenerate orbits *for all three algebras in Item 2* of Theorem 6.2 and of remaining *four algebras in Item 3* is contradictory. □

8. Spherical orbits of algebras 17 and 1357D

In this section we show that the algebras 17 and $1357D$ have only spherical 7-dimensional orbits in the space $\mathbb{C}^4$.

Proposition 8.1. *Levi non-degenerate orbits of the algebra* 17 *with the relations* $[e_1, e_2] = [e_3, e_4] = [e_5, e_6] = e_7$ *are only spherical surfaces (quadrics)*

$$\mathrm{Im}\, z_4 = |z_1|^2 + |z_2|^2 \pm |z_3|^2$$

up to a holomorphic equivalence.

Proof. For a complete consideration it is sufficient to consider only three of eight 4-dimensional Abelian ideals, for instance,

$$I'_4 =< e_2, e_4, e_5, e_7 >, \qquad I''_4 =< e_2, e_4, e_6, e_7 >, \qquad I'''_4 =< e_2, e_3, e_6, e_7 > . \tag{8.1}$$

By Proposition 7.1 it is sufficient to describe all non-degenerate surfaces being the orbits of the discussed algebra under the flattening of each of these ideals separately. We also note that a symmetry of relations (6.5) allows us to discuss only one such ideal.

Indeed, the passage from I'_4 to I''_4 or from I''_4 to I'''_4 means just a re-indexation of the pair of the fields involved in the only commutation relation of the algebra 17 containing this pair. It does not change qualitatively a final picture with the orbits of this algebra.

Thus, we consider the ideal $I''_4 =< e_2, e_4, e_6, e_7 >$ assuming that it is flattened after some holomorphic change of the coordinates. Commutations relations (6.5) for each of three remaining basis fields in the 7-dimensional algebra g with flattened fields e_2, e_4, e_6, e_7 allow us to simplify essentially the form of the fields e_1, e_3, e_5. By these relations, all of them turn out to be of a same structure:

$$\begin{aligned} e_1 &= (A_1, B_1, C_1, -z_1 + D_1), \\ e_3 &= (A_3, B_3, C_3, -z_2 + D_3), \\ e_5 &= (A_5, B_5, C_5, -z_3 + D_5), \end{aligned} \tag{8.2}$$

where A_k, B_k, C_k, D_k, $k = 1, 3, 5$, are some complex constants.

By the shifts of the variables it is easy to remove the constants D_k in the last components of these three fields. And considering instead of the obtained fields their combinations with the quadruple of flattened fields, we can suppose that all constants A_k, B_k, C_k are pure imaginary. We introduce the notations

$$A_k = ia_k, \qquad B_k = ib_k, \qquad C_k = ic_k, \qquad a_k, b_k, c_k \in \mathbb{R},$$

and consider one more matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_5 & b_5 & c_5 \end{pmatrix}. \tag{8.3}$$

By the complete rank condition of the discussed realization of the algebra 17 this matrix is non-degenerate.

Remark 8.1. *We can obtain additional restrictions for the entries of this matrix taking into consideration extra three relations* $[e_1, e_3] = [e_3, e_5] = [e_1, e_5] = 0$ *not used yet. But in fact we do not need such restrictions.*

We proceed from fields (8.2) to a system of partial differential equations $\mathrm{Re}\,(e_k(\Phi)_{|M}) = 0$, $k = 1, 3, 5$, describing the basis of the tubular orbit of the algebra 17. In the matrix form this system can be written as

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_5 & b_5 & c_5 \end{pmatrix} \cdot \begin{pmatrix} \partial F/\partial y_1 \\ \partial F/\partial y_2 \\ \partial F/\partial y_3 \end{pmatrix} = \begin{pmatrix} -y_1 \\ -y_2 \\ -y_3 \end{pmatrix}.$$

The non-degeneracy of the matrix (8.3) allows us to rewrite the latter equation in a form resolved with respect to the partial derivates of the function $F(y_1, y_2, y_3)$:

$$\frac{\partial F}{\partial y_1} = l_1(y_1, y_2, y_3), \qquad \frac{\partial F}{\partial y_2} = l_2(y_1, y_2, y_3), \qquad \frac{\partial F}{\partial y_3} = l_3(y_1, y_2, y_3)$$

with some linear functions $l_k(y_1, y_2, y_3)$.

