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LOCAL BOUNDARY VALUE PROBLEMS FOR A LOADED EQUATION OF PARABOLIC-HYPERBOLIC TYPE DEGENERATING INSIDE THE DOMAIN

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Abstract. In the beginning of 21st century, boundary value problems for non-degenerating equations of hyperbolic, parabolic, hyperbolic-parabolic and elliptic-hyperbolic types were studied. Recently this direction is intensively developed since rather important problems in mathematical physics and biology lead to boundary value problems for non-degenerate loaded partial differential equations. Boundary value problems for second order degenerating equation of a mixed type were not studied before. This is first of all because of the fact that there is no representation for the general solution to this equations. On the other hand, such problems are reduced to poorly studied integral equations with a shift. The present work is devoted to formulating and studying local boundary value problems for loaded equation of parabolic-hyperbolic type degenerating inside the domain.

In the present work we find a new approach for obtaining a representation for the general solution to a degenerating loaded equation of a mixed type. The uniqueness of the formulated problem is proved by the methods of energy integrals. The existence of solutions to the formulated problems is equivalently reduced to a second order integral Fredholm and Volterra equations with a shift. We prove the unique solvability of the obtained integral equations.

Keywords: loaded equation of parabolic-hyperbolic type, loaded equation with a degeneration, representation of general solution, method of energy integrals, extremum principle, integral equation with a shift.

Mathematics Subject Classification: 35M10, 35M12, 35L10, 35K10

1. Introduction

First results on model equation of mixed type, containing parabolic-hyperbolic operators, on constructing solutions, studying their properties and boundary value problems, were obtain in paper by I.M. Gelfand [1]. Later they were developed in works by G.M. Struchina [2], Ya.S. Uflyand [3] and L.A. Zolina [4].

Apart of these papers, in the end of the twentieth century, many papers by their pupils are appeared [5]–[9]; in these works there were studied the Tricomi problem and its generalizations, problems with shifts, problem of Bitsadze-Samarskii type and other non-local problems for parabolic and hyperbolic equations as well as for mixed parabolic-hyperbolic and elliptic-hyperbolic second order equations.

In works [10]–[13], on the base of the methods of the spectral analysis, boundary value problems for the mixed second order equations were studied in a rectangular domain.

Boundary value problems for loaded equations arise in studying many important problems in mathematical physics and biology [14], especially problem of a long forecasting and controlling ground water [15], modelling processes of particles transfer [16], problems on heat and mass

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transfer with a finite speed, modeling the filtration of a liquid in porous media [17], inverse problem [18], many problems on optimal control of agroecosystem [19].

A notion "loaded equation" has appeared first in works by A. Kneser [20]. The definition of loaded equations nowadays commonly used in the scientific literature was given by A.M. Nakhushev in 1976. In its work [21], he provided most general definitions and classification of various loaded equations, namely, loaded differential, integral, integral-differential, functional equations as well as their various applications.

At present, the class of the considered equations for non-degenerate loaded hyperbolic, parabolic, hyperbolic-parabolic and elliptic-parabolic equations is essentially enlarged; here we mention works [22]–[27]. The theory of boundary value problems for loaded second order integral-differential operator was developed in works [28], [29]. In works [30], [31] local and nonlocal boundary value problems were studied for degenerating hyperbolic and mixed type equations of second and third order.

To the best of the authors' knowledge, the boundary value problems for degenerating mixed type equation of second order were studied relatively little. We mention works by A.M. Nakhushev [32], B. Islomov and F. Juraev [33], R.R. Ashurov and S.Z. Jamalov [34]. First of all this due to the absence of a representation for the general solution of such equations; on the other hand, such problems are reduced to little-studied integral equations.

The present work is devoted to formulation and studying local boundary value problems for a loaded parabolic-hyperbolic equation degenerating inside the domain.

2. Formulation of problem

Let Ω be a bounded simply connected domain in the plane of variables x, y enveloped by the curves:

$$\begin{aligned}
&S_1 = \{(x,y) : x = 1, \quad 0 < y < 1\}, \qquad S_2 = \{(x,y) : \ x = -1, \quad 0 < y < 1\},\\
&S_3 = \{(x,y) : 0 < x < 1, \quad y = 1\}, \qquad S_4 = \{(x,y) : -1 < x < 0, \quad y = 1\};\\
&\Gamma_1 = \left\{(x,y) : \ x - \frac{2}{2-m}(-y)^{\frac{2-m}{2}} = 0, \quad y \leqslant 0\right\},\\
&\Gamma_2 = \left\{(x,y) : \ x + \frac{2}{2-m}(-y)^{\frac{2-m}{2}} = 0, \quad y \leqslant 0\right\},\\
&\Gamma_3 = \left\{(x,y) : \ x + \frac{2}{2-m}(-y)^{\frac{2-m}{2}} = 1, \quad y \leqslant 0\right\},\\
&\Gamma_4 = \left\{(x,y) : \ x - \frac{2}{2-m}(-y)^{\frac{2-m}{2}} = -1, \quad y \leqslant 0\right\}, \quad m < 0.
\end{aligned}$$

We introduce the notations:

$$\begin{aligned}
&\Omega_1^+ = \Omega \cap \{(x,y) : x > 0, \quad y > 0\}, \qquad \Omega_2^+ = \Omega \cap \{(x,y) : x < 0, \quad y > 0\},\\
&\Omega_1^- = \Omega \cap \{(x,y) : x > 0, \quad y < 0\}, \qquad \Omega_2^- = \Omega \cap \{(x,y) : x < 0, \quad y < 0\},\\
&I_1 = \{(x,y) : 0 < x < 1, \quad y = 0\}, \qquad I_2 = \{(x,y) : -1 < x < 0, \quad y = 0\},\\
&I_3 = \{(x,y) : x = 0, \quad 0 < y < 1\}, \quad \Omega_j = \Omega_j^+ \cup \Omega_j^- \cup J_j, (j = 1,2), \quad \Omega_3 = \Omega_1^+ \cup \Omega_2^+ \cup J_3,\\
&A_j((-1)^{j+1},0) = \bar{I}_j \cap \bar{S}_j, \quad C_j\left((-1)^{j+1}\frac{1}{2}; -\left((-1)^{j+1}\frac{2-m}{4}\right)^{2/(2-m)}\right) = \bar{\Gamma}_j \cap \bar{\Gamma}_{j+2},\\
&O(0,0) = \bar{I}_1 \cap \bar{I}_2, \qquad B_1(1,1) = \bar{S}_1 \cap \bar{S}_3, \qquad B_2(-1,1) = \bar{S}_2 \cap \bar{S}_4, \qquad B_0(0,1) = \bar{S}_3 \cap \bar{S}_4.
\end{aligned}$$

In the domain Ω we consider the equation

$$0=\begin{cases} u_{xx}-|x|^p u_y-\rho_j u\left(x,0\right), & (x,y)\in\Omega_j^+, \\ u_{xx}-(-y)^m u_{yy}+\mu_j u\left(x,0\right), & (x,y)\in\Omega_j^-, \end{cases} \tag{2.1}$$

where m, p, ρ_j, μ_j $(j=1,2)$ are arbitrary real numbers and

$$m<0, \qquad p>0, \qquad \rho_j>0, \qquad \mu_j>0, \qquad j=1,2. \tag{2.2}$$

In the domain Ω, we study the following boundary value problems for equation (2.1).

