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THE STRUCTURE OF FOLIATIONS WITH INTEGRABLE EHRESMANN CONNECTION

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Abstract. We study foliations of arbitrary codimension q on n-dimensional smooth manifolds admitting an integrable Ehresmann connection. The category of such foliations is considered, where isomorphisms preserve both foliations and their Ehresman connections. We show that this category can be considered as that of bifoliations covered by products. We introduce the notion of a canonical bifoliation and we prove that each foliation (M, F) with integrable Ehresmann connection is isomorphic to some canonical bifoliation. A category of triples is constructed and we prove that it is equivalent to the category of foliations with integrable Ehresmann connection. In this way, the classification of foliations with integrable Ehresman connection is reduced to the classification of associated diagonal actions of discrete groups of diffeomorphisms of the product of manifolds. The classes of foliations with integrable Ehresmann connection are indicated. The application to G-foliations is considered.

Keywords: foliation, integrable Ehresmann connection for a foliation, global holonomy group, canonical bifoliation.

Mathematics Subject Classification: 57R30, 53C12

1. Introduction. Main results

Ehresmann connection for a smooth foliation (M, F) of a codimension q on an n-dimensional manifold M is defined as a transversal q-dimensional distribution $\mathfrak{M}$ on M possessing a property of vertical-horizontal homotopy; an exact definition is provided in Section 2.1. The notion of Ehresmann connection is of a global nature. The Ehresmann connection allows one to translation the integral curves of the distribution $\mathfrak{M}$ called horizontal along curves located in corresponding leaves of the foliations called vertical [1]. If the distribution $\mathfrak{M}$ is tangential to some q-dimensional foliation (M, F^t), $TF^t = \mathfrak{M}$, then the Ehresmann connection is called integrable, while (M, F, F^t) is called bifoliation.

We consider a category of foliations with an integrable Ehresmann connection, where the isomorphisms preserve not only foliations, but also their Ehresmann connections. In fact, the study of foliations (M, F) with integrable Ehresmann connection is equivalent to studying bifoliations (M, F, F^t). By applying the Kashiwabara theorem [2], we show that the space of the universal covering $f : \widetilde{M} \to M$ is the product of manifolds $\widetilde{M} = L \times N$, while the leaves of the induced bifoliation on $\widetilde{M}$ are formed by the leaves of the product $L \times N$. Such foliations are called covered product. An opposite is true as well, if (M, F_1, F_2) is a bifoliation covered by a product, then TF_2 is an integrable Ehresmann connection for the foliation (M, F_1), while TF_1 is an integrable Ehresmann connection for (M, F_2). Thus, we identify the category of foliations with an integrable Ehresmann connection with the category $\mathfrak{BiF}$ of bifoliation covered by a product.

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We stress that the foliations covered by a product arise naturally in solving various problems, see Section 5, in particular, in studying various G-structures on manifolds. For instance, on compact Kähler manifolds [3], on holomorphic Poisson manifolds [4], on projection manifolds [5], in studying bi-Lagrange foliations on symplectic manifolds [6].

We develop the method by Ya.L. Shapiro [7], [8], which he used for describing the structure of complete reducible Riemannian manifolds, and we apply this method in the category of foliations with integrable Ehresmann connection. We introduce the notation of a canonical bifoliation, see Section 3.3, and we prove the following theorem.

Theorem 1.1. *Let (M, F_1) be a foliation with an integrable Ehresmann connection $\mathfrak{M}_1$ and (M, F_2) be a foliation such that $TF_2 = \mathfrak{M}_1$. Then $\mathfrak{M}_2 = TF_1$ is an integrable Ehresmann connection for (M, F_2) and*

1. *the bifoliation (M, F_1, F_2) is covered by the product;*
2. *there exists a $\mathfrak{M}_i$-holonomic covering X_i for the leaves of the foliation (M, F_i), where $i = 1, 2$;*
3. *on the product of the manifolds $X_1 \times X_2$, a free proper discontinuous diagonal action of a group of diffeomorphisms Ψ is well-defined and this group is isomorphic to a quotient group $G/(G_{11} \times G_{22})$ of the fundamental group $G = \pi_1(M)$ over the product of normal subgroups G_{ii} isomorphic to fundamental groups $\pi_1(X_i)$;*
4. *in the category $\mathfrak{BiF}$, the bifoliation (M, F_1, F_2) is isomorphic to some canonical bifoliation $((X_1 \times X_2)/\Psi, F_1, F_2)$.*

Remark 1.1. *If in X_2 there exists a point fixed with respect to the action of only the unit element in the group Ψ, then X_1 is a leaf in the foliation (M, F_1) with a trivial group of $\mathfrak{M}_1$-holonomy. If a foliation (M, F_1) possesses a quasianalytic holonomy pseudogroup, then the manifold X_1 is diffeomorphic to each fiber of the foliation (M, F_1) with a trivial germ holonomy. As it is known* [9]*, the set of such leaves is a dense G_δ-subset in M.*

We introduce a category $\mathfrak{T}$ of triples (X_1, X_2, Ψ) and we prove the following theorem.

Theorem 1.2. *Let $\xi = (M, F_1, F_2)$ and $\xi' = (M', F_1', F_2')$ be bifoliations covered by product and isomorphic (in $\mathfrak{BiF}$) to canonical bifoliations $((X_1 \times X_2)/\Psi, F_1, F_2)$ and $((X_1' \times X_2')/\Psi', F_1', F_2')$, respectively. Then the bifoliations ξ and ξ' are isomotphic in the category $\mathfrak{BiF}$ if and only if the triples (X_1, X_2, Ψ) and (X_1', X_2', Ψ') are isomorphic in the category $\mathfrak{T}$.*

Corollary 1.1. *An equivalence class $[(X_1, X_2, \Psi)]$ of the triples in the category $\mathfrak{T}$ is a complete invariant of the bifoliation $\xi = (M, F_1, F_2)$. In particular, the structural group Ψ of the bifoliation ξ is its algebraic invariant.*

Thus, owing to Theorem 1.2, the classification of the foliations with an integrable Ehresmann connection is reduced to the classification of triples (X_1, X_2, Ψ) defined in the aforementioned way by these foliations and their Ehresmann connections, that is, to the classification of the associated diagonal actions of discrete groups of the diffeomorphisms Ψ on the product of manifolds $X_1 \times X_2$.

2. Holonomy groups of foliations with Ehresmann conection

2.1. Ehresmann connection for foliations. The notion of the Ehresmann connection for foliations (M, F) was introduced by R.A. Blumenthal and J.J. Hebda [1] as a natural generalization of the Ehresmann connection for submersions.

Let (M, F) be a foliation of codimension q and $\mathfrak{M}$ be a smooth q-dimensional distribution on n-dimensional smooth manifold M transversal to the foliation (M, F). This means that at each point $x \in M$ the tangential vector space T_xM to M is decomposed into a direct sum of vector subspaces $T_xM = T_xF \oplus \mathfrak{M}_x$, where T_xF is a tangential space to a leaf of the foliation (M, F), and $\mathfrak{M}_x$ is the value of the distribution $\mathfrak{M}$ at a point x. Piece-wise smooth integral curves of the distribution $\mathfrak{M}$ are called *horizontal,* while piece-wise smooth curves in the leaves of the foliation are called *vertical.* Let $I_1 = I_2 = I = [0, 1]$. A continuous mapping of the square $I_1 \times I_2$ into M is called piece-wise smooth if there exist partitions of the segments I_1 and I_2: $0 = s_0 < s_1 < \ldots < s_{m-1} < s_m = 1$ and $0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = 1$, respectively, such that for each $i = \overline{1, m}$, $j = \overline{1, k}$, the restriction $H|_{[s_{i-1}, s_i] \times [t_{j-1}, t_j]}$ is a smooth mapping. A piece-wise smooth mapping H of the square $I_1 \times I_2$ into M

is called *vertical-horizontal homotopy*, if the curve $H|_{\{s\}\times I_2}$ is vertical for each $s \in I_1$ and the curve $H|_{I_1\times\{t\}}$ is horizontal for each $t \in I_2$, see Figure 2.1. In this case the pair of paths $(H|_{I_1\times\{0\}}, H|_{\{0\}\times I_2})$ is called a *base* of H. It is known that there exists at most one vertical-horizontal homotopy with a given base.

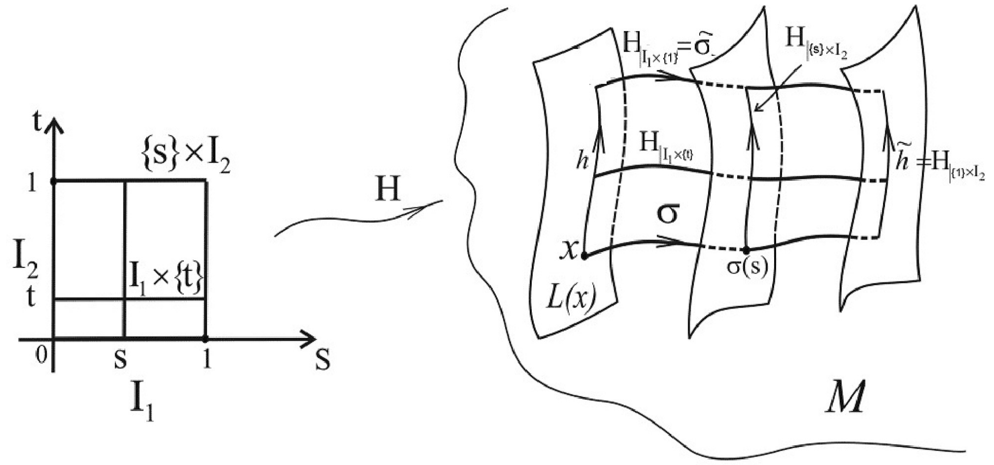


Figure 2.1: Mapping H is a vertical-horizontal homotopy with base (σ, h)

Remark 2.1. *We follow the terminology of R. Hermann and use the term vertical-horizontal homotopy. S. Kashiwabara* [2] *call such mapping latticed, while R.A. Blumenthal and J.J. Hebda* [1] *call it rectangle.*

A pair of paths (σ, h) in M with a common initial point $\sigma(0) = h(0)$ is called *admissible* if σ is a horizontal curve, and h is a vertical curve.