It is clear that the solutions of such system of equations (under the validity of the matching conditions) are some quadratic forms

$$F = Q(y_1, y_2, y_3).$$

Returning back to the complex variables in the space $\mathbb{C}^4$, we necessarily obtain the equations of possible orbits of the discussed realization of Heisenberg algebra in the form $\mathrm{Im}\, z_4 = Q(\mathrm{Im}\, z_1, \mathrm{Im}\, z_2, \mathrm{Im}\, z_3)$. By removing holomorphic and anti-holomorphic terms in the right hand side of the latter equation reduces it to the form

$$\mathrm{Im}\, z_4 = H(z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3)$$

with some Hermitian form in the right hand side. Interesting only in non-degenerate orbits and using linear transformations of complex variables, we get only two possibilities for the orbits of the Heisenberg algebra under realization associated with flattening the ideal I_4'':

$$\mathrm{Im}\, z_4 = |z_1|^2 + |z_2|^2 \pm |z_3|^2.$$

It remains to note that both these spherical surfaces are known homogeneous surfaces and their homogeneous property is realized by simple linear shifts along the first triple of the variables and square, with respect to the shift parameters, transformation of the variable z_4. The Lie algebra corresponding to such group of transformations is exactly the Heisenberg algebra.

Completing the proof of Proposition 8.1, we once again recall on the symmetricity of this algebra and on the literal reproducing of the obtained conclusions under the flattening of each two other ideals I_4', I_4'''. □

Remark 8.2. *After such consideration of the algebra* 17*, it is easy to show the absence of non-degenerate* 7*-dimensional orbits in* $\mathbb{C}^4$ *of the algebra* 157 *defined by the relations*

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = [e_2, e_4] = [e_5, e_6] = e_7.$$

Indeed, after the permutation $e_2 \leftrightarrow e_3$, three of four commutation relations in the last algebra coincide with formulae (6.5) for the algebra 17, while the additional fourth relation is rewritten as $[e_1, e_3] = e_2$. Abelian ideals (8.1) are also present in the algebra 157. The flattening of each of them leads to similar form (8.2) of the fields complement to these ideals. As it is easy to confirm, the fourth relation for these fields leads to contradictions in all cases.

Proposition 8.2. *Each Levi non-degenerate orbit of the algebra* $1357D$ *in* $\mathbb{C}^4$ *is holomorphically equivalent to an indefinite quadrics*

$$\operatorname{Im} z_4 = |z_1|^2 + |z_2|^2 - |z_3|^2.$$

Proof. For the algebra $1357D$ from Item 5 in Theorem 6.2 the flattening of the bases of two ideals $I'_4 =< e_3, e_5, e_6, e_7 >$ and $I''_4 =< e_4, e_5, e_6, e_7 >$ leads to contradictions after applying the remark on the commutator of the fields containing the constants in a fixed component. But for the ideal $I'''_4 =< e_1, e_3, e_5, e_7 >$ we obtain a quadruple of flattened fields

$$e_1 = \frac{\partial}{\partial z_1}, \qquad e_3 = \frac{\partial}{\partial z_2}, \qquad e_5 = \frac{\partial}{\partial z_3}, \qquad e_7 = \frac{\partial}{\partial z_4} \tag{8.4}$$

and, after consideration of extra 12 commutators of the flattened fields with the remaining triple, we also get simplified representations

$$\begin{aligned} e_2 &= (A_2, z_1 + B_2, -z_2 + C_2, -z_3 + D_2), \\ e_4 &= (A_4, B_4, C_4, z_2 + D_4), \\ e_6 &= (A_6, B_6, C_6, z_1 + D_6). \end{aligned} \tag{8.5}$$

At this stage it remains to check three commutation relations:

$$[e_2, e_4] = e_6, \qquad [e_2, e_6] = [e_4, e_6] = 0.$$

By these relations we have (after the shifts of each of three complex variables z_1, z_2, z_3) the following specifications of the form of the triple of fields (8.5):

$$\begin{aligned} e_2 &= (A_2, z_1 + B_2, -z_2, -z_3), \\ e_4 &= (0, -A_2, B_2, z_2 + D_4), \\ e_6 &= (0, 0, -A_2, z_1). \end{aligned} \tag{8.6}$$