Problem 1. Find a function $u(x,y)$ possessing the following properties:
1) $u(x,y)\in C\left(\bar{\Omega}\right)\cap C^1\left(\Omega\right)\cap C^{2,1}_{x,y}\left(\Omega_1^+\cup\Omega_2^+\right)\cap C^2\left(\Omega_1^-\cup\Omega_2^-\right)$;
2) $u(x,y)$ is a regular solution of equation (2.1) in the domains Ω_j^+ and Ω_j^- $(j=1,2)$;
3) $u\left(x,y\right)$ satisfies the boundary conditions

$$u|_{S_j}=\varphi_j(y), \qquad 0\leqslant y\leqslant 1, \tag{2.3}$$

$$u|_{\Gamma_j}=\psi_j(x), \qquad 0\leqslant(-1)^{j+1}x\leqslant\frac{1}{2}, \qquad j=1,2; \tag{2.4}$$

4) on the curve of degeneration I_i, $(i=\overline{1,3})$, the matching conditions are satisfied:

$$\lim_{y\to+0}u_y(x,y)=\lim_{y\to-0}u_y(x,y), \qquad (x,0)\in I_j, \qquad j=1,2, \tag{2.5}$$

$$\lim_{x\to+0}u_x(x,y)=\lim_{x\to-0}u_x(x,y), \qquad (x,0)\in I_3; \tag{2.6}$$

where $\varphi_1(y)$, $\varphi_2(y)$, $\psi_1(x)$, $\psi_2(x)$ are given functions and $\psi_1(0)=\psi_2(0)$,

$$\varphi_j(y)\in C\left[0,1\right]\cap C^1\left(0,1\right), \qquad j=1,2, \tag{2.7}$$

$$\psi_1(x)\in C^1\left[0,\frac{1}{2}\right]\cap C^3\left(0,\frac{1}{2}\right), \qquad \psi_2(x)\in C^1\left[-\frac{1}{2},0\right]\cap C^3\left(-\frac{1}{2},0\right). \tag{2.8}$$

Problem 2(3). Find a function $u(x,y)$ possessing all properties of Problem 1 except of conditions (2.4), which are replaced by the conditions

$$u|_{\Gamma_1}=g_1(x), \qquad 0\leqslant x\leqslant\frac{1}{2}, \qquad u|_{\Gamma_4}=g_2(x), \qquad -1\leqslant x\leqslant-\frac{1}{2}, \tag{2.9}$$

$$\left(u|_{\Gamma_2}=f_1(x), \qquad -\frac{1}{2}\leqslant x\leqslant 0, \qquad u|_{\Gamma_3}=f_2(x), \qquad \frac{1}{2}\leqslant x\leqslant 1\right), \tag{2.10}$$

where $g_1(x)$, $g_2(x)$, $(f_1(x)$, $f_2(x))$ are given functions and $g_1(-1)=\varphi_2(0)$, $(f_2(1)=\varphi_2(0))$,

$$g_1(x)\in C^1\left[0,\frac{1}{2}\right]\cap C^3\left(0,\frac{1}{2}\right), \qquad g_2(x)\in C^1\left[-1,-\frac{1}{2}\right]\cap C^3\left(-1,-\frac{1}{2}\right), \tag{2.11}$$

$$\left(f_1(x)\in C^1\left[-\frac{1}{2},0\right]\cap C^3\left(-\frac{1}{2},0\right), \qquad f_2(x)\in C^1\left[\frac{1}{2},1\right]\cap C^3\left(\frac{1}{2},1\right)\right). \tag{2.12}$$

3. Uniqueness of solution to Problem 1

If Conditions 1) and 2) in Problem 1 are satisfied, then each regular solution to equation (2.1) can be represented as [22]:

$$u\left(x,y\right)=\upsilon\left(x,y\right)+\omega\left(x\right), \tag{3.1}$$

where

$$\upsilon\left(x,y\right)=\begin{cases} \upsilon_j\left(x,y\right), & (x,y)\in\Omega_j^+, \\ w_j\left(x,y\right), & (x,y)\in\Omega_j^-, \end{cases} \tag{3.2}$$

$$\omega\left(x\right)=\begin{cases}\omega_j^{+}(x), & (x,0)\in\bar{I}_j,\\ \omega_j^{-}(x), & (x,0)\in\bar{I}_j,\end{cases}\tag{3.3}$$

where $\upsilon_j\left(x,y\right)$ and $w_j\left(x,y\right)\left(j=1,2\right)$ are regular solutions of the equation

$$L\upsilon_j\equiv\upsilon_{jxx}-\left|x\right|^{p}\upsilon_{jy}=0,\qquad (x,y)\in\Omega_j^{+},\tag{3.4}$$

$$Lw_j\equiv w_{jxx}-\left(-y\right)^{m}w_{jyy}=0,\qquad (x,y)\in\Omega_j^{-}\qquad \left(j=1,2\right),\tag{3.5}$$

while $\omega_j^{+}(x)$ and $\omega_j^{-}(x)$, $j=1,2$, are arbitrary two continuously differentiable solutions of the equation

$$\omega_j^{+\prime\prime}\left(x\right)-\rho_j\omega_j^{+}\left(x\right)=\rho_j\upsilon_j\left(x,0\right),\qquad (x,0)\in I_j,\tag{3.6}$$

$$\omega_j^{-\prime\prime}\left(x\right)+\mu_j\omega_j^{-}\left(x\right)=-\mu_j w_j\left(x,0\right),\qquad (x,0)\in I_j.\tag{3.7}$$