Definition 2.1. *A distribution $\mathfrak{M}$ is called an Ehresmann connection for a foliation (M, F) if for each admissible pair of paths (σ, h) in M there exists a vertical-horizontal homotopy H with the base (σ, h).*

We recall that a smooth q-dimensional distribution $\mathfrak{M}$ on a n-dimensional manifold M is called *integrable* if through each point $x \in M$, a q-dimensional integral manifold of this distribution passes. As it is known, a distribution $\mathfrak{M}$ is integrable if and only if it satisfies the Frobenius condition, that is, at each point in M there exists a neighbourhood U and q smooth vector fields $X_1, \ldots, X_q$, which form a basis in the tangential vector space $\mathfrak{M}_x$ at each point $x \in U$ and satisfies the condition

$$[X_i, X_j] = C_{ij}^k X_k, \qquad i, j, k = 1, \ldots, q,$$

where C_{ij}^k are smooth functions on U.

Definition 2.2. *Let (M, F) be a foliation admitting an Ehresmann connection $\mathfrak{M}$. If the distribution $\mathfrak{M}$ is integrable then $\mathfrak{M}$ is called an integrable Ehresmann connection for the foliation (M, F).*

2.2. Groups of $\mathfrak{M}$-holonomy. Let (M, F) be a foliation with an Ehresmann connection $\mathfrak{M}$. Let us consider an admissible pair of paths (δ, τ). We say that the curve $\widetilde{\delta}$ is obtained *by a translation of the path δ along τ with respect to the Ehresmann connection $\mathfrak{M}$* if $\widetilde{\delta} := H|_{I\times\{1\}}$. We denote this translation by $\delta \overset{\tau}{\to}> \widetilde{\delta}$.

We denote by Ω_x, $x \in M$, the set of horizontal curves with origin at the point x. We define the action of a fundamental group $\pi_1(L, x)$ of a leaf $L = L(x)$ on the set Ω_x as follows: $\Phi_x : \pi_1(L, x) \times \Omega_x \to \Omega_x : ([h], \sigma) \mapsto \tilde{\sigma}$, where $[h] \in \pi_1(L, x)$, and $\tilde{\sigma}$ is the translation of the curve $\sigma \in \Omega_x$ along h with respect to $\mathfrak{M}$, see Figure 2.2. We stress that the action Φ_x is well-defined since the result depends only on the homotopic class of the loop h. Let $K_{\mathfrak{M}}(L, x)$ be the kernel of the action Φ_x, that is,

$K_{\mathfrak{M}}(L,x) = \{\alpha \in \pi_1(L,x) \,|\, \alpha(\sigma) = \sigma \;\; \forall \sigma \in \Omega_x\}$. The quotient group $H_{\mathfrak{M}}(L,x) = \pi_1(L,x)/K_{\mathfrak{M}}(L,x)$ is called a *group of* $\mathfrak{M}$*-holonomy* of the leaf L [1]. By the linear connectivity of the leaves, the groups of $\mathfrak{M}$-holonomy at various points of the same leaf are isomorphic. Let $\Gamma(L,x)$ be the germ holonomy group of the leaf L conventionally used in the foliation theory [10]. Then the epimorphism of the groups $\chi : H_{\mathfrak{M}}(L,x) \to \Gamma(L,x)$ is well-defined which satisfies the identity

$$\chi \circ \beta = \gamma, \tag{2.1}$$

where $\beta : \pi_1(L,x) \to H_{\mathfrak{M}}(L,x)$ is a quotient map and $\gamma([h]) :=< h >$ is the germ of a local holonomic diffeomorphism of a transversal q-dimensional disc along the loop h at the point x.

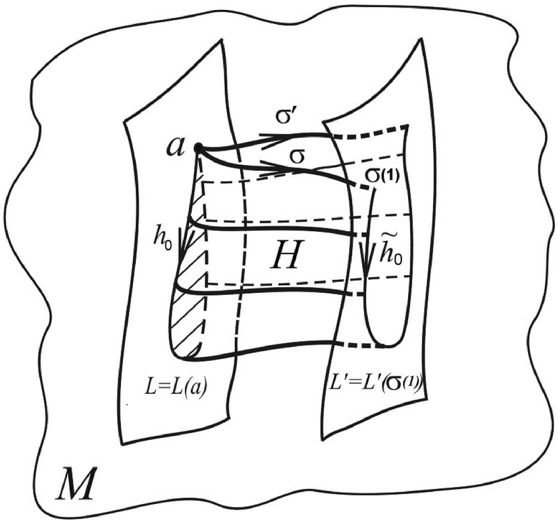


Figure 2.2: Curve $\sigma' \in \Omega_a$, which is the result of the action of an element $[h_0] \in \pi_1(L,a)$ on a curve $\sigma \in \Omega_a$

2.3. Criterion of isomorphism of holonomy groups of foliations with Ehresmann connection to germ holonomy groups. Let (M,F) be a smooth regular foliation of codimension q on an n-dimensional smooth manifold M. Assume that (M,F) admits an Ehresmann connection $\mathfrak{M}$. At each point $a \in M$ two holonomy groups are defined, a germ group $\Gamma(L,a)$ and a group of $\mathfrak{M}$-holonomy $H_{\mathfrak{M}}(L,a)$ of foliation (M,F) with the Ehresmann connection $\mathfrak{M}$. Our aim is to find necessary and sufficient conditions ensuring that the natural epimorphism of the groups

$$\chi_a \colon H_{\mathfrak{M}}(L,a) \to \Gamma(L,a)$$

satisfying a commutative diagram

$$\begin{array}{ccc} & \pi_1(L,a) & \\ {}^{\beta_a}\swarrow & & \searrow^{\gamma_a} \\ H_{\mathfrak{M}}(L,a) & \xrightarrow{\quad \chi_a \quad} & \Gamma(L,a), \end{array} \tag{2.2}$$

where β_a and γ_a are corresponding quotient mappings, is an isomorphism.

Defining of foliation by Haefliger cocyle. Let T be a smooth q-dimensional, probably non-connected, manifold. A T*-cocycle* or *Haefliger cocycle* is a family $\theta = \{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$, satisfying the conditions:

(H_1) $\{U_i, i \in J\}$ is an open cover of the manifold M;

(H_2) $f_i \colon U_i \to T$ are submersions;

(H_3) if $U_i \cap U_j \neq \emptyset$, then the diffeomorphism $\gamma_{ij} \colon f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$, is well-defined, which satisfies the identity $f_i(x) = \gamma_{ij} \circ f_j(x)$ for all $x \in f_j(U_i \cap U_j)$;

(H_4) if $U_i \cap U_j \cap U_k \neq \emptyset$, then for each $x \in f_k(U_i \cap U_j \cap U_k)$ the identity $(\gamma_{ij} \circ \gamma_{jk})(x) = \gamma_{ik}(x)$ holds and moreover, $\gamma_{ii} = id_{U_i}$.

Two T-cocycles are called *equivalent* if their union is also a T-cocycle. An equivalence class of T-cocycles is called a *foliation of codimension* q *on the manifold* M. Each T-cocycle $\theta = \{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$

belongs to a unique equivalence class of T-cocycles and this is why it defines a foliation on M. We say that a foliation *is defined by the cocycle* θ. A set Σ of all leaves of the submersions f_i in the equivalence class of a T-cocycle is a base of some topology τ_F on M, which is called *leaf topology*. The components of the linear connectivity of a topology space (M, τ_F) form a partition $F := \{L_\alpha \,|\, \alpha \in \mathcal{A}\}$ of the manifold M, which is called *a foliation of codimension* q and is denoted by (M, F).

Pseudogroup of holonomy of foliation. Let T be a smooth manifold, which is not supposed to be connected.

We recall that a *pseudogroup* $\mathcal{G}$ *of the transformations of the manifold* T is a family of diffeomorphisms $g\colon U \to V$, where U and V are open subsets in T obeying the following conditions:

1) if $g \in \mathcal{G}$, then $g^{-1} \in \mathcal{G}$;

2) if $g\colon U \to V$ and $g'\colon U' \to V'$ belong to $\mathcal{G}$, then $g' \circ g\colon U \cap g'^{-1}(V') \to V'$ also belong to the family $\mathcal{G}$;

3) $id_T \in \mathcal{G}$;

4) if the family $\mathcal{G}$ contains $g\colon U \to V$, it also contains the restriction of $g|_{U'}$ to each open subset U' in U;

5) if $h\colon U \to V$ is a diffeomorphism between the open subsets U and V in T, coinciding in the vicinity of each point in U with some element in $\mathcal{G}$, then $h \in \mathcal{G}$.

A family of local diffeomorphisms $B = \{\gamma_i\}_{i\in J}$ of the manifold T containing id_T is said to generate a pseudogroup of diffeomorphisms $\mathcal{G}$ if it belongs to $\mathcal{G}$ and each element in $\mathcal{G}$ is obtained from the elements in B in one of the following ways: applying the inverse mapping, restriction to an open subset, composition or continuation, that is, as in 5). Then we say that the pseudogroup $\mathcal{G}$ is generated by set B and we denote $\mathcal{G} = \langle\gamma_i\rangle_{i\in J}$.

Assume that a foliation (M, F) is defined by a T-cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j\in J}$. A pseudogroup $\mathcal{G} = \langle\gamma_{ij}\rangle_{i,j\in J}$ of diffeomorphisms of a transversal manifold T is called a *pseudogroup of the holonomy of the foliation* (M, F).

The domain of $g \in \mathcal{G}$ is denoted by $\mathcal{O}(g)$. The set

$$\mathcal{G}.x := \{g(x) \mid g \in \mathcal{G}, \quad x \in \mathcal{O}(g)\}$$

is called an orbit of a point x with respect to a pseudogroup $\mathcal{G}$. Let

$$\mathcal{G}_x := \{g \in \mathcal{G} \mid g(x) = x,\, x \in \mathcal{O}(g)\}.$$

By the symbol $\{g\}_x$, $g \in \mathcal{G}$, we denote the germ of local diffeomorphisms at a point $x \in T$, while $\Gamma\mathcal{G}_x$ stands for the group of germs $\{\{g\}_x,\, g \in \mathcal{G}_x\}$ in x of local diffeomorphisms in $\mathcal{G}_x$, each being defined in some neighbourhood of the point x.