Then we need to integrate the algebra of vector fields $1357D$ with simplified basis (8.4), (8.6). As it has been mentioned above, the presence of the quadruple of fields (8.4) indicated a tubular structure of the discussed hypersurface M. This is why we can suppose that it is defined by the equation $\Phi(y_1, y_2, y_3, y_4) = 0$ or, in view of the non-degeneracy of M we are interesting in, it can be written as resolved with respect to one of the variables

$$y_4 = F(y_1, y_2, y_3). \tag{8.7}$$

In this case we have to integrate the system of three partial differential equations corresponding to fields (8.6). We note that by considering, instead of the fields e_2, e_4, e_6, the flattening of their combinations with the fields in the flattened quadruple, we can suppose that the constants A_2, B_2, D_4 are pure imaginary. Moreover, for zero A_2 we get a Levi degenerate situation with six zeroes in the first components of the basis fields of the algebras and this is we assume that $A_2 \neq 0$.

For a practical integration we employ a formal writing expressing that an arbitrary field e_k is tangential to the studied surface M as the equation

$$\operatorname{Re}\left(e_k(\Phi)_{|M}\right) \equiv 0, \qquad k = 1, \dots, 7.$$

In the variables y_1, y_2, y_3, in view of simplifying notations $A_2 = ia$, $B_2 = ib$, $D_2 = id$ we then obtain the following system of equations:

$$a\frac{\partial F}{\partial y_1} + (y_1 + b)\frac{\partial F}{\partial y_2} + y_2\frac{\partial F}{\partial y_3} = y_3, \qquad -a\frac{\partial F}{\partial y_2} + b\frac{\partial F}{\partial y_3} = y_2 + d, \qquad -a\frac{\partial F}{\partial y_3} = y_1. \tag{8.8}$$

Starting with the simplest equation and successively solving others in this system, in view of the notation $\lambda = 1/a$ we arrive at the formulae:

$$F = \lambda y_1 y_3 + G(y_1, y_2), \qquad \frac{\partial G}{\partial y_2} = \lambda^2 b y_1 - \lambda(y_2 + d), \qquad G = \lambda^2 b y_1 y_2 - \frac{1}{2}\lambda(y_2 + d)^2 + H(y_1).$$

For the function $H(y_1)$ we get the equation

$$H' = -\lambda^3 b y_1^2 + m y_1 + n$$

with some coefficients m, n in the right hand side. Then a final formula describing all solutions of system (8.8) reads as

$$y_4 = \lambda y_1 y_3 + \lambda^2 b y_1 y_2 - \frac{1}{2}\lambda(y_2 + d)^2 - \frac{1}{3}\lambda^3 b y_1^3 + P_2(y_1), \tag{8.9}$$

where $P_2(y_1)$ is some second order polynomial.

We note that under the condition $b = 0$ equation (8.9) contains no third order terms and this is why it describes, up to affine transformations of the variables, an indefinite quadrics

$$v = y_1 y_3 + y_2^2 \tag{8.10}$$

in the space $\mathbb{R}^4_y$.

In the case $b \neq 0$, by means of affine transformations

$$y_1^* = y_1 + \alpha, \qquad y_2{*} =, \; y_3^* = y_3 + \lambda b y_2 + m y_1 + n, \qquad y_4^* = \frac{1}{\lambda} y_4 + Q y_2 + M y_1 + N,$$

with appropriate coefficients this equation is transformed to the form

$$y_4 = y_1 y_3 + y_2^2 + y_1^3. \tag{8.11}$$

However, returning back to complex variables and using at most square transformations, it is easy to reduce both equations (8.10) and (8.11) (see Appendix) to a common form

$$\operatorname{Im} z_4 = |z_1|^2 + |z_2|^2 - |z_3|^2$$

of an indefinite quadrics in the space $\mathbb{C}^4$. □

9. Non-spherical orbits of algebras $137D$ and $1357A$

Let us discuss now *most interesting algebras* $137D$ *and* $1357A$ *in Item 4 of Theorem 6.2.*