Taking into consideration that the function $ax+b$ solves equations (3.4) and (3.5), arbitrary functions $\omega_j^{+}(x)$ and $\omega_j^{-}(x)$, $(j=1,2)$, can be obeyed the conditions

$$\omega_j^{+}((-1)^{j+1})=\omega_j^{+\prime}((-1)^{j+1})=0,\tag{3.8}$$

$$\omega_j^{-}(0)=\omega'_{j-}(0)=0\quad (j=1,2).\tag{3.9}$$

The solutions to Cauchy problems (3.6), (3.8) and (3.7), (3.9) are respectively of the form:

$$\omega_j^{+}\left(x\right)=\sqrt{\rho_j}\int\limits_{(-1)^{j+1}}^{x}\tau_j(t)\sinh\sqrt{\rho}_j(x-t)dt,\qquad (x,0)\in\bar{I}_j,\tag{3.10}$$

$$\omega_j^{-}(x)=-\sqrt{\mu_j}\int\limits_{0}^{x}\tau_j(t)\sinh\sqrt{\mu_j}(x-t)dt,\qquad (x,0)\in\bar{I}_j,\tag{3.11}$$

where

$$\tau_j\left(x\right)\equiv\upsilon_j\left(x,0\right)=w_j\left(x,0\right),\qquad (x,0)\in\overline{I}_j.\tag{3.12}$$

By (2.1), (2.3), (2.4), (3.2), (3.3), (3.8), (3.9), Problem 1 is reduced to Problem 1* for the equation

$$0=\begin{cases}L\upsilon_j, & (x,y)\in\Omega_j^{+},\\ Lw_j, & (x,y)\in\Omega_j^{-}\end{cases}\tag{3.13}$$

subject to the boundary conditions

$$\upsilon_j|_{S_j}=\varphi_j(y),\qquad 0\leqslant y\leqslant 1,\tag{3.14}$$

$$w_j|_{\Gamma_j}=\psi_j\left(x\right)-\omega_j^{-}(x),\qquad 0\leqslant(-1)^{j+1}x\leqslant\frac{1}{2},\tag{3.15}$$

where $\omega_j^{-}\left(x\right)$ are determined by (3.11).

In order to prove the uniqueness of the solution to Problem 1, we first prove the same for Problem 1* for equations (3.13).

The following lemma plays an important role in the proof of the uniqueness of the solution to Problem 1*.

Lemma 3.1. *If* $\varphi_1(y)\equiv\varphi_2(y)\equiv 0$ *as* $y\in[0,1]$, $\psi_1(x)\equiv 0$ *as* $x\in\left[0,\frac{1}{2}\right]$ *and* $\psi_2(x)\equiv 0$ *as* $x\in\left[-\frac{1}{2},0\right]$, *then*

$$\tau_j(x)\equiv 0\qquad \text{as}\quad x\in\bar{I}_j,\qquad j=1,2,\tag{3.16}$$

where $\tau_j(x)$, $j=1,2$, *are determined by (3.12).*

Proof. We are going to prove this lemma by means of the method of energy integrals. Let $w_j(x,y)$ be a twice continuously differentiable solution to the homogeneous problem 1^* in domains Ω_j^- and $\Omega_{j\varepsilon}^-$, where $\Omega_{j\varepsilon}^-$ is the domain with boundary $\partial\Omega_{j\varepsilon}^- = \bar I_{j\varepsilon} \cup \bar\Gamma_{j\varepsilon} \cup \bar\Gamma_{(j+2)\varepsilon}$ located strictly in the domain Ω_j^-, $(j=1,2)$, and ε is a sufficiently small positive number.

Let $j=1$. We integrate the identity

$$\begin{aligned} 0 =& x^p(-y)^{-m}w_1(w_{1xx} - (-y)^m\, w_{1yy}) \\ =& \frac{\partial}{\partial x}(x^p(-y)^{-m}w_1w_{1x}) - \frac{\partial}{\partial y}(x^p w_1 w_{1y}) \\ &- x^p\left[(-y)^{-m}w_{1x}^2 - w_{1y}^2\right] - px^{p-1}(-y)^{-m}w_1w_{1x} \end{aligned} \tag{3.17}$$

over the domain $\Omega_{1\varepsilon}^-$ and apply the Green formula. Then we get:

$$\begin{aligned} \int\limits_{\bar\Gamma_{1\varepsilon}\cup\bar\Gamma_{3\varepsilon}\cup\bar J_{1\varepsilon}} x^p(-y)^{-m}w_1w_{1x}dy + x^pw_1w_{1y}dx =& \iint\limits_{\Omega_{1\varepsilon}^-} x^p\left((-y)^{-m}w_{1x}^2 - w_{1y}^2\right)dxdy \\ &+ p\iint\limits_{\Omega_{1\varepsilon}^-} x^{p-1}(-y)^{-m}w_1w_{1x}dxdy. \end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ and taking into consideration Condition 1) in Problem 1 as in [35, Ch. 5], we obtain:

$$\begin{aligned} \int\limits_0^1 x^p\tau_1(x)\nu_1(x)dx =& \int\limits_{\bar\Gamma_3} x^p(-y)^{-\frac{m}{2}}w_1dw_1 - \int\limits_{\bar\Gamma_1} x^p(-y)^{-\frac{m}{2}}w_1dw_1 \\ &- \iint\limits_{\Omega_1^-} x^p\left((-y)^{-m}w_{1x}^2 - w_{1y}^2\right)dxdy \\ &- p\iint\limits_{\Omega_1^-} x^{p-1}(-y)^{-m}w_1w_{1x}dxdy, \end{aligned} \tag{3.18}$$

where

$$\tau_1(x) = w_1(x,0), \qquad (x,0)\in\bar I_1, \qquad \nu_1(x) = w_{1y}(x,0), \qquad (x,0)\in I_1. \tag{3.19}$$

In order to calculate the right hand side in identity (3.18) we pass to characteristic coordinates

$$\xi = x + \frac{2}{2-m}(-y)^{\frac{2-m}{2}}, \qquad \eta = x - \frac{2}{2-m}(-y)^{\frac{2-m}{2}}. \tag{3.20}$$

Then the domain Ω_1^- is mapped into a triangle Δ_1^- with sides O_1C_{11}, $C_{11}A_{11}$ and $A_{11}O_1$ located on the straight lines $\eta=0$, $\xi=1$ and $\eta=\xi$.