Lemma 2.1. *Let (M, F) be a foliation defined by a cocycle $\{U_i, f_i, \gamma_{ij}\}_{i,j\in J}$ and $\mathcal{G} = \langle\gamma_{ij}\rangle_{i,j\in J}$ be its pseudogroup of holonomy. Let $h\colon I \to L(a)$ be a piece-wise smooth loop at a point a and $\xi = \{U_{i_1}, \ldots, U_{i_m}\}$ be a chain covering the curve $h(I)$, $U_{i_{k-1}} \cap U_{i_k} \neq \emptyset$, $k = 2, \ldots, m$, where $a \in U_{i_1} \cap U_{i_m}$. Then in some neighbourhood of the point $b := f_{i_1}(a)$ a composition of mappings $\gamma := \gamma_{i_1 i_m} \circ \gamma_{i_{m-1} i_m} \circ \ldots \circ \gamma_{i_2 i_1} \in \mathcal{G}$ is well-defined and $\gamma(b) = b$. The mapping*

$$\mu_a\colon \Gamma(L, a) \to \Gamma\mathcal{G}_b\colon \{H_h\}_a \mapsto \{\gamma\}_b,$$

where γ corresponds to the chain ξ covering $h(I)$ is independent of the choice of the covering ξ and is an isomorphism of the groups.

Proof. Let $\xi' = \{U_{j_1}, \ldots, U_{j_k}\}$ be a refinement of the covering ξ, that is, a covering $h(I)$ such that each $U_{j_r} \in \xi'$ is a subset of some $U_{i_r} \in \xi$. It is easy to confirm that under the refinement of the covering ξ, as well as under the adding of new elements to the covering ξ, the germs $\{H_h\}_a$ and $\{\gamma\}_b$ do not change. If $\eta = \{V_{l_1}, \ldots, V_{l_s}\}$ is another covering of the curve $h(I)$, then by passing to a joint refinement for both covering ξ and η we see that μ_a is well-defined, that is, it is independent on the choice of the chain ξ.

Since the composition of the germs is defined via the compositions of the corresponding diffeomorphisms, then μ_a is a homomorphism of the groups.

It follows from the definition μ_a that $\{\gamma \circ f_{i_m}\}_a = \{f_{i_1} \circ H_h\}_a$, and therefore, if $\{\gamma\}_b = \{id\}_b$, then $\{\gamma\}_b = \{\gamma_{i_1 i_1}\}_b$. This is why $\{H_h\}_a = \{id\}_a$, that is, $\ker\mu_a = \{id\}_a$ and μ_a is a monomorphism of the groups.

Let us show that μ_a is an epimorphism of the groups. Let $\gamma \in \mathcal{G}$, $\gamma(b) = b$. Then it follows from the definition of the pseudogroup of the holonomy $\mathcal{G} = \langle \gamma_{ij} \rangle_{i,j \in J}$ that in some neighbourhood of the point b, $\gamma(b) = b$, the transformation γ coincides with the composition of the generators and therefore, it corresponds to some chain ξ in the aforementioned way. Thus, μ_a is surjective. □

Corollary 2.1. *If $\{H_h\}_a = \{id\}_a$, where $H_h \colon D_a \to D_a$ is a holonomic diffeomorphism along the loop h, then for any another transversal disk D'_a the holonomic diffeomorphism H'_h of some neighbourhood $V_a \subset D'_a$ in D'_a satisfies the identity $\{H'_h\}_a = \{id\}_a$.*

Quasianalytic pseudogroups of transformations. We shall say that the diffeomorphism $\gamma \colon U \to V$ of open sets U and V is *quasianalytic* if the existence of a connected open subset $U_0 \subset U$ such that $\gamma|_{U_0} = id_{U_0}$ yields $\gamma = id_{U'}$, where U' is the connected component of U containing U_0.

A pseudogroup of local diffeomorphisms $\mathcal{G}$ of a probably non-connected manifold T is said to be *quasianalytic* if each transformation $\gamma \in \mathcal{G}$ is quasianalytic.

We prove the following theorem for arbitrary foliations with Ehresmann connection not using the notion of paths and foliations consistent with system of paths in contrast to the proof by N.I. Zhukova in [11].

Theorem 2.1. *Let (M, F) be a foliation with Ehresmann connection. A natural epimorpism of the groups of holonomy*

$$\chi_a \colon H_{\mathfrak{M}}(L, a) \to \Gamma(L, a)$$

is an isomorphism for each point $a \in M$ if and only if a holonomic pseudogroup of diffeomorphisms $\langle \gamma_{ij} \rangle_{i,j \in J}$ of the foliation (M, F) is quasianalytic.

Proof. Sufficiency. Assume that $\mathfrak{M}$ is an Ehresmann connection for the foliation (M, F) defined by the cocycle $\theta = \{U_i, f_i, \gamma_{ij}\}_{i,j \in J}$ with a quasianalytic pseudogroup of diffeomorphisms $\mathcal{G} = \langle \gamma_{ij} \rangle_{i,j \in J}$ and $\chi_a \colon H_{\mathfrak{M}}(L, a) \to \Gamma(L, a)$ is the natural epimorphism satisfying diagram (2.2) and $[h] \in \beta_a^{-1}(\ker \chi_a)$. This means that the loop h in the leaf $L = L(a)$ induces a holonomic mapping H_h coinciding with the identity mapping in some neighbourhood V_a of the transversal disk D_a to the foliation (M, F). We denote the germ of the diffeomorphism H_h at the point $a = h(0) = h(1)$ by $\{H_h\}_a$, while the trivial germ, that is the germ of the identity diffeomorphism is denoted by $\{H_h\}_a = \{id\}_a$. We take an arbitrary horizontal curve $\sigma \in \Omega_a$ and we let $\sigma \overset{h}{\longrightarrow\!\!>} \tilde{\sigma}$ and $h \overset{\sigma|_{[0,s]}}{\longrightarrow\!\!>} h_s$. We introduce $N := \{s \in [0, 1] \mid h_s(0) = h_s(1), \{H_{h_s}\}_{\sigma(s)} = \{id\}_{\sigma(s)}\}$, that is, N is the set of points in the segment $[0, 1]$, where the path h_s generates the trivial germ $\{H_{h_s}\}$ at the point $h_s(0) = \sigma(s)$. We observe that at each point $\sigma(s)$ there exists a transversal disk $D_{\sigma(s)}$ and a number $\delta = \delta(s) > 0$ such that $\sigma((s-\delta, s+\delta)) \subset D_{\sigma(s)}$. Assume that $s \in N$, then by the definition of N, the germ $\{H_{h_s}\}_{\sigma(s)}$ is trivial. According to Corollary 2.1, this germ is independent of the choice of the transversal disk at a point $\sigma(s)$ and this is why $\{H'_{h_s}\}_{\sigma(s)} = \{id\}_{\sigma(s)}$, where H'_{h_s} is a holonomic diffeomorphism along the path h_s in some neighbourhood $V_0 \subset D_{\sigma(s)}$ of the point $\sigma(s)$ of the transversal disk $D_{\sigma(s)}$. We suppose that $V_0 = D_{\sigma(s)}$ since otherwise we can achieve this by lessening the disk. The definition of a holonomic diffeomorphism H'_{h_s} implies that for each $s' \in (s-\delta, s+\delta)$ the identity $H'_{h'_s} = H'_{h_s}$ holds and this is why $\{H'_{h'_s}\}_{\sigma(s')} = \{id\}_{\sigma(s')}$. This yields $(s-\delta, s+\delta) \subset N$, where $\delta = \delta(s)$. Thus, N is an open subset of the segment $[0, 1]$.

Let us show that N is closed in $[0, 1]$. Let s_0 belong to the closure $\overline{N}$ in $[0, 1]$ and $\{s_n\}$ be a sequence in N converging to s_0 as $n \to \infty$. Since

$$\tilde{\sigma}(s_n) = h_{s_n}(0) = h_{s_n}(1) = \sigma(s_n),$$

then owing to the continuity of $\tilde{\sigma}$ and σ, the identity holds $\tilde{\sigma}(s_0) = \sigma(s_0)$, which means that h_{s_0} is a loop at the point $\sigma(s_0)$. Let $D_0 = D_{\sigma(s_0)}$ be a transversal disk at the point $\sigma(s_0)$, and $H'_{h_{s_0}}$ be a holonomic diffeomorphism defined in the neighbourhood $W_0 \subset D_0$ of the point $\sigma(s_0)$. Then there exists a number $\delta_0 > 0$ such that $\sigma(s) \in W_0$ for all $s \in (s_0 - \delta_0, s_0 + \delta_0)$. Since $s_0 \in \overline{N}$, then there exists $s_1 \in (s_0 - \delta_0, s_0 + \delta_0)$, for which $\{H_{h_{s_1}}\}_{\sigma(s_1)} = \{H'_{h_{s_1}}\}_{\sigma(s_1)} = \{id\}_{\sigma(s_1)}$. At that, according with Lemma 2.1, $\mu_{\sigma(s_0)}(\{H'_{h_{s_1}}\}_{\sigma(s_1)}) = \{\gamma\}_{\sigma(s_1)} = \{id\}_{\sigma(s_1)}$. By the quasianalyticity $\gamma \in \mathcal{G}$ this yields that $\gamma = id_V$ everywhere in a connected component of the domain of V. Therefore, $\{H'_{h_{s_0}}\}_{\sigma(s_0)} = \{\gamma\}_{\sigma(s_0)} =$

$\{id\}_{\sigma(s_0)}$, that is, $s_0 \in N$ and $N = \overline{N}$. By the choice of h, the set N contains the zero. Thus, N is a non-empty open and closed subset of the connected segment $[0,1]$ and hence $N = [0,1]$. This means that $\tilde{\sigma} = \sigma$ and $\beta_a^{-1}(\ker \chi_a) = \ker \beta_a$, that is, the kernel $\ker \chi_a$ is trivial and χ_a is the isomorphism of the groups.