We begin with writing 4-dimensional Abelian ideals in these algebras. In view of descriptions (6.7), we confirm easily that such ideals corresponding to coding $(1, 1, 2)$-triangles are

$$137D \;:\; I_4' =< e_4, e_5, e_6, e_7 >, \quad I_4'' =< e_2, e_5, e_6, e_7 >, \quad I_4''' =< e_3, e_4, e_6, e_7 >,$$
$$1357A \;:\; I_4' =< e_4, e_5, e_6, e_7 >, \quad I_4'' =< e_3, e_5, e_6, e_7 >, \quad I_4''' =< e_2, e_4, e_5, e_7 > .$$

At the same time, two first ideals I_4', I_4'' for each of these algebras can be flattened under the existence of at least one non-degenerate orbit of these algebras. For instance, by the scheme described above, for the basis fields of the algebra $137D$, complement to the ideal I_4', we obtain:

$$e_1 = (A_1, B_1, -z_1 + C_1, D_1), \qquad e_2 = (A_2, B_2, C_2, -z_1 + D_2).$$

But then their commutator should obey the identity $[e_1, e_2] = e_5 = (0, 1, 0, 0)$ being inconsistent with the facts that the second components of these fields are constants.

Under the flattening of the ideal I_4'' in the same algebra we have:

$$e_1 = (A_1, -z_1 + B_1, C_1, -z_3 + D_1), \qquad e_4 = (A_4, B_4, C_4, z_1 + D_4).$$

Such form of these fields contradicts to the condition $[e_1, e_4] = e_6 = (0, 0, 1, 0)$.

For the algebra $1357A$, under the flattening of the basis in I'_4, we have similar simplified representations:

$$e_1 = (A_1, B_1, C_1, -z_2 + D_1), \qquad e_4 = (A_4, B_4, C_4, -z_1 + D_4)$$

contradicting the relation $[e_1, e_4] = e_5 = (0, 1, 0, 0)$. And flattening the basis I''_4, we get: $e_1 = (A_1, -z_1 + B_1, C_1, D_1)$, $e_2 = (A_2, B_2, C_2, -z_3 + D_2)$, and this contradicts the relation $[e_1, e_2] = e_4 = (1, 0, 0, 0)$.

We proceed to flattening the ideal I'''_4 for the algebra $1357A$. Reproducing in this case a previous scheme of discussing all 21 commutation relations, we arrive at non-contradictory descriptions of the triples of the fields

$$\begin{aligned} e_1 &= (ia, -z_1, -z_2, -z_3), \\ e_3 &= (0, -ia, z_1, z_2 + id_3), \\ e_6 &= (0, 0, -ia, z_1 + id_6). \end{aligned} \tag{9.1}$$

Here a, d_3, d_6 are some real coefficients. We can suppose that $a \neq 0$ since as $a = 0$, the first components of the six basis fields of the algebra $1357A$ turn out to be zero and this is impossible for independent fields tangential to a non-degenerate hypersurface M.

Now let us consider the ideal I'''_4 for the algebra $137D$. The above described procedures of specifying the form of the triples of basis fields e_1, e_2, e_5 complement to I'''_4 give rise to the following non-contradictory formulae (as in (9.1), here $a \neq 0$):

$$\begin{aligned} e_1 &= (ia, ib, -z_2 + ic, -z_3), \\ e_2 &= (0, 2ia, -z_1, -z_2), \\ e_5 &= (0, 0, ia, -z_1 + id). \end{aligned} \tag{9.2}$$

The integration of the algebras $1357A$ and $137D$ with simplified bases is made by the scheme described in the previous section. In each of two cases it is reduced to solving a system of three partial differential equations.

We first discuss a system corresponding to fields (9.1) of the algebra $1357A$. Defining an integral surface of this algebra by the equation

$$y_4 = F(y_1, y_2, y_3), \tag{9.3}$$

we have three relations for the partial derivatives of the function F:

$$\begin{aligned} &a\frac{\partial F}{\partial y_1} - y_1\frac{\partial F}{\partial y_2} - y_2\frac{\partial F}{\partial y_3} = -y_3, \\ &-a\frac{\partial F}{\partial y_2} + y_1\frac{\partial F}{\partial y_3} = y_2 + d_3, \\ &a\frac{\partial F}{\partial y_3} = y_1 + d_6. \end{aligned} \tag{9.4}$$