By (3.11), (3.15), as $\psi_1(x)=0$, in view of (3.20) and canonical form of equation (3.5) as $j=1$, that is, $\upsilon_{\xi\eta} = \frac{\beta}{\xi-\eta}(\upsilon_\xi - \upsilon_\eta)$, it follows from the right hand side of identity (3.18) that

$$\begin{aligned} \int\limits_{\bar\Gamma_1} x^p(-y)^{-\frac{m}{2}}w_1dw_1 =& \left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{-2\beta}\left(\omega_1^-\left(\frac{1}{2}\right)\right)^2 \\ &- \frac{p-2\beta}{2}\left(\frac{1}{2}\right)^p\left(\frac{2-m}{4}\right)^{-2\beta}\int\limits_0^1 \xi^{p-2\beta-1}w_1^2(\xi,0)d\xi, \end{aligned} \tag{3.21}$$

$$\int\limits_{\bar{\Gamma}_3} x^p(-y)^{-\frac{m}{2}} w_1 dw_1 = -\left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{-2\beta}\left(\omega_1^-\left(\frac{1}{2}\right)\right)^2$$
$$-\left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{-2\beta} p\int\limits_0^1 (1+\eta)^{p-1}(1-\eta)^{-2\beta} w_1^2(1,\eta)d\eta \tag{3.22}$$
$$+\left(\frac{1}{2}\right)^{p}\left(\frac{2-m}{4}\right)^{-2\beta} \beta\int\limits_0^1 (1+\eta)^{p}(1-\eta)^{-2\beta-1} w_1^2(1,\eta)d\eta,$$

$$\iint\limits_{\Omega_1^-} x^p\big(w_{1y}^2-(-y)^{-m}w_{1x}^2\big)dxdy = \left(\frac{1}{2}\right)^{p}\left(\frac{2-m}{4}\right)^{-2\beta}\left(\left(\omega_1^-\left(\frac{1}{2}\right)\right)^2\right.$$
$$-\frac{p-\beta}{2}\int\limits_0^1 \frac{w_1^2(\xi,0)d\xi}{\xi^{1+2\beta-p}} - \beta\int\limits_0^1 \frac{(1+\eta)^p w_1^2(1,\eta)d\eta}{(1-\eta)^{2\beta+1}}$$
$$+p\int\limits_0^1 \frac{(1+\eta)^{p-1} w_1^2(1,\eta)d\eta}{(1-\eta)^{2\beta}} \tag{3.23}$$
$$\left.-p(p-1)\iint\limits_{\triangle_1} \frac{(\xi+\eta)^{p-2}w_1^2(\xi,\eta)}{(\xi-\eta)^{2\beta}}d\xi d\eta\right),$$

$$\iint\limits_{\Omega_1^-} x^{p-1}(-y)^{-m} w_1 w_{1x} dxdy = -\left(\frac{1}{2}\right)^{p+1}\left(\frac{2-m}{4}\right)^{-2\beta}\left(\int\limits_0^1 \xi^{p-2\beta-1}w_1^2(\xi,0)d\xi\right.$$
$$\left.-\int\limits_0^1 \frac{(1+\eta)^{p-1}}{(1-\eta)^{-2\beta}} w_1^2(1,\eta)d\eta\right) - \left(\frac{1}{2}\right)^{p}\left(\frac{2-m}{4}\right)^{-2\beta} \tag{3.24}$$
$$\cdot (p-1)\iint\limits_{\triangle_1} \frac{(\xi+\eta)^{p-2}}{(\xi-\eta)^{2\beta}} w_1^2(\xi,\eta)d\xi d\eta,$$

where $2\beta = -\frac{m}{2-m}$, $0 < m < 1$, and

$$0 < -\beta < \frac{1}{2}, \qquad 0 < p-2\beta < 1. \tag{3.25}$$

Substituting (3.22), (3.23) and (3.24) into (3.18), in view of (2.2) and (3.25) we get

$$\int\limits_0^1 x^p \tau_1(x)\nu_1(x)dx = \frac{p-\beta}{2^{p+1}}\left(\frac{2-m}{4}\right)^{-2\beta}\int\limits_0^1 \xi^{p-2\beta-1}w_1^2(\xi,0)d\xi \geqslant 0. \tag{3.26}$$

Let $j = 2$. Then as above, we integrate identity (3.17) over the domain Ω_2^-, we get

$$\int\limits_{-1}^0 (-x)^p \tau_2(x)\nu_2(x)dx = \frac{p-\beta}{2^{p+1}}\left(\frac{2-m}{4}\right)^{-2\beta}\int\limits_{-1}^0 (-\xi)^{p-2\beta-1}w_2^2(\xi,0)d\xi \geqslant 0, \tag{3.27}$$

where

$$\tau_2(x) = w_2(x,0), \qquad (x,0)\in \bar{I}_2, \qquad \nu_2(x) = w_{2y}(x,0), \qquad (x,0)\in I_2. \tag{3.28}$$

By Condition 1) in Problem 1 and by the continuity of $\omega(x)$, in view of (3.1), (3.2), (3.3), (3.19), (3.28) we have:

$$w_j(x,-0) = \upsilon_j(x,+0) = \tau_j(x), \qquad (x,0)\in \bar{I}_j, \tag{3.29}$$

$$w_{jy}(x,-0) = \upsilon_{jy}(x,+0) = \nu_j(x), \quad (x,0)\in I_j \quad (j=1,2). \tag{3.30}$$

Owing to the assumptions of Problem 1, we pass to the limit as $y\to+0$ in equation (3.4) and in view of (3.29) and (3.30) we obtain:

$$\tau_j''(x) - |x|^p\,\nu_j(x) = 0. \tag{3.31}$$

Then by the assumptions of Lemma 1, (3.31) and $\tau_j(0)=\tau_j((-1)^{j+1})=0$ we find:

$$\int\limits_0^{(-1)^{j+1}} |x|^p\,\tau_j(x)\nu_j(x)dx + \int\limits_0^{(-1)^{j+1}} \tau'^2_j(x)dx = 0, \quad j=1,2. \tag{3.32}$$

Comparing (3.26), (3.27) and (3.32), we obtain:

$$\int\limits_0^{(-1)^{j+1}} |x|^p\tau_j(x)\nu_j(x)dx = 0$$

or

$$\int\limits_0^{(-1)^{j+1}} \tau_j'^2(x)dx = 0, \quad j=1,2.$$

By the conditions $\tau_j(0)=\tau_j((-1)^{j+1})=0$ this implies:

$$\tau_j(x)\equiv 0 \quad \text{as} \quad x\in\bar{I}_j, \quad j=1,2. \tag{3.33}$$

□

By (3.33), (3.10), (3.11), (3.3) we get:

$$\omega(x)\equiv 0, \qquad \text{for all} \quad x\in\bar{I}_1\cup\bar{I}_2. \tag{3.34}$$

Theorem 3.1. *If the assumptions of Lemma 3.1 and (3.34) are satisfied, then in the domain* Ω *Problem* 1^* *for equation (3.13) can have at most one solution.*