To prove the necessity we assume that $\chi_a\colon H_{\mathfrak{M}}(L,a) \to \Gamma(L,a)$ is an isomorphism of the groups, that is, the kernel $\ker \chi_a$ is trivial and $\ker \beta_a = \ker \gamma_a$, where β_a and γ_a are epimorphisms satisfying commutative diagram (2.2). Without loss of generality we assume that the foliation (M,F) with the Ehresmann connection $\mathfrak{M}$ is defined by a cocycle $\theta = \{U_i, f_i, \gamma_{ij}\}_{i,j\in J}$, where $\mathfrak{M}_i := \mathfrak{M}|_{U_i}$ is the Ehresmann connection for the submersion $f_i\colon U_i \to V_i := f_i(U_i)$.

Let us show first that each composition $\gamma = \gamma_{kk-1} \circ \ldots \circ \gamma_{32} \circ \gamma_{21}\colon U \to V$ of the generators of the pseudogroup $\mathcal{G}$ is quasianalytic. Suppose that there exists a connected open subset $U_0 \subset U$ such that $\gamma|_{U_0} = id_{U_0}$. We take a point $b \in U_0$. For simplicity, the connected component U containing U_0 is denoted by the same symbol U. Then $b \in U_0$ can be connected with an arbitrary point $x \in U$ by a smooth curve ϕ in U, $\phi(0) = b$, $\phi(1) = x$. We note that $U \subset f_1(U_1)$, $V \subset f_k(U_k)$. Since γ_{i-1i}, $i = 1,\ldots,k-1$, are well-defined, then $U_{i-1} \cap U_i \neq \emptyset$ and this is why the neighbourhood $U_1, U_2, \ldots, U_k$ in the cocycle θ form a chain. Taking into consideration that $\gamma(b) = b$, we obtain $U_1 \cap U_k \neq \emptyset$. Therefore, there exists a point $a \in f_1^{-1}(b) \cap f_k^{-1}(b)$. Let σ be a lift of the path ϕ to the point a with respect to the Ehresmann connectivity for the submersion $\mathfrak{M}|_{U_1}$. Then $f_1 \circ \sigma = \phi$, $\sigma \in \Omega_a$. We denote $b_1 = b$, $b_i := \gamma_{ii-1}(b_{i-1})$, $i = 2,\ldots,k$, and at that, $b_k = b_1 = b$. Moreover, there is a path h in the leaf $L = L(a)$ belonging to the union of local leaves $\cup_{i=1}^k f_i^{-1}(b_i)$ closed at the point a. We note that a holonomic diffeomorphism H_h of some neighbourhood of the point a in the transversal disk D_a satisfies the identity $f_k \circ H_h = \gamma \circ f_1$. Then, taking into consideration that $\{\gamma\}_b = \{id\}_b$, we get $\{H_h\}_a = \{id\}_a$; therefore, $[h] \in \ker \gamma_a$. According to the definition, $\ker \beta_a = \ker \gamma_a$, and this is why $[h] \in \ker \beta_a$ and the lift of σ along h does not change σ, that is, if $\sigma \overset{h}{\longrightarrow}> \tilde{\sigma}$, then $\tilde{\sigma} = \sigma$. Let $h \overset{\sigma}{\longrightarrow}> h_1$ and $h_1(1) = \sigma(1) = c$. At that, $f_1(c) = f_1(\sigma(1)) = \phi(1) = x$.

Let us show that $[h_1] \in \ker \beta_c$. We assume the opposite, then there exists a curve $\sigma \in \Omega_c$ possessing the property $\sigma' \overset{h_1}{\longrightarrow}> \tilde{\sigma}' \neq \sigma'$. Then $\sigma\sigma' \overset{h}{\longrightarrow}> \sigma\tilde{\sigma}' \neq \sigma\sigma'$, where $\sigma\sigma'$ and $\sigma\tilde{\sigma}'$ are the products of the corresponding paths and this contradicts to the belonging of $[h]$ to the kernel $\ker \beta_a$. Thus, $[h_1] \in \ker \beta_c$. By assumption, $\ker \beta_c = \ker \gamma_c$, and therefore, $[h_1] \in \ker \gamma_c$. This means that $\{H_{h_1}\}_c = \{id\}_c$, where H_{h_1} is a holonomic diffeomorphism along the loop h_1 of some neighbourhood of the point c in the transversal disk D_c. Taking into consideration that the curve $h_1(I)$ is covered by the same chain of the neighbourhoods $U_1, \ldots, U_k$, as $h(I)$ and applying Lemma 2.1, we obtain the identity $\{\gamma\}_{x=f_1(c)} = \{id\}_x$. Thus, γ coincides with id_U. The proven fact implies the quasianalyticity of each generator γ_{ij} of the holonomic pseudogroup $\mathcal{G}$.

Now let γ be an arbitrary element in $\mathcal{G}$. If γ is an inverse element to the generator, then its quasianalyticity is a direct implication of the quasianalyticity of the generator.

Let γ be the union of the generators, that is, at each point in the domain of γ there exists a neighbourhood, in which γ coincides with one of its generators. Assume that V is a connected component of the domain of γ and there exists a neighbourhood $V_0 \subset V$ such that $\gamma|_{V_0} = id_{V_0}$. We connect a point $b \in V_0$ with an arbitrary point $y \in V$ by a smooth curve ψ in V and we consider a finite chain of neighbourhoods $V_1, \ldots, V_m$, $V_i \cap V_{i+1} \neq \emptyset$, covering $\psi(I)$ and possessing a property that $\gamma|_{V_i}$ coincides with one of the generators $\mathcal{G}$. We denote by $\gamma^{(i)}$ the generator in $\mathcal{G}$, which coincides with $\gamma|_{V_i}$, it is possible that different i are associated with the same generator. Since $V_1 \cap V_0$ is an open subset, on which $\gamma^{(1)}$ coincides with the identity mapping, that is, the quasianalyticity $\gamma^{(1)}$ implies $\gamma^{(1)}|_{V_1} = id_{V_1}$. Since $\gamma^{(1)}|_{V_1\cap V_2} = \gamma^{(2)}|_{V_1\cap V_2}$, owing to the quasianalyticity of $\gamma^{(2)}$, this implies $\gamma^{(2)}|_{V_2} = id_{V_2}$. Continuing these arguing, we see that $\gamma|_{V_m} = id_{V_m}$. By the arbitrariness of $y \in V$ this yields $\gamma|_V = id_V$. This completes the checking of quasianalyticity of the pseudogroup $\mathcal{G}$. □

3. Bifoliations covered by product

3.1. Kashiwabara theorem. Let (M,F_1) and (M,F_2) be foliations of codimension q_1 and q_2, respectively, and $q_1 + q_2 = n$. If these foliations are transversal, that is, $T_xM = T_xF_1 \oplus T_xF_2$ at each point $x \in M$, then this pair of foliation is *called bifoliation* and is denoted by (M,F_1,F_2), or, for brevity, by (F_1,F_2).

Let (M, F) be a foliation admitting an integrable Ehresmann connection $\mathfrak{M}$ on an n-dimensional manifold M. The set of maximal integral manifolds of the distribution $\mathfrak{M}$ forms a foliation (M, F^t). At that, (M, F, F^t) is a bifoliation.

We denote by $\mathfrak{N}$ a distribution tangential to the leaves of the foliation (M, F). We stress that if $\mathfrak{M}$ is an integrable Ehresmann connection for the foliation (M, F), then $\mathfrak{N}$ is an integrable Ehresmann connection for the foliation (M, F^t).

We recall that the foliation (M, F) has an integrable Ehresmann connection $\mathfrak{M}$ if and only if the bifoliation (M, F, F^t) possesses the following property. For each admissible pair of paths (σ, h) there exists a piece-wise smooth mapping $H\colon I_1 \times I_2 \to M$ such that for each fixed $s \in I_1$ the restriction $H|_{\{s\}\times I_2}$ is a curve in the leaf $L(\sigma(s))$ of the foliation (M, F) passing the point $\sigma(s)$ and for each fixed $t \in I_2$ the restriction $H|_{I_1\times\{t\}}$ is a curve in the leaf $N(h(t))$ of the foliation (M, F^t) passing the point $h(t)$.

Let $L \times N$ be the product of the manifolds M and N, then the triple $(L \times N, \mathcal{F}_1, \mathcal{F}_2)$, where $\mathcal{F}_1 := \{L \times \{y\},\ y \in N\}$, $\mathcal{F}_2 := \{\{z\} \times N,\ z \in L\}$, is called *canonical bifoliations of the product* $L \times N$.

S. Kashiwabara in [2] proved a theorem, which we formulate here in a form convenient for us. By $\mathfrak{Fol}$ we denote a category of foliations, the morphisms of which are smooth mappings transforming the leaves of one foliation into ones of another.

Theorem (Kashiwabara theorem)**.** *Let $\mathfrak{M}$ be an integrable Ehresmann connection for the foliation (M, F) and (M, F^t) is a foliation such that $TF^t = \mathfrak{M}$. Let $L = L(x)$, $N = N(x)$ be leaves of the foliations (M, F) and (M, F^t), respectively, passing an arbitrary point $x \in M$. If the manifold M is simply-connected, then there exists a diffeomorphism*

$$\Theta\colon M \to L \times N$$

being an isomorphism in the category $\mathfrak{Fol}$ as of foliations (M, F) and $(L \times N, \mathcal{F}_1)$, so of foliations (M, F^t) and $(L \times N, \mathcal{F}_2)$, where $(L \times N, \mathcal{F}_1, \mathcal{F}_2)$ is the canonical bifoliation of the product $L \times N$.

Definition 3.1. *A bifoliation (M, F, F^t) is called a bifoliation covered by a product if there exists a covering $\kappa\colon X_1 \times X_2 \to M$ of the product of the manifolds $X_1 \times X_2$ onto M, which maps the leaves of the product into the corresponding leaves of the foliations (M, F) and (M, F^t).*

Applying Kashiwabara theorem, it is easy to show that the following statement is true.