Proposition 9.1. *As $a \neq 0$, each tubular hypersurface in the space $\mathbb{C}^4$ described by equations* (9.3)-(9.4) *is holomorphically equivalent to the surface*

$$v = (z_1\bar{z}_3 + z_3\bar{z}_1) + |z_2|^2 + |z_1|^4.$$

Proof. Solving equations in system (9.4) in order of increasing its complexity, we obtain (for instance, by means of computer-assisted calculations) the following formulae:

$$F = -\frac{1}{a}y_3(y_1 + d_6) + G(y_1, y_2), \qquad G(y_1, y_2) = -\frac{1}{2a}y_2^2 - \frac{1}{a^2}y_1y_2(y_1 + d_6) - \frac{d_3}{a}y_2 + H(y_1).$$

At the same time, for the function $H(y_1)$ we obtain the equation

$$aH'(y_1) = \frac{y_1}{a}(y_1^2 + d_6 y_1 + d_3 a).$$

Finally, the general solution to system (9.4) is described by the formula:

$$\begin{aligned} y_4 = & -\frac{1}{12a^3}(3y_1^4 + 4d_6 y_1^3) - \frac{1}{a^2} y_2 y_1^2 \\ & - \frac{y_1}{2a^2}(2\, d_6 y_2 + d_3 y_1 + 2\, y_3 a) - \frac{1}{2a}\left({y_2}^2 + 2d_3 y_2 + 2d_6 y_3\right). \end{aligned} \tag{9.5}$$

We temporarily complicate this equation selecting interesting for us terms and introducing some new, less essential coefficients:

$$\begin{aligned} y_4 = & -\frac{1}{4a^3}\left(y_1 + \frac{1}{3}d_6\right)^4 - \frac{1}{a^2}y_2\left(y_1 + \frac{1}{3}d_6\right)^2 - \frac{1}{a}\left(y_1 + \frac{1}{3}d_6\right)(y_3 + my_1 + ny_2) \\ & - \frac{1}{2a}y_2^2 + (\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3) + \beta. \end{aligned} \tag{9.6}$$

Then we employ an affine transformation of the coordinates:

$$y_1^* = y_1 + \frac{1}{3}d_6, \qquad y_3^* = \frac{1}{2a}(y_3 + my_1 + ny_2), \qquad y_4^* = y_4 - (\alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 + \beta). \tag{9.7}$$

Then the equations of the obtained orbits are simplified to

$$y_4 = -\frac{1}{4a^3}y_1^4 - \frac{1}{a^2}y_2 y_1^2 - y_1 y_3 - \frac{1}{2a}y_2^2. \tag{9.8}$$

Remark 9.1. *Tubular hypersurface* (9.8) *is holomorphically equivalent in the space* $\mathbb{C}^4$ *(see Appendix below) to a generalization of the Winkelmann surface:*

$$v = (z_1\bar{z}_3 + z_3\bar{z}_1) + |z_2|^2 + |z_1|^4. \tag{9.9}$$

This remark completes the proof. □

We proceed to similar considering the orbits of the algebra $137D$. In this case, the set of fields (9.2) produces the following system of partial differential equations for defining functions (9.3) of such orbits:

$$\begin{aligned} & a\frac{\partial F}{\partial y_1} + b\frac{\partial F}{\partial y_2} + (c - y_2)\frac{\partial F}{\partial y_3} = -y_3, \\ & 2a\frac{\partial F}{\partial y_2} - y_1\frac{\partial F}{\partial y_3} = -y_2, \\ & a\frac{\partial F}{\partial y_3} = d - y_1. \end{aligned} \tag{9.10}$$

Proposition 9.2. *As* $a \neq 0$, *each tubular hypersurface in the space* $\mathbb{C}^4$ *described by equations* (9.3), (9.10) *is holomorphically equivalent to the surface*

$$v = (z_1\bar{z}_3 + z_3\bar{z}_1) + |z_2|^2 - |z_1|^4.$$

Proof. As for system (9.4), the solution of system (9.10) can be obtained by a step-by-step study of separate equations. For the function $F = F(y_1, y_2, y_3)$ describing the orbits of the algebra $137D$ we have

$$F = -\frac{1}{a}(b + y_1)y_3 + G(y_1, y_2); \qquad G = -\frac{1}{4a^2}y_2^2 - \frac{1}{2a^2}(b + y_1)y_1 y_2 + H(y_1),$$

where $H(y_1)$ solves an ordinary differential equation

$$aH' = \frac{b{y_1}^2}{2a^2} + \frac{(2ac + b^2)\, y_1}{2a^2} + \frac{bc}{a}.$$

This means that a general solution to system (9.10) reads as

$$y_4 = -\frac{1}{a}(b+y_1)y_3 - \frac{1}{4a^2}y_2^2 - \frac{1}{2a^2}(b+y_1)y_1y_2 + \frac{b}{6a^3}y_1^3 + (\alpha y_1^2 + \beta y_1 + \gamma).$$