Proof. According to the maximum principle for parabolic equations [6], [36], [37], by (3.33), boundary value problem 1^* for equation (3.13) in the domain $\bar{\Omega}_3$ subject to homogeneous conditions (3.12), (3.14) has no non-zero solutions, that is, $\upsilon_j(x,y)\equiv 0$ in $\bar{\Omega}_j^+$, $(j=1,2)$. Then it follows from (3.15), (3.3), (3.34) that

$$\omega_j^-(x)\equiv 0, \quad (x,0)\in\bar{I}_j, \quad j=1,2. \tag{3.35}$$

By the uniqueness of the solution to the Cauchy problem with homogeneous conditions

$$w_j(x,y)|_{y=0} = 0, \quad (x,0)\in\bar{I}_j, \quad w_{jy}(x,y)|_{y=0} = 0, \quad (x,0)\in I_j$$

for equation (3.13) in the domain Ω_j^- and owing to (3.34) and (3.35) we obtain $w_j(x,y)\equiv 0$ in $\bar{\Omega}_j^-$. Hence, by (3.2) we have:

$$\upsilon(x,y)\equiv 0, \qquad (x,y)\in\bar{\Omega}. \tag{3.36}$$

Now (3.36) yields the uniqueness of the solution to Problem 1^* for equation (3.13). □

Theorem 3.2. *If the assumptions of Theorem 3.1 are satisfied, then in the domain* Ω *Problem 1 for equation (2.1) can have at most one solution.*

Proof. By (3.34), (3.36) it follows from (3.1) that

$$u(x,y)\equiv 0, \qquad (x,y)\in\bar{\Omega}. \tag{3.37}$$

This proves the uniqueness of solution to Problem 1 for equation (2.1). □

4. EXISTENCE OF SOLUTION TO PROBLEM 1

Theorem 4.1. *If conditions (2.2), (2.7), (2.8) and (3.25) are satisfied, then Problem 1 is solvable in the domain* Ω*.*

In the proof of Theorem 4.1, the following problem play an important role; these problems are also of an independent interest.

Problem 1_j. Find a solution $u(x,y)\in C(\bar{\Omega}_j)\cap C^1(\Omega_j)\cap C^2(\Omega_j^+\cup\Omega_j^-)$ $(j=1,2)$ to equation (2.1) satisfying conditions (2.3), (2.4) and

$$u(0,y)=\tau_3(y), \qquad (0,y)\in\bar{I}_3, \tag{4.1}$$

where $\tau_3(y)$ is a given function and

$$\tau_3(y)\in C(\bar{I}_3)\cap C^1(I_3). \tag{4.2}$$

Problem 1_3. Find a solution $u(x,y)\in C(\bar{\Omega}_3)\cap C^1(\Omega_3\cup I_1\cup I_2)\cap C^{2,1}_{x,y}(\Omega_1^+\cup\Omega_2^+)$ to equation (2.1) satisfying conditions (2.3) and

$$u(x,y)\big|_{y=0}=\tau_j(x)+\omega_j^+(x), \quad (x,0)\in\bar{I}_j \quad (j=1,2),$$

where $\tau_j(x)$ and $\omega_j^+(x)$ are defined respectively by (3.29) and (3.10).

4.1. Study of Problem 1_j $(j=1,2)$.

Theorem 4.2. *If conditions (2.2), (2.7), (2.8), (3.25) and (4.2) are satisfied, then Problem* 1_j *is uniquely solvable in the domain* Ω_j*.*

Proof. By Lemma 3.1 and the extremum principle for degenerating parabolic-hyperbolic equations [37] we see that a solution $u(x,y)$ to Problem 1_j as $\psi_j(x)\equiv 0$ attains its positive maximum and negative minimum in the closed domain $\bar{\Omega}_j^+$ only on $\overline{\Gamma}_j\cup\overline{I}_3$, $(j=1,2)$.

According to the extremum principle, homogeneous Problem 1_j, that is, problem with zero boundary conditions, has no non-zero solution. This implies a uniqueness of solution to Problem 1_j.

We proceed to proving the existence of solution to Problems 1_j and 1_j^* subject to Conditions (3.14), (3.15) and $\upsilon_j(0,y)=\tau_3(y)$, $(0,y)\in\bar{I}_3$.

By the properties of solutions to Cauchy problem [33] for equation (3.13) in domain Ω_j^- $(j=1,2)$ and in view of (3.15) we have:

$$\begin{aligned}\psi_1\left(\frac{x}{2}\right)-\omega_1^-\left(\frac{x}{2}\right)=&\gamma_1 x^{1-2\beta}\Gamma(\beta)\,D_{0x}^{-\beta}x^{\beta-1}\tau_1(x)\\&-\gamma_2\Gamma(1-\beta)\,D_{0x}^{\beta-1}x^{-\beta}\nu_1(x), \quad (x,0)\in I_1,\end{aligned}\tag{4.3}$$

$$\begin{aligned}\psi_2\left(\frac{x}{2}\right)-\omega_2^-\left(\frac{x}{2}\right)=&\gamma_1(-x)^{1-2\beta}\Gamma(\beta)\,D_{x0}^{-\beta}(-x)^{\beta-1}\tau_2(x)\\&-\gamma_2\Gamma(1-\beta)\,D_{x0}^{\beta-1}(-x)^{-\beta}\nu_2(x), \quad (x,0)\in I_2,\end{aligned}\tag{4.4}$$

where $\tau_j(x)$ and $\nu_j(x)$ are defined by (3.29) and (3.30), respectively,

$$\gamma_1=\frac{\Gamma(2\beta)}{\Gamma^2(\beta)}, \qquad \gamma_2=\frac{1}{2}\left(\frac{4}{2-m}\right)^{2\beta}\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)},$$

while $D_{0x}^{-\alpha}(\cdot)$ and $D_{x0}^{-\alpha}(\cdot)$ are integral operators of fractional order $\alpha\,(\alpha>0)$ [38]:

$$D_{ax}^{-\alpha}\phi_j(t)=\frac{1}{\Gamma(\alpha)}\int\limits_a^{(-1)^{j+1}x}\frac{\phi_j(t)dt}{(x-t)^{1-\alpha}},\quad (-1)^{j+1}x>a,\quad \operatorname{Re}\alpha>0. \tag{4.5}$$