Proposition 3.1. *A bifoliation (M, F, F^t) is covered by a product if and only if the distribution $\mathfrak{M} = TF^t$ is an integrable Ehresmann connection for the foliation (M, F).*

In [7], [8] Ya.L. Shapiro call M with the bifoliation (F, F^t) covered by a product as a reducible manifold, while the pair (F, F^t) is called its two-sheeted structure.

3.2. Lemmata. Let G be a group acting smoothly on a manifold X on the left as the group of diffeomorphism. An action G is called *proper discontinuous* if it satisfies the following conditions:

1) if x and y do not belong to the same orbit of the group G, then they posses respectively neighbourhoods U_x and U_y such that $g(U_x) \cap U_y = \emptyset$ for each $g \in G$;

2) a stationary subgroup G_x at each point $x \in X$ is finite;

3) for each point $x \in X$ there exists a neighbourhood U satisfying the identities $g(U) = U$ for all $g \in G_x$ and $g(U) \cap U = \emptyset$ for all $g \in G \setminus G_x$.

We recall that the group G acts X freely if $G_x = \{id_X\}$ for each $x \in X$.

If G is a proper discontinuous group of diffeomorphisms acting freely on a differentiable manifold X, then the quotient space X/G possesses the structure of differentiable manifold and the natural projection $\pi\colon X \to X/G$ is a smooth regular covering mapping.

The following lemma can be proved easily.

Lemma 3.1. *Let G be a proper discontinuous group of the diffeomorphisms of a manifold X acting freely and H be its normal divisor. Then on the quotient manifold $X' := X/H$, the group of diffeomorphisms Ψ, isomorphic to the quotient group G/H, acts freely and properly discontinuous. There exists* *a diffeomorphism $\Xi\colon X/G \to X'/\Psi$ obeying a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & X/H \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\ X/G & \xrightarrow{\Xi} & X'/\Psi, \end{array}$$

where f, π, π' are natural projections being regular covering mappings with the groups of covering transformations H, G, Ψ, respectively.

A diffeomorphism $\tilde{f}$ of a manifold $\tilde{X}$ is said to lie over the diffeomorhpism f of the manifold X with respect to the mapping $\pi\colon \tilde{X} \to X$ if the following diagram is commutative:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ X & \xrightarrow{f} & X. \end{array}$$

Lemma 3.2. *Let $\kappa\colon \tilde{X} \to X$ be the universal covering mapping and Γ be a group of covering transformations κ. Then*

1) over each diffeomorphism g of the manifold X, some diffeomorphism $\tilde{g}$ of the manifold $\tilde{X}$ lies;

2) a diffeomorphism $\tilde{g}$ of the manifold $\tilde{X}$ lies over some diffeomorphism of the manifold X if and only if $\tilde{g} \circ \Gamma = \Gamma \circ \tilde{g}$;

3) the set of all diffeomorphisms of the manifold $\tilde{X}$ lying over the diffeomorphisms in the group G forms a group $\tilde{G}$ and the quotient group $\tilde{G}/G$ is isomorphic to the group Γ.

Proof. This statement can be proved similar to the proofs of Theorems 28.10 and 28.7 for Buzeman G-spaces in [12]. □

If $\tilde{G}$ and G are related by property 3) in Lemma 3.2, then we say that the group $\tilde{G}$ lies over the group G with respect to $\kappa\colon \tilde{X} \to X$.

Lemma 3.3. *Let Ψ be a proper discontinuous group of diffeomorphisms of a manifold X acting freely and $\kappa\colon X \to X/\Psi$ be its natural projection. Let $\tilde{\kappa}\colon \tilde{X} \to X$ be the universal covering mapping and H be the group of covering transformations of the covering $\tilde{\kappa}$. Then the group G of covering transformations of the universal covering $\kappa \circ \tilde{\kappa}\colon \tilde{X} \to X/\Psi$ lies over the group Ψ with respect to $\tilde{\kappa}\colon \tilde{X} \to X/\Psi$ and the group Ψ is isomorphic to the quotient group G/H.*

Proof. The group G of covering transformations of the universal covering $\kappa \circ \tilde{\kappa}$ is isomorphic to a fundamental group of the manifold X/Ψ, while the group of covering transformations H of the universal covering $\tilde{\kappa}$ is isomorphic to the fundamental group of the manifold X.

It follows from Lemma 3.2 that the diffeomorphisms of the manifold $\tilde{X}$ lying over the diffeomorphisms of the group Ψ with respect $\tilde{\kappa}$ form a group, which we denote by $\tilde{\Psi}$, and the group Ψ is isomorphic to the quotient group $\tilde{\Psi}/H$.

We are going to show that G coincides with $\tilde{\Psi}$. Each transformation $\tilde{\psi} \in \tilde{\Psi}$ lies over some $\psi \in \Psi$, that is, $\tilde{\kappa} \circ \tilde{\psi} = \psi \circ \tilde{\kappa}$. It is easy to confirm that Ψ is a group of covering transformations of the covering κ and this is why for each $\psi \in \Psi$ the identity $\kappa \circ \psi = \kappa$ holds true. This implies the chain of identities:

$$(\kappa \circ \tilde{\kappa}) \circ \tilde{\psi} = \kappa \circ (\tilde{\kappa} \circ \tilde{\psi}) = \kappa \circ (\psi \circ \tilde{\kappa}) = (\kappa \circ \psi) \circ \tilde{\kappa} = \kappa \circ \tilde{\kappa}$$

or

$$(\kappa \circ \tilde{\kappa}) \circ \tilde{\psi} = \kappa \circ \tilde{\kappa}.$$

Thus, each $\tilde{\psi} \in \tilde{\Psi}$ lies over the identity diffeomorphism with respect to $\kappa \circ \tilde{\kappa}$, that is, $\tilde{\Psi} \subset G$.

Let us show that the inverse inclusion holds. Let H be the group of covering transformations of the covering $\tilde{\kappa}$. It is obvious that $H \subset G$. A natural projection $\kappa\colon X \to X/\Psi$ is a regular covering mapping and this is why H is a normal divisor of the group G, that is, $g \circ H = H \circ g$. By the second statement of Lemma 3.2 this implies that each diffeomorphism $g \in G$ lies over some diffeomorphism g^* of the manifold X, that is, $\tilde{\kappa} \circ g = g^* \circ \tilde{\kappa}$. Hence, $\kappa \circ \tilde{\kappa} \circ g = \kappa \circ g^* \circ \tilde{\kappa}$, and taking into consideration the identity $\kappa \circ \tilde{\kappa} \circ g = \kappa \circ \tilde{\kappa}$, we get $\kappa \circ g^* \circ \tilde{\kappa}(\tilde{x}) = \kappa \circ \tilde{\kappa}(\tilde{x})$ for each $\tilde{x} \in \tilde{X}$. Since $x = \tilde{\kappa}(\tilde{x})$ ranges in the entire X as $\tilde{x}$ ranges over all $\tilde{X}$, then the previous identity implies $\kappa \circ g^*(x) = \kappa(x)$ for each $x \in X$, that is, $\kappa \circ g^* = \kappa$ and g^* is a covering transformation for κ. This is why $g^* \in \Psi$ and $g \in \tilde{\Psi}$, and therefore, $G \subset \tilde{\Psi}$.

Thus, $G = \tilde{\Psi}$. The proof is complete. □

We recall that a G_δ-subset of the manifold X is the intersection of a countable family of open dense in X subsets. Since on each manifold X there exists a compete Riemann metrics, according to Hopf-Rinov theorem, X can be regarded as a complete metric space. Therefore, by the Baire theorem, each G_δ-subset of X is dense in X.

We shall employ the following statement, which can be proved easily.

Lemma 3.4. *Let $\kappa\colon \tilde{X} \to X$ be a smooth universal covering mapping for the manifold X, and G be the group of the diffeomorphisms of the manifold $\tilde{X}$ lying over the group Ψ. Then, if one of the groups G and Ψ possesses one of the following properties:*

1) there exists a point fixed only with respect to the identity diffeomorphism;

2) the group acts freely;

3) the set of the fixed points with respect to each diffeomorphism of the group except for the identity one is dense G_δ-subset;

4) the group is completely discontinuous;

then the other group possesses the same property.

In what follows we apply the following lemma [13, Lm. 1].

Lemma 3.5 (Shapiro's lemma). *Let X_1, X_2 be some sets, $X = X_1 \times X_2$ and G be some group of transformations of X preserving its structure of the product. Then let pr_1 and pr_2 be canonical projections of the set X on X_1 and X_2, while G_1 and G_2 be the groups of the transformations of X_1 and X_2 induced on them (by means of pr_1 and pr_2) by group G and $q_1\colon G \to G_1$, $q_2\colon G \to G_2$ be natural homomorphisms. Let G'_{11} and G'_{22} be the kernels of the homomorphisms q_1, q_2 and $G_{11} := q_1(G'_{22})$, $G_{22} := q_2(G'_{11})$. Then*

1) there exists an isomorphism of quotient groups

$$\Theta\colon G_1/G_{11} \to G_2/G_{22},$$

such that if $\{g_i\} \in G_i/G_{ii}$, $i = 1, 2$, where $g_i \in G_i$, $\{g_i\}$ is a corresponding to g_i class and $\{g_2\} = \Theta(\{g_1\})$, then G is formed by the set of pairs $(g_1, g_2) = g_1 \circ g_2$, where $g_1(x_1, x_2) := (g_1(x_1), x_2)$, $g_2(x_1, x_2) := (x_1, g_2(x_2))$, $x_i \in X_i$;

2) let G_i be groups of transformations of X_i and G_{ii} be their aforementioned normal divisors and there exists an isomorphism $\Theta\colon G_1/G_{11} \to G_2/G_{22}$, then the quintuple $(G_1, G_2, G_{11}, G_{22}, \Theta)$ determines uniquely the group of transformations G on $X_1 \times X_2$ preserving the structure of the product, with respect to which the mentioned quintuple plays the role indicated in the first part of the lemma.