Simplifying this equation by means of affine transformations of the variables y_k similarly as this was done for the solutions of system (9.4), we easily reduce it to the form

$$y_4 = y_1y_3 - \frac{1}{4a^2}y_2^2 - \frac{1}{2a^2}y_1^2y_2 + \frac{b}{6a^3}y_1^3. \tag{9.11}$$

In its turn, such tubular algebraic third order surface in the space $\mathbb{C}^4$ is holomorphically equivalent (see Appendix) to a fourth order surface:

$$v = (z_1\bar{z}_3 + z_3\bar{z}_1) + |z_2|^2 - |z_1|^4. \tag{9.12}$$

The proof is complete. □

Proven Propositions 6.1, 6.2, 7.1, 7.2, 8.1, 8.2, 9.1, 9.2 allow us to regard the proof of Theorem 6.2 as completed.

We note that surfaces (9.9) and (9.12) are two holomorphically non-equivalent generalization of the known Winkelmann surface

$$v = (z_1\bar{z}_3 + z_3\bar{z}_1) + |z_1|^4$$

in the 3-dimensional complex space. As it was shown in [8], these surfaces are non-spherical and posses richest 13-dimensional algebras of symmetries among all homogeneous non-spherical hypersurfaces in $\mathbb{C}^4$. The considered subalgebras $1357A$ and $137D$ are 7-dimensional subalgebras of the complete algebras of symmetries for surfaces (9.9) and (9.12).

10. Appendix: transformations of algebraic tubes

In this section we provide calculations relating with three different fragments of the papers but having a common nature. We consider holomorphic transformations of three families of algebraic tubular hypersurfaces in the space $\mathbb{C}^4$ described by «similar» equations ($A, B, C, D \in \mathbb{R}$):

$$y_4 = y_1y_3 + Ay_2^2 + Cy_1^3, \tag{10.1}$$

$$y_4 = y_1y_3 + Ay_2^2 + By_1^2y_2 + Cy_1^3, \tag{10.2}$$

$$y_4 = y_1y_3 + Ay_2^2 + By_1^2y_2 + Dy_1^4. \tag{10.3}$$

Proposition 10.1. *As $A \neq 0$, independent of the value of the coefficient C, the following statements hold:*
1) surface (10.1) *is holomorphically equivalent to an indefinite quadrics*

$$\operatorname{Im} z_4 = |z_1|^2 + |z_2|^2 - |z_3|^2;$$

2) as $B \neq 0$, surface (10.2) *is equivalent to the generalization of the Winkelmann surface*

$$\operatorname{Im} z_4 = z_1\bar{z}_3 + z_3\bar{z}_1 + |z_2|^2 - |z_4|^2;$$

3) as $6AD - B^2 = 0$, surface (10.3) *is spherical, while as $6AD - B^2 \neq 0$, it is equivalent to one of two generalizations of the Winkelmann surface:*

$$\operatorname{Im} z_4 = z_1\bar{z}_3 + z_3\bar{z}_1 + |z_2|^2 \pm |z_4|^2.$$

Proof. We observe that we can discuss three equations (10.1)–(10.3) as a single general equation

$$\operatorname{Im} z_4 = x_1x_3 + Ax_2^2 + Bx_1^2x_2 + Cx_1^3 + Dx_1^4, \tag{10.4}$$

in which we have not passed yet from the imaginary parts of the complex variables z_1, z_2, z_3 to their real parts by means of the change $z_k \to iz_k^*$.

By dilatations of two variables $z_2 = Az_2^*$, $z_4 = Az_4^*$, the non-zero coefficient A in (10.4) becomes one, while the coefficients B, C, D are replaced by B/A, C/A, D/A, respectively.