Applying differential operators, $\frac{d}{dx}D_{0x}^{-\beta}\ldots\equiv D_{0x}^{1-\beta}\ldots$ and $-\frac{d}{dx}D_{x0}^{-\beta}\ldots\equiv D_{x0}^{1-\beta}\ldots$ to both sides of the identities (4.3), (4.4) and employing formulae [38]

$$\begin{aligned}
&D_{0x}^{1-\beta}D_{0x}^{\beta-1}\nu_1(x)=\nu_1(x),\\
&D_{x0}^{1-\beta}\ D_{x0}^{\beta-1}\nu_2(x)=\nu_2(x),\\
&D_{0x}^{1-\beta}x^{1-2\beta}D_{0x}^{-\beta}x^{\beta-1}\tau_1(x)=x^{-\beta}D_{0x}^{1-2\beta}\tau_1(x),\\
&D_{x0}^{1-\beta}(-x)^{1-2\beta}D_{x0}^{-\beta}(-x)^{\beta-1}\tau_2(x)=(-x)^{-\beta}D_{x0}^{1-2\beta}\tau_2(x),
\end{aligned}$$

we obtain functional relations between $\tau_j(x)$ and $\nu_j(x)$ transferred from the domain Ω_j^- to I_j, $(j=1,2)$:

$$\begin{aligned}\nu_1(x)=&\frac{\gamma_1\Gamma(\beta)}{\gamma_2\Gamma(1-\beta)}D_{0x}^{1-2\beta}\tau_1(x)\\
&+\frac{x^\beta}{\gamma_2\Gamma(1-\beta)}D_{0x}^{1-\beta}\omega_1^-\left(\frac{x}{2}\right)-\frac{x^\beta}{\gamma_2\Gamma(1-\beta)}D_{0x}^{1-\beta}\psi_1\left(\frac{x}{2}\right),\qquad (x,0)\in I_1,\end{aligned} \tag{4.6}$$

$$\begin{aligned}\nu_2(x)=&\frac{\gamma_1\Gamma(\beta)}{\gamma_2\Gamma(1-\beta)}D_{x0}^{1-2\beta}\tau_2(x)\\
&+\frac{(-x)^\beta}{\gamma_2\Gamma(1-\beta)}D_{x0}^{1-\beta}\omega_2^-\left(\frac{x}{2}\right)-\frac{(-x)^\beta}{\gamma_2\Gamma(1-\beta)}D_{x0}^{1-\beta}\ \psi_2\left(\frac{x}{2}\right),\qquad (x,0)\in I_2.\end{aligned} \tag{4.7}$$

Bearing in mind conditions in Problem 1, we pass to the limit as $y\to+0$ in equation (3.4) and in view of (3.29) and (3.30) we get (3.31) with conditions

$$\tau_1(0)=\tau_3(0)=\psi_1(0),\quad \tau_1(1)=\varphi_1(0), \tag{4.8}$$

$$\tau_2(-1)=\varphi_2(0),\quad \tau_2(0)=\tau_3(0)=\psi_2(0). \tag{4.9}$$

Solving problem (3.31) and (4.8), (4.9), we obtain a functional relation for $\tau_j(x)$ and $\nu_j(x)$ transferred from the domain Ω_j^+ to I_j:

$$\tau_j(x)=(-1)^{j+1}\int\limits_0^{(-1)^{j+1}}G_j(x,t)\left((-1)^{j+1}t\right)^p\nu_j(t)dt+f_j(x),\qquad (x,0)\in\bar{I}_j, \tag{4.10}$$

where

$$G_1(x,t)=\begin{cases}(t-1)x, & 0\leqslant x\leqslant t,\\ (x-1)t, & t\leqslant x\leqslant 1,\end{cases} \tag{4.11}$$

$$G_2(x,t)=\begin{cases}(x+1)t, & -1\leqslant x\leqslant t,\\ (t+1)x, & t\leqslant x\leqslant 0,\end{cases} \tag{4.12}$$

$$f_j(x)=\psi_j(0)+(-1)^{j+1}x\left(\varphi_j(0)-\psi_j(0)\right). \tag{4.13}$$

Excluding $\tau_j(x)$ from (4.6), (4.7) and (4.10), in view of (3.11) we obtain an integral equation for $\nu_j(x)$, $(j=1,2)$:

$$\nu_1(x)-\int\limits_0^1 K_1(x,t)t^p\nu_1(t)dt=\Psi_1(x),\qquad (x,0)\in I_1, \tag{4.14}$$

$$\nu_2(x) + \int\limits_{-1}^{0} K_2(x,t)(-t)^p \nu_2(t)dt = \Psi_2(x), \qquad (x,0) \in I_2, \tag{4.15}$$

where

$$\begin{aligned} K_j(x,t) =& \frac{\gamma_1 \Gamma(\beta)}{\gamma_2 \Gamma(1-\beta)} A_{jx}^{1-2\beta} G_j(x,t) - \frac{\left((-1)^{j+1}x\right)^{\beta}}{2\gamma_2 \Gamma\left(1-\beta\right)} A_{jx}^{1-\beta} \\ & \cdot \int\limits_0^x \sin \frac{\sqrt{\mu_1}(x-z)}{2} G_j\left(\frac{z}{2}, t\right) dz, \end{aligned} \tag{4.16}$$

$$\begin{aligned} \Psi_j(x) =& \frac{\gamma_1 \Gamma(\beta)}{\gamma_2 \Gamma(1-\beta)} A_{jx}^{1-2\beta} f_j(x) - \frac{((-1)^{j+1}x)^{\beta}}{\gamma_2 \Gamma\left(1-\beta\right)} A_{jx}^{1-\beta} \psi_j\left(\frac{x}{2}\right) \\ & - \frac{\left((-1)^{j+1}x\right)^{\beta}}{2\gamma_2 \Gamma\left(1-\beta\right)} A_{jx}^{1-\beta} \int\limits_0^x \sin \frac{\sqrt{\mu_1}(x-z)}{2} f_j\left(\frac{z}{2}\right) dz, \quad (x,0) \in I_j, \end{aligned} \tag{4.17}$$

$$A_{jx}^{\alpha} g(x) = \begin{cases} D_{0x}^{\alpha} g(x), & j=1, \\ D_{x0}^{\alpha} g(x), & j=2. \end{cases} \tag{4.18}$$

By (2.2), (2.7), (2.8) and (3.25), the properties of the operator of integro-differentiation and of Beta and hypergeometric functions [38, Ch. 1] and the function $G_j(x,t)$ $(j=1,2)$, it follows from (4.16), (4.17) that the kernel and the right hand side of equations (4.14) and (4.15) admit the estimates:

$$|K_j(x,t)| \leqslant c_1, \tag{4.19}$$

$$|\Psi_j(x)| \leqslant const\, |x|^{2\beta-1}, \quad c_j = const > 0. \tag{4.20}$$

By (2.7), (2.8), (4.20) we hence conclude that $\Psi_j(x) \in C^2\left(I_j\right)$ and the function $\Psi_j(x)$ can possess a singularity of order less than $1-2\beta$ as $|x| \to 0$ and it is bounded as $|x| \to 1$.