3.3. Category of twofoliations covered by product. We denote by $\mathfrak{BiF}$ a category, the objects of which are bifoliations (M, F_1, F_2) covered by product. Morphisms of two objects (M, F_1, F_2) and (M', F'_1, F'_2) are differentiable mappings $f\colon M \to M'$ being the morphisms of foliations (M_i, F_i), $i = 1, 2$, and (M'_i, F'_i) in the category of foliations $\mathfrak{Fol}$.

Let X_1 and X_2 be manifolds of dimension n_1 and n_2, respectively. Assume that we are given

$$\Phi_1\colon \Psi \times X_1 \to X_1 \quad \text{and} \quad \Phi_2\colon \Psi \times X_2 \to X_2,$$

which are left actions of the group Ψ such that the induced action of the group Ψ,

$$\Phi\colon \Psi \times X_1 \times X_2 \to X_1 \times X_2\colon (\psi, (x_1, x_2)) \mapsto (\psi \cdot x_1, \psi \cdot x_2) \qquad \forall \psi \in \Psi,$$

is free and proper discontinuous. Therefore, the projection

$$\kappa\colon X_1 \times X_2 \to (X_1 \times X_2)/\Psi$$

on the space of orbits is a regular covering mapping with the group Ψ as the group of covering transformations. This is why on $(X_1 \times X_2)/\Psi$, a structure of a smooth manifold is induced, with respect to which κ becomes a smooth covering mapping. Since the action Φ of the group Ψ on $X_1 \times X_2$

preserves the structure of the product, on $(X_1 \times X_2)/\Psi$ the bifoliation $(\mathcal{F}_1, \mathcal{F}_2)$ is induced covered by the product $X_1 \times X_2$.

Definition 3.2. *A quotient manifold $(X_1 \times X_2)/\Psi$ is called canonical, and the triple $((X_1 \times X_2)/\Psi, \mathcal{F}_1, \mathcal{F}_2)$ obtained in this way is called canonical bifoliation.*

Hereinafter in this section the indices i and j take the values 1 and 2, and $i \neq j$.

Proposition 3.2. *Each bifoliation (M, F_1, F_2) covered by the product is isomorphic in the category $\mathfrak{BiF}$ to some canonical bifoliation $((X_1 \times X_2)/\Psi, \mathcal{F}_1, \mathcal{F}_2)$.*

Proof. Let $f\colon \tilde{X}_1 \times \tilde{X}_2 \to M$ be an universal covering, then by the Kashiwabara theorem $\tilde{\mathcal{F}}_i = f^*\mathcal{F}_i$ are standard foliations of the product. Since the group G of covering transformations of the covering f is a group of automorphisms of $(\tilde{X}_1 \times \tilde{X}_2, \tilde{\mathcal{F}}_1, \mathcal{F}_2)$ in the category $\mathfrak{BiF}$, then for each $x_i \in \tilde{X}_i$ and $g \in G$ the identities

$$g(\{x_1\} \times \tilde{X}_2) = \{x_1'\} \times \tilde{X}_2 \quad \text{and} \quad g(\tilde{X}_1 \times \{x_2\}) = \tilde{X}_1 \times \{x_2'\},$$

hold, where $x_i' \in \tilde{X}_i$. This is why the mappings $g_i := pr_i \circ g$ are well-defined, where pr_1 and pr_2 are canonical projections of the product $\tilde{X}_1 \times \tilde{X}_2$ onto the factors and

$$g(x_1, x_2) = (g_1(x_1), g_2(x_2)) \qquad \forall (x_1, x_2) \in \tilde{X}_1 \times \tilde{X}_2.$$

Let $G_i := \{g_i = pr_i \circ g \mid g \in G\}$ and $q_i\colon G \to G_i$ be a natural epimorphism of the groups, $G_{ii} := q_i(q_j^{-1}(1_j))$, where 1_j is the unit of the group G_j. At the same time, G_{ii} is a normal subgroph of the group G_i and the quotient groups G_i/G_{ii} are well-defined. According to the Shapiro lemma, there exists an isomorphism $\Theta\colon G_1/G_{11} \to G_2/G_{22}$ satisfying the following commutative diagram

$$\begin{array}{ccccc} G_1 & \xleftarrow{q_1} & G & \xrightarrow{q_2} & G_2 \\ {\scriptstyle h_1}\downarrow & & & & \downarrow{\scriptstyle h_2} \\ G_1/G_{11} & & \xrightarrow{\quad\Theta\quad} & & G_2/G_{22}, \end{array}$$

where $h_i\colon G_i \to G_i/G_{ii}$ are natural projections.

We take an arbitrary point $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ in $f^{-1}(x)$. Let

$$N_i(\tilde{x}_j) := \{g \in G \mid g(\tilde{X}_i \times \{\tilde{x}_j\}) = \tilde{X}_i \times \{\tilde{x}_j\}\} \quad \text{and} \quad N_i(\tilde{x}) := q_i N_i(\tilde{x}_j).$$

Since the group of covering transformations G acts on $\tilde{X}_1 \times \tilde{X}_2$ freely and properly discontinuous, the groups $N_i(\tilde{x}_j)$ and $N_i(\tilde{x})$ act on $\tilde{X}_i \times \{\tilde{x}_j\}$ and $\tilde{X}_i$, respectively, freely and properly discontinuous. It is easy to confirm that the restriction $f|_{\tilde{X}_i \times \{\tilde{x}_j\}}\colon \tilde{X}_i \times \{\tilde{x}_j\} \to L_i(x)$ is a regular covering on the leaf $L_i(x) \in F_i$ with the group of covering transformations $N_i(\tilde{x}_j)$. Since $G_{ii} \subset N_i(\tilde{x}_j)$, then G_{ii} acts on $\tilde{X}_i$ freely and properly discontinuous and this is why the quotient manifold $X_i := \tilde{X}_i/G_{ii}$ is well-defined. The direct product $G_{11} \times G_{22}$ of normal subgroups G_{ii} of the groups G_i is a normal subgroup of the group G, therefore, a smooth quotient manifold $(\tilde{X}_1 \times \tilde{X}_2)/(G_{11} \times G_{22})$ is well-defined, which is diffeomorphic to the product of manifolds $(\tilde{X}_1/G_{11}) \times (\tilde{X}_2/G_{22})$. According to Lemma 3.1, on the product of manifolds $X_1 \times X_2$, the group Ψ acts freely and properly discontinuous and this group is isomorphic to the quotient group $G/(G_{11} \times G_{22})$ and there exists a differomophism $\Theta\colon M \to (X_1 \times X_2)/\Psi$ obeying the commutative diagram

$$\begin{array}{ccc} \tilde{X}_1 \times \tilde{X}_2 & \xrightarrow{f_1 \times f_2} & X_1 \times X_2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \kappa} \\ (\tilde{X}_1 \times \tilde{X}_2)/G = M & \xrightarrow{\Theta} & (X_1 \times X_2)/\Psi, \end{array} \tag{3.1}$$

where $f_i\colon \tilde{X}_i \to X_i = \tilde{X}_i/G_{ii}$ and $\kappa\colon X_1 \times X_2 \to (X_1 \times X_2)/\Psi$ are quotient mappings on the spaces of orbits. It follows from the Shapiro lemma that the group Ψ is isomorphic to each quotient group G_1/G_{11} and G_2/G_{22}. Since the group Ψ preserves the structure of the product, on the quotient manifold $(X_1 \times X_2)/\Psi$ a canonical bifoliation $(\mathcal{F}_1, \mathcal{F}_2)$ is well-defined and Θ is an isomoprhism in the category $\mathfrak{BiF}$. The proof is complete. $\square$

Definition 3.3. *The abstract group Ψ obeying Proposition 3.2 is called a structural group of the bifoliation covered by a product.*

The next statement follows the proof of Proposition 3.2.

Corollary 3.1. *If $\xi = ((X_1 \times X_2)/\Psi, \mathcal{F}_1, \mathcal{F}_2)$ is a canonical bifoliation, then the structural group Ψ is isomoprhic to each of quotient groups $G/(G_{11} \times G_{22})$, G_1/G_{11}, G_2/G_{22}, where the group G is isomorphic to the fundamental group $\pi_1(M)$ of the quotient manifold $M \cong (X_1 \times X_2)/\Psi$, while the group G_{ii}, $i = 1, 2$, is isomorphic to the fundamental group $\pi_1(X_i)$ of the manifold X_i.*

3.4. Interpretation of canonical bifoliation.

Lemma 3.6. *Let $\kappa\colon \tilde{M} \to M$ be a smooth regular covering mapping for a manifold M. If h and h' are paths originating at the points $y = h(0)$ and $y' = h'(0)$ covering a piece-wise smooth path σ in M, then there exists a unique covering transformation ψ satisfying the identity $\psi \circ h = h'$.*

Proof. We denote by Ψ the group of covering transformations of the regular covering κ. We recall that Ψ acts simply transitively on each fiber of $\kappa^{-1}(x)$, $x \in M$.

Since $\sigma = \kappa \circ h = \kappa \circ h'$ for each $s \in I$, there exist a segment $I_s = [s - \delta, s + \delta] \subset I$ and a covering transformation ψ_s such that $\psi_s \circ h|_{I_s} = h'|_{I_s}$. By the compactness of the segment I there exists its partition $0 = t_0 < t_1 < \ldots < t_{m+1} = 1$ possessing the property $\psi_i \circ h|_{I_i} = h'|_{I_i}$, where $I_i = [t_i, t_{i+1}]$, $i = 0, \ldots, m$, $\psi_i \in \Psi$. Since $\psi_i(t_{i+1}) = \psi_{i+1}(t_{i+1})$, we have $\psi_1 = \psi_2 = \ldots = \psi_m = \psi \in \Psi$ and $\psi \circ h = h'$. □

Definition 3.4. *Let (M, F) be a foliation with Ehresmann connection $\mathfrak{M}$. A regular covering mapping $f : L_0 \to L$ on the fiber L of this foliation is called $\mathfrak{M}$-holonomic if the group of all its covering transformation is isomorphic to the group of $\mathfrak{M}$-holonomy $H_{\mathfrak{M}}(L)$.*

According Proposition 3.2, each bifoliation (M, F_1, F_2) is isomorphic in the category $\mathfrak{BiF}$ to a canonical bifoliation $((X_1{\times}X_2)/\Psi, \mathcal{F}_1, \mathcal{F}_2)$. The next statement provides a geometric meaning of the manifolds X_1 and X_2.