We pass to complex variables by substituting the formulae $x_k = (z_k + \bar{z}_k)/2$ for $k = 1, 2, 3$ into (10.4). Opening then the brackets in the right hand of equation (10.4), we note that the arising sum $\varphi(z) + \overline{\varphi(z)}$ of holomorphic and antiholomorphic terms can be removed. This can be done by means of the change $w^* = w - 2i\varphi(z)$ followed by omitting the asterisk. Hence, instead of (10.4) we get

$$\begin{aligned}\operatorname{Im} z_4 =& \frac{1}{4}(z_1\bar{z}_3 + z_3\bar{z}_1) + \frac{1}{4}|z_2|^2 + \frac{B}{8A}(z_1^2\bar{z}_2 + 2z_1z_2\bar{z}_1 + z_2\bar{z}_1^2 + 2z_1\bar{z}_1\bar{z}_2)\\ &+ \frac{3C}{8A}(z_1^2\bar{z}_1 + z_1\bar{z}_1^2) + \frac{D}{16A}(4z_1^3\bar{z}_1 + 6|z_4|^2 + 4z_1\bar{z}_1^3).\end{aligned}$$

We employ one more change of coordinates

$$z_2^* = z_2 + \frac{B}{2A}z_1^2, \qquad z_3^* = z_3 + \frac{B}{A}z_1z_2 + \frac{3C}{2A}z_1^2 + \frac{D}{A}z_1^3, \qquad z_4^* = 4z_4,$$

under which equation (10.4) becomes

$$\operatorname{Im} z_4 = (z_1\bar{z}_3 + z_3\bar{z}_1) + |z_2|^2 + \left(\frac{3D}{2A} - \frac{B^2}{4A^2}\right)|z_1|^4.$$

Now it is clear that each surface with equation (10.4) is transformed by holomorphic changes either into an indefinite quadrics

$$\operatorname{Im} z_4 = (z_1\bar{z}_3 + z_3\bar{z}_1) + |z_2|^2 \qquad (\text{or} \qquad \operatorname{Im} z_4 = |z_1|^2 - |z_3|^2 + |z_2|^2),$$

if the condition $6AD - B^2 = 0$ holds or into the surface

$$\operatorname{Im} z_4 = (z_1\bar{z}_3 + z_3\bar{z}_1) + |z_2|^2 + N|z_1|^4 \tag{10.5}$$

with a non-zero real $N = (6AD - B^2)/4A^2$. It remains to note that for a non-zero N one more change

$$z_2 \to \sqrt{|N|}z_2, \qquad z_3 \to |N|z_3, \qquad z_4 \to |N|z_4$$

turns this coefficient into $sgn(N) = \pm 1$.

Completing the proof, we take into consideration that $B = D = 0$ in equation (10.1) and this is why it satisfies the sphericity condition $6AD - B^2 = 0$. For equation (10.2) the coefficient $N = (6AD - B^2)/4A^2$ is obviously negative. And for equation (10.3) this coefficient can be an arbitrary real number, positive or negative or zero. This completes the proof. □

Returning back to the orbits of the above considered algebras, we conclude on the sphericity in the case of the algebra $1357D$ in Proposition 8.2.

In its turn, for the orbits of the algebras $137D$ and $1357A$ described by equations (9.8) and (9.11), respectively, the sets of the parameters (A, B, C, D) are of the form

$$\left(\frac{1}{2a}, \frac{1}{a^2}, 0, \frac{1}{4a^3}\right) \qquad \text{and} \qquad \left(-\frac{1}{4a^2}, -\frac{1}{2a^2}, \frac{b}{6a^3}, 0\right).$$

For the first set the parameter $N = (6AD - B^2)/4A^2$ is positive, while for the second it is negative and this leads to two different equations (10.5).

Remark 10.1. *Equations of obtained in* [4] *homogeneous surfaces*

$$y_4 = y_1y_3 + y_2^2 + y_1^2y_2 + Dy_1^4 \qquad \text{and} \qquad y_4 = y_1y_3 + y_2^2 + x_1y_1y_2 + Dy_1^4, \tag{10.6}$$

are reduced by similar calculations to (9.9) *and* (9.12) *for* $D \neq 1/12$. *And the sphericity of the first surface in* (10.6) *for* $D = 1/12$ *was proved in* [17].

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