By (2.2), (4.19) and (4.20), equations (4.14) and (4.15) are integral Fredholm equation of second kind. According to the theory of integral Fredholm equations [39] and by the uniqueness of solution to Problem 1_j we conclude that integral equations (4.14) and (4.15) are uniquely solvable in the class $C^2\left(I_j\right)$ and $\nu_j(x)$ can have a singularity of order less than $1-2\beta$ as $|x| \to 0$ and is bounded as $|x| \to 1$; the solutions are given by the formula:

$$\nu_j(x) = \Psi_j(x) + \int\limits_0^{(-1)^{j+1}} K_j^*(x,t)\Psi_j(t)dt, \quad (x,0) \in I_j \quad (j=1,2), \tag{4.21}$$

where $K_j^*(x,t)$ is the resolvent of the kernel $K_j(x,t)$.

Substituting (4.21) into (4.10), we find:

$$\tau_j(x) \in C(\bar{I}_j) \cap C^2(I_j) \quad (j=1,2). \tag{4.22}$$

Therefore, Problem 1_j^* is uniquely solvable by its equivalence to integral Fredholm equations of second kind (4.14) and (4.15).

Thus, the solution to Problem 1_j^* can recovered in the domain Ω_j^+ as a solution to the Dirichlet problem for equations (3.4) [40], while in Ω_j^- it is recovered as a solution to the Cauchy problem for equation (3.5).

This completes the studying of solvability of Problem 1_j^* for equation (3.13).

By (4.10), (4.21), (3.10), (3.11), (3.1), (3.2), (3.3) we determine the functions $\omega_j^+(x)$ and $\omega_j^-(x)$. Then the solution to Problem 1_j in the domain Ω_j^+ can be found as

$$u(x,y) = \upsilon_j(x,y) + \omega_j^+(x), \tag{4.23}$$

where $\upsilon_j(x,y)$ is the solution to the Dirichlet problem for equation (3.4) [37], [40], while in the domains Ω_j^+ it reads as

$$u(x,y) = w_j(x,y) + \omega_j^-(x) \quad (j=1,2)\,, \tag{4.24}$$

where $w_j(x,y)$ is the solution of the Cauchy problem for equation (3.5) in the domain Ω_j^- $(j=1,2)$ [33].

Thus, Problem 1_j is uniquely solvable in the domain Ω_j. □

4.2. Study of Problem 1_3.

Theorem 4.3. *Let conditions (2.2, (2.7), (3.25) and (4.22) be satisfied. Then Problem* 1_3 *is uniquely solvable in the domain* Ω_3.

Proof. The solution to the Dirichlet boundary value problem subject to conditions (3.14), (4.1) for equation (3.4) in the domain Ω_j^+ reads as [40]

$$\begin{aligned}\upsilon_j(x,y) =& (-1)^{j+1}\left(\int\limits_0^{(-1)^{j+1}} R_j(x,t,y;\alpha)\left((-1)^{j+1}t\right)^p \tau_j(t)dt\right.\\ &\left.+\frac{\partial}{\partial y}\int\limits_0^y R_j^{(1)}(x,y-t;\alpha)\,\tau_3(t)dt+\frac{\partial}{\partial y}\int\limits_0^y R_j^{(2)}(x,y-t;\alpha)\,\varphi_j(t)dt\right)\end{aligned} \tag{4.25}$$

and belongs to the class $u(x,y)\in C(\bar{\Omega}_j^+)\cap C^1(\Omega_j\cup I_j)\cap C^{2,1}_{x,y}(\Omega_j^+)$ if conditions (2.7), (4.2), (4.22) are satisfied. Here $R_j(x,t,y;\alpha)$ is the Green function of Dirichlet problem for equation (3.13) in the domain Ω_j^+, $(j=1,2)$:

$$\begin{aligned}R_j(x,\xi,y;\alpha) =& \sum_{k=0}^{\infty}\exp\left(-\frac{\lambda_k^2 y}{4}\right)\frac{(1-\alpha)\sqrt{x\xi}}{J_{2-\alpha}^2(\lambda_k)}J_{1-\alpha}\left(\lambda_k(1-\alpha)\left((-1)^{j+1}x\right)^{\frac{1}{2(1-\alpha)}}\right)\\ &\cdot J_{1-\alpha}\left(\lambda_k(1-\alpha)\left((-1)^{j+1}\xi\right)^{\frac{1}{2(1-\alpha)}}\right),\end{aligned} \tag{4.26}$$

$$\begin{aligned}R_j^{(1)}(x,y;\alpha) =& 1+(-1)^j(1-\alpha)^{2(1-\alpha)}x\\ &-\int\limits_0^{(-1)^{j+1}}\left(1+(-1)^j(1-\alpha)^{2(1-\alpha)}\xi\right)R_j(x,t,y;\alpha)\left((-1)^{j+1}\xi\right)^p d\xi,\end{aligned} \tag{4.27}$$

$$\begin{aligned}R_j^{(2)}(x,y;\alpha) =& (-1)^{j+1}(1-\alpha)^{2(1-\alpha)}x\\ &-\int\limits_0^{(-1)^{j+1}} R_j(x,t,y;\alpha)\left((-1)^{j+1}(1-\alpha)^{2(1-\alpha)}\xi\right)\left((-1)^{j+1}\xi\right)^p d\xi,\end{aligned} \tag{4.28}$$

where

$$J_\theta(z)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\theta+1)}\left(\frac{z}{2}\right)^{\theta+2k}$$

is the Bessel function of the first kind, λ_k are positive roots of the equation $J^2_{1-\alpha}(\lambda_k) = 0$, $k \in \mathbb{N} \cup \{0\}$, $\alpha = \frac{p+1}{p+2}$, and

$$\frac{1}{2} < \alpha < 1. \tag{4.29}$$

Differentiating (4.25) with respect to x and passing to the limit as $x \to 0$, we get

$$\nu_3(y) = \frac{\partial}{\partial y} \int\limits_0^y N_j(y-t;\alpha)\tau_3(t)dt + \Phi_j(y), \quad (0,y) \in I_3, \tag{4.30}$$

where $\upsilon_{jx}(0,y) = \nu_3(y)$, $(0,y) \in I_3$,

$$\begin{aligned}\Phi_j(y) =& \lim_{x\to 0}(-1)^{j+1}\frac{\partial}{\partial x}\left(\int\limits_0^1 R_j(x,t,y;\alpha)\left((-1)^{j+1}t\right)^p \tau_j(t)dt\right.\\ &\left.+\frac{\partial}{\partial y}\int\limits_0^y R_j^{(2)}(x,y-t;\alpha)\,\varphi_j(t)dt\right),\end{aligned} \tag{4.31}$$

$$\begin{aligned}N_j(y-t;\alpha) \equiv& (1-\alpha)^{2\alpha-1}(-1)^{j+1}\lim_{x\to 0}\frac{\partial}{\partial x}\left(R_j^{(1)}(x,y-t;\alpha)\right)\\ =& (-1)^j\left((1-\alpha) + \sum_{k=0}^{\infty}\exp\left(-\frac{\lambda_k^2(y-t)}{4}\right)\frac{2^{2\alpha}\lambda_k^{-2\alpha}}{\Gamma^2(1-\alpha)J^2_{2-\alpha}(\lambda_k)}\right).\end{aligned}$$

Owing to the properties of the function $J_\theta(z)$, the function $N_j(y-t;\alpha)$ can be represented as [40]

$$N_j(y-t;\alpha) = \frac{(-1)^j}{\Gamma(1-\alpha)}(y-t)^{\alpha-1} + B_j(y-t), \tag{4.32}$$

where $B_j(y-t)$, $(j = 1,2)$, are continuously differentiable functions as $y \geqslant t$.