Proposition 3.3. *Let $(M, F_1, F_2) \in Ob(\mathfrak{BiF})$ and $((X_1{\times}X_2)/\Psi, \mathcal{F}_1, \mathcal{F}_2)$ be an isomorphic canonical bifoliation, $i, j = 1, 2$, $i \neq j$. Then the manifold X_i is a space of $\mathfrak{M}_i$-holonomic covering mapping for each leaf of the foliation F_i, where $\mathfrak{M}_i := TF_j$.*

Proof. We identify M with $(X_1 \times X_2)/\Psi$ by the diffeomorphism Θ satisfying commutative diagram 3.1. Let $L_1 = L_1(x)$ be an arbitrary leaf of the foliation F_1, $(x_1, x_2) \in \kappa^{-1}(x)$, then $\kappa^1 := \kappa|_{X_1 \times \{x_2\}}\colon X_1 \times \{x_2\} \to L_1$ is a covering mapping. We keep the above introduced notations and let

$$\Psi(x_2) := \{\psi \in \Psi \mid \psi(X_1 \times \{x_2\}) = X_1 \times \{x_2\}\},$$

at that, $\Psi(x_2)$ acts freely and properly discontinuous on $X_1 \times \{x_2\}$. Let Ψ_1 and Ψ_2 be the group of diffeomorphisms X_1 and X_2 induced by the group Ψ, then $\psi = (\psi_1, \psi_2)$, where $\psi_i \in \Psi_i$, and the canonical projections $q_i\colon \Psi \to \Psi_i$ are isomorphisms of the groups. Since $\psi = (\psi_1, \psi_2) \in \Psi(x_2)$ if and only if $\psi_2(x_2) = x_2$, then $q_i|_{\Psi(x_2)}\colon \Psi(x_2) \to \Psi_{x_2}$ is an isomorphism onto the stationary subgroup of the group Ψ at a point $x_2 \in X_2$. Since $L_1 = (X_1 \times \{x_2\})/\Psi(x_2)$, the covering mapping κ^1 is regular, and $\Psi(x_2)$ is its group of covering transformations. We need to show that the group $\Psi(x_2)$ is isomorphic to the group of holonomy $H_{\mathfrak{M}_2}(L_1, x)$.

Since κ^1 is a regular covering, the image of the induced homomorphism

$$\kappa^1_*\colon \pi_1(X_1 \times \{x_2\}, (x_1, x_2)) \to \pi_1(L_1, x)$$

is a normal subgroup of ρ and the group $\Psi(x_2)$ is isomorphic to the quotient group $\pi_1(L_1, x)/\rho$. Let

$$\nu\colon \pi_1(L_1, x) \to \Psi(x_2)$$

be the natural epimorphism on the mentioned quotient group. We denote by

$$\beta\colon \pi_1(L_1, x) \to H_{\mathfrak{M}_2}(L_1, x)$$

a natural epimorphism, the kernel of which $\ker\beta = K_{\mathfrak{M}_2}(L_1, x)$ is formed by the elements $[h] \in \pi_1(L_1, x)$ acting trivially on the set of horizontal curves Ω_x, by the definition of the group of $\mathfrak{M}$-holonomy. Let us recall how the translation of an arbitrary path $\sigma \in \Omega_x$ along a path h is defined. Let

$\sigma \overset{h}{\longrightarrow}> \tilde{\sigma}$ and $y = (x_1, x_2)$. We denote by υ, k, $\tilde{\upsilon}$ the paths in $X_1 \times X_2$ originating at y and covering respectively σ, h and $\tilde{\sigma}$. Then a standard vertical-horizontal homotopy with respect to the product $X_1 \times X_2$ defines a translation $\upsilon \overset{k}{\longrightarrow}> \tilde{\upsilon}$.

Let us show that $\ker \nu \subset \ker \beta$. We take an arbitrary element $[h] \in \ker \nu = \rho$, then the path $\tilde{h}$ covering h and originating at y is a loop at the point y. For each $\sigma \in \Omega_x$, by $\tilde{\sigma}$ we denote the result of the translation of σ along h with respect to an integrable Ehresmann connection $\mathfrak{M}_2$. Let υ and $\tilde{\upsilon}$ be the paths covering σ and $\tilde{\sigma}$, respectively. Since $\tilde{h}$ is a loop, then $\upsilon = \tilde{\upsilon}$, therefore, $\sigma = \tilde{\sigma}$ and $[h] \in \ker \beta$.

Vice versa, let $[h] \in \ker \beta$. Suppose that $[h] \notin \ker \nu$. We note that the inclusion $j\colon L_1 \to M$ induces an isomorphism $j_*\colon \rho \to G_{11}$ onto a normal subgroup of the fundamental group $\pi_1(M, x)$, considered as a group of covering transformations G of the covering $f : \tilde{X}_1 \times \tilde{X}_2 \to M$, and G_{11} coincides with the transformations in G inducing $id_{\tilde{X}_2}$ on $\tilde{X}_2$. The relation $[h] \notin \rho$ implies that $[h]$ induces a non-identity transformation $\psi_2 \in \Psi_{x_2}$. There exists $z \in X_2$, for which $z' := \psi_2(z) \neq z$. We connect x_2 with a point z in X_2 by a piece-wise smooth curve γ, then the curve $\tilde{\gamma} := \psi_2 \circ \gamma$ connects x_0 with z' in X_2. Let υ be a curve in the leaf $X_1 \times \{x_2\}$ of the product $X_1 \times X_2$ such that $pr_2 \circ \upsilon = \gamma$. Since $[h] \notin \rho$, the path $\tilde{h}$ originating at y and covering h is not closed. Let $\upsilon \overset{\tilde{h}}{\longrightarrow}> \tilde{\upsilon}$ and $pr_2 \circ \upsilon = \gamma$. Since $[h] \in \ker \beta$, then $\sigma \overset{h}{\longrightarrow}> \tilde{\sigma} = \sigma$, where $\sigma := k \circ \upsilon$. This is why υ and $\tilde{\upsilon}$ are the paths originating at the points y and $\upsilon := \tilde{h}(1)$ covering the same path σ. According Lemma 3.6, the element $\psi = (\psi_1, \psi_2)$ satisfies identity $\tilde{\upsilon} = \psi \circ \upsilon$. Hence, taking into consideration that $\psi_2 = pr_2 \circ \psi$, we get $\gamma = \psi_2 \circ \gamma = \tilde{\gamma}$, and therefore, $z = z'$. This means that $\psi_2 = id_{X_2}$. This contradiction that the inclusion $\ker \nu \supset \ker \beta$ holds.

Thus, we have $\ker \nu = \ker \beta$ and this implies the existence of the isomorphism of the groups

$$\Xi\colon \Psi(x_2) \to H_{\mathfrak{M}_2}(L_1, x),$$

obeying the identity $\Xi \circ \nu = \beta$.

Thus, X_1 is an $\mathfrak{M}_1$-holonomic covering for each fiber of the foliation (M, F_1). Similarly, X_2 serves as an $\mathfrak{M}_2$-holonomic covering space for the leaves of the foliation (M, F_2). □

4. Category of triples equivalent to category $\mathfrak{BiF}$

4.1. Category of triples $\mathfrak{T}$. Let (X_1, X_2, Ψ) be a triple consisting of smooth manifolds X_i, $i = 1, 2$, of an arbitrary dimension n_i, a group Ψ acting effectively on X_i and diagonally on the product of manifolds $X_1 \times X_2$ by the rule

$$\Phi\colon \Psi \times X_1 \times X_2 \to X_1 \times X_2\colon (\psi, (x_1, x_2)) \mapsto (\psi \cdot x_1, \psi \cdot x_2) \qquad \forall \psi \in \Psi,$$

and the group Ψ acts on $X_1 \times X_2$ freely and properly discontinuous.

If (X_1, X_2, Ψ) and (X_1', X_2', Ψ') are two such triples, then a morphism of the first triple into the second triple is a pair (h, μ), where h is a smooth mapping $X_1 \times X_2 \to X_1' \times X_2'$ preserving the structure of the product, while $\mu\colon \Psi \to \Psi'$ is a homomorphism of the group satisfying the identity $h \circ \psi = \mu(\psi) \circ h$ for all $\psi \in \Psi$.

The obtained category of triples (X_1, X_2, Ψ) is denoted by $\mathfrak{T}$.

Proposition 4.1. *Let (M, F_1, F_2) be a bifoliation covered by a product. Assume that*

$$\Theta_1\colon M \to (X_1 \times X_2)/\Psi \quad \text{and} \quad \Theta_2\colon M \to (Y_1 \times Y_2)/\Psi'$$

are isomorphisms in the category $\mathfrak{BiF}$ of the bifoliation (M, F_1, F_2) and canonical bifoliations $((X_1 \times X_2)/\Psi, \mathcal{F}_1, \mathcal{F}_2)$, $((Y_1 \times Y_2)/\Psi', \mathcal{F}_1', \mathcal{F}_2')$. Then the triples (X_1, X_2, Ψ) and (Y_1, Y_2, Ψ') are isomorphic in the category $\mathfrak{T}$.

Proof. The uniqueness of the universal covering for M, definition of the canonical bifoliation and Lemma 3.1 imply the existence of an isomorphism of the groups $\mu : \Psi \to \Psi'$ and a diffeomorphism

$$h\colon (X_1 \times X_2)/\Psi \to (Y_1 \times Y_2)/\Psi'$$

implementing the equivalence of the actions of the groups Ψ and Ψ'. At that, the pair (h, μ) is an isomorphism of the triples (X_1, X_2, Ψ) and (Y_1, Y_2, Ψ') in the category $\mathfrak{T}$. □

4.2. Equivalence of categories $\widetilde{\mathfrak{BiF}}$ and $\widetilde{\mathfrak{T}}$. Let $\mathfrak{B}$ be some category. By $\widetilde{\mathfrak{B}}$ we denote a category, the objects of which are the classes of isomorphic objects of the category $\mathfrak{B}$, while the morphisms are the classes of isomorphic morphisms in the category $\mathfrak{B}$. If $B \in Ob(\mathfrak{B})$, then by $[B]$ we denote a corresponding object in the category $\widetilde{\mathfrak{B}}$ and a similar notation is used for morphisms.