Substituting (4.32) into (4.30), we obtain a functional relation for $\tau_3(y)$ and $\nu_3(y)$ transferred from the domain Ω_j^+ to I_3 :

$$\nu_3(y) = \frac{(-1)^j}{\Gamma(1-\alpha)}\frac{\partial}{\partial y}\int\limits_0^y (y-t)^{\alpha-1}\tau_3(t)dt + \frac{\partial}{\partial y}\int\limits_0^y B_j(y-t)\tau_3(t)dt + \Phi_j(y).$$

Then by formula (4.5) we have

$$\nu_3(y) = \frac{(-1)^j\Gamma(\alpha)}{(1-\alpha)}D^{1-\alpha}_{0y}\tau_3(y) + B_j(0)\tau_3(y) + \int\limits_0^y B_j'(y-t)\tau_3(t)dt + \Phi_j(y). \tag{4.33}$$

Excluding $\nu_3(y)$ from relations (4.33) for $j = 1$ and for $j = 2$ and applying the integral operator $D^{\alpha-1}_{0y}(\centerdot)$, in view of the identities $\tau_3(0) = 0$ and $D^{\alpha-1}_{0y}D^{1-\alpha}_{0y}\tau_3(y) = \tau_3(y)$, we obtain

$$\tau_3(y) = \int\limits_0^y M(y,t)\tau_3(t)dt + \Phi(y), \quad (0,y) \in \bar{I}_3, \tag{4.34}$$

where

$$M(y,t) = \frac{1}{2\Gamma(\alpha)}\left(\frac{B_2(0) - B_1(0)}{(y-t)^\alpha} - \int\limits_t^y \frac{B_2'(z-t) - B_1'(z-t)}{(y-z)^\alpha}dz\right), \tag{4.35}$$

$$\Phi(y) = \frac{\Gamma(1-\alpha)}{2(\alpha)} D_{0y}^{\alpha-1} \left(\Phi_2(y) - \Phi_1(y)\right); \tag{4.36}$$

here $\Phi_j(y)$, $(j = 1, 2)$, is determined by (4.31).

By (2.2), (2.7), (3.25), (4.22), (4.29), properties of the function $B_j(y-t)$, (4.26), (4.27), (4.28), (4.31), (4.35), (4.36) we see that:

1) the kernels $M(y,t)$ are continuous in $\{(y,t) : 0 \leqslant t < y \leqslant 1\}$ and as $y \to t$, they admit the estimate

$$|M(y,t)| \leqslant const(y-t)^{-\alpha}; \tag{4.37}$$

2) the function $\Phi(y)$ belongs to the class $C\left(\bar{I}_3\right) \cap C^1\left(I_3\right)$ and admits the estimate

$$|\Phi(y)| \leqslant const\ y^{1-\alpha}. \tag{4.38}$$

It follows from (4.37) and (4.38) that integral equation (4.34) is an integral Volterra equation of second kind with a weak singularity.

According to the theory of integral Volterra equations of second kind [39] we conclude that integral equation (4.34) is uniquely solvable in the class $C\left(\bar{J}_3\right) \cap C^1\left(J_3\right)$ and its solution is given by the formula

$$\tau_3(y) = \int\limits_0^y M^*(y,t)\Phi(t)dt + \Phi(y), \qquad (0,y) \in \bar{I}_3, \tag{4.39}$$

where $M^*(y,t)$ is the resolvent of the kernel $M(y,t)$.

Substituting (4.39) into (4.33) and taking into consideration (4.37), (4.38), we define a function $\nu_3(y)$

$$\nu_3(y) \in C^1\left(I_3\right), \tag{4.40}$$

and $\nu_3(y)$ can have a singularity of order less than $1-\alpha$ as $y \to 0$ and is bounded as $y \to 1$.

Therefore, problem 1_3^* is uniquely solvable.

Thus, the solution to Problem 1_3^* can be recovered in the domain Ω_j^+, $(j = 1, 2)$, as the solution to the Dirichlet problem for equation (3.4) [40]. This completes the study of the solvability of Problem 1_3^* for equation (3.4) in the domain Ω_3.

By (4.10), (4.21), (3.10), (3.1), (3.2), (3.3) we determine the functions $\omega_j^+(x)$. Then the solution to Problem 1_3 in the domain Ω_3 can be found as

$$u(x,y) = \upsilon_j(x,y) + \omega_j^+(x), \tag{4.41}$$

where $\upsilon_j(x,y)$ is the solution to the Dirichlet problem for equation (3.4), see(4.25).

Therefore, Problem 1_3 is uniquely solvable. □

We proceed to proving the solvability of Problem 1.

Proof. Let $u(x,y)$ be the solution to Problem 1 in the domain Ω subject to conditions (2.3)–(2.6). Then employing the results on Problems 1_i, $(i = \overline{1,3})$, see Sections 4.1 and 4.2, Problem 1 is equivalently reduced to Problems 1_1 and 1_2 for equation (2.1), where $\tau_3(y)$ is defined by formula (4.39). □

The unique solvability of Problems 1_1 and 1_2 is implied by Theorem 4.2. Therefore, there exists a solution to Problem 1 in the domain Ω. This completes the studying of Problem 1 for equation (2.1).

The following statements hold true.

Theorem 4.4. *If conditions (2.2), (2.7), (2.11), (3.25) and (4.29) are satisfied, then Problem 2 is uniquely solvable in the domain Ω.*

Theorem 4.5. *If conditions (2.2), (2.7), (2.12), (3.25) and (4.29) are satisfied, then Problem 3 is uniquely solvable in the domain* Ω.

The proof of Theorems 4.4 and 4.5 follow the same lines as that of Theorems 3.2 and 4.1.

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