Theorem 4.1. *There exists a covariant functor $\Upsilon\colon \widetilde{\mathfrak{BiF}} \to \widetilde{\mathfrak{T}}$ realizing the equivalence of the categories $\widetilde{\mathfrak{BiF}}$ and $\widetilde{\mathfrak{T}}$.*

Proof. We construct the functor $\Upsilon\colon \widetilde{\mathfrak{BiF}} \to \widetilde{\mathfrak{T}}$ as follows.

Let $[\xi] = [(M, F_1, F_2)] \in Ob(\widetilde{\mathfrak{BiF}})$ and $((X_1 \times X_2)/\Psi, \mathcal{F}_1, \mathcal{F}_2)$ be the canonical bifoliation isomorphic to $\xi = (M, F_1, F_2)$ according to Proposition 3.2. Then $(X_1, X_2, \Psi) \in \mathfrak{T}$. We define $\Upsilon([\xi]) := [(X_1, X_2, \Psi)]$. Proposition 4.1 ensures that this definition is well-defined.

Let $f : \xi \to \xi'$ be a morphism of the bifoliation $\xi = (M, F_1, F_2)$ into $\xi' = (M', F_1', F_2')$ in the category $\mathfrak{BiF}$ and

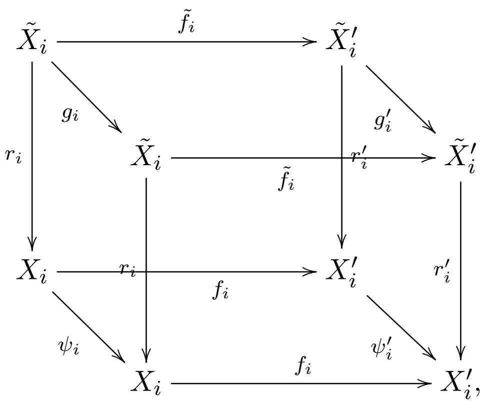
$$\Theta\colon M \to (X_1 \times X_2)/\Psi, \qquad \Theta'\colon M' \to (X_1' \times X_2')/\Psi'$$

be the morphisms ξ and ξ' to the associated canonical bifoliations $\eta = ((X_1 \times X_2)/\Psi, \mathcal{F}_1, \mathcal{F}_2)$ and $\eta' = ((X_1' \times X_2')/\Psi', \mathcal{F}_1', \mathcal{F}_2')$, respectively. According to Proposition 3.2, $[\xi] = [\eta]$, $[\xi'] = [\eta']$. This is why we can assume that $[f]$ is a morphism of $[\eta]$ and $[\eta']$. Without loss of generality, we let $\xi = \eta$ and $\xi' = \eta'$. Let $r_i\colon \tilde{X}_i \to X_i$ and $r_i'\colon \tilde{X}_i' \to X_i'$ be smooth universal coverings. We denote by $\kappa\colon \tilde{X}_1 \times \tilde{X}_2 \to M$ and $\kappa'\colon \tilde{X}_1' \times \tilde{X}_2' \to M'$ universal covering of the corresponding bifoliations on M and M', preserving the structure of the product. We take arbitrary points $(x_1, x_2) \in \tilde{X}_1 \times \tilde{X}_2$ and $(y_1, y_2) \in \tilde{X}_1' \times \tilde{X}_2'$ such that $f \circ \kappa(x_1, x_2) = \kappa'(y_1, y_2)$. Then f induces a differentiable mapping $\tilde{f}\colon \tilde{X}_1 \times \tilde{X}_2 \to \tilde{X}_1' \times \tilde{X}_2'$, which maps (x_1, x_2) into (y_1, y_2) and satisfies the identity $\kappa' \circ \tilde{f} = f \circ \kappa$. At that, $\tilde{f}$ defines diffeomorphisms $\tilde{f}_i\colon \tilde{X}_i \to \tilde{X}_i'$ possessing the properties:

1) $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$,

2) for each $g_i \in G_i$ there exists $g_i' \in G_i'$, satisfying the identity $g_i' \circ \tilde{f}_i = g_i \circ \tilde{f}_i$.

We define $\nu_i\colon G_i \to G_i'\colon g_i \mapsto g_i'$, then ν_i is a homomorphism of these groups. We note that $\nu_i(G_{ii}) \subset G_{ii}'$, where $G_{ii} := q_i(q_j^{-1}(1_j))$, $G_{ii}' := q_i'({q'}_j^{-1}(1_j'))$. This is why the homomorphisms ν_i induce homomorphisms of the quotient groups $\mu_i\colon G_i/G_{ii} \to G_i'/G_{ii}'$. Since $\Psi_i = G_i/G_{ii}$, $\Psi_i' = G_i'/G_{ii}'$, then $\mu_i : \Psi_i \to \Psi_i'$ is a homeomorphism. Applying Lemma 3.1, it is easy to show that $\tilde{f}_i$ induces a mapping $f_i\colon \tilde{X}_i/G_{ii} = X_i \to \tilde{X}_i'/G_{ii}' = X_i'$ satisfying the following commutative diagram:



where $r_i\colon \tilde{X}_i \to X_i = \tilde{X}_i/G_{ii}$ and $r_i'\colon \tilde{X}_i' \to X_i' = \tilde{X}_i'/G_{ii}'$ are quotient mappings onto the orbit spaces. The mapping $\mu_i\colon \Psi_i \to \Psi_i'\colon \psi_i \mapsto \psi_i'$ is a homomorphism of the groups such that the pair (h, μ), where $\mu = (\mu_1, \mu_2)$, $h = (f_1, f_2)$, is a morphism of the triples (X_1, X_2, Ψ) and (X_1', X_2', Ψ'). We let $\Upsilon([f]) := [(h, \mu)]$. A straightforward checking based on Proposition 4.1 shows that Υ is a covariant functor from the category $\widetilde{\mathfrak{BiF}}$ into the category $\widetilde{\mathfrak{T}}$.

Let us check that the constructed functor Υ realizes the equivalence of the categories $\widetilde{\mathfrak{BiF}}$ and $\widetilde{\mathfrak{T}}$. We are going to the show the existence of the inverse functor $\Lambda\colon \widetilde{\mathfrak{T}} \to \widetilde{\mathfrak{BiF}}$. We consider an arbitrary object $[(X_1, X_2, \Psi)]$ in the category $\widetilde{\mathfrak{T}}$, where $(X_1, X_2, \Psi) \in \mathfrak{T}$. It follows from the definition of the category $\mathfrak{T}$ that the canonical bifoliation $((X_1 \times X_2)/\Psi, \mathcal{F}_1, \mathcal{F}_2)$ covered by the product $X_1 \times X_2$ is well-defined. We define $\Lambda[(X_1, X_2, \Psi)] := [((X_1 \times X_2)/\Psi, \mathcal{F}_1, \mathcal{F}_2)]$.

Let

$$(h, \mu) : (X_1, X_2, \Psi) \to (X_1', X_2', \Psi')$$

be a morphism in the category $\mathfrak{T}$. It induces a differentiable mapping

$$f\colon (X_1 \times X_2)/\Psi \to (X_1' \times X_2')/\Psi' : (x_1, x_2)\Psi \mapsto h(x_1, x_2)\Psi'$$

satisfying the identity

$$p' \circ h = f \circ p,$$

where $p\colon X_1 \times X_2 \to (X_1 \times X_2)/\Psi$ and $p'\colon X_1' \times X_2' \to (X_1' \times X_2')/\Psi'$ are natural projections onto the spaces of orbits. Since h, p, $p' \in Mor(\mathfrak{BiF})$, the mentioned identity implies that f is a morphism of canonical bifoliations on manifolds $(X_1 \times X_2)/\Psi$ and $(X_1' \times X_2')/\Psi'$, that is, f is a morphism in the category $\mathfrak{BiF}$. We let $\Lambda[(h, \mu)] := [f]$. It is easy to confirm that Λ is a well-defined covariant functor satisfying the identities

$$\Lambda \circ \Upsilon = Id_{\widetilde{\mathfrak{BiF}}} \quad \text{and} \quad \Upsilon \circ \Lambda = Id_{\widetilde{\mathfrak{T}}}.$$

Thus, Υ realizes the equivalence of the categories $\widetilde{\mathfrak{BiF}}$ and $\widetilde{\mathfrak{T}}$. □

4.3. Proof of Theorems 1.1 and 1.2. Theorem 1.1 is implied by Propositions 3.2 and 3.3.

Theorem 1.2 is implied by Theorem 4.1 on equivalence of categories $\widetilde{\mathfrak{BiF}}$ and $\widetilde{\mathfrak{T}}$.

5. Classes of foliations with integrable Ehresmann connection

5.1. Foliations with integrable Ehresmann connection of codimension $q = 1$. Let (M, F) be a foliation with the Ehresmann connection $\mathfrak{M}$ of codimension. Since each one-dimensional distribution is integrable, the distribution $\mathfrak{M}$ is integrable and determines a foliation. Therefore, such foliations are in the class of the studied foliations.

5.2. Suspension foliations. The construction of suspension foliation was suggested by A. Haefliger and was described in details in [14]. Let (M, F) be a foliation of codimension q on an n-dimensional manifold M defined by the suspension of a homomorphism

$$\rho : \pi_1(B, b) \to \mathrm{Diff}(T)$$

of the fundamental group $\pi_1(B, b)$ of a manifold B into the group of the diffeomorphisms of a q-dimensional manifold T. Then there exists a simple q-dimensional foliation (M, F^t) formed by the fibres of a locally trivial bundle $p : M \to B$ with a standard fibre T such that the bifoliation (M, F, F^t) is covered by product. Therefore, the distribution $\mathfrak{M} = TF^t$ is an integrable Ehresmann connection for the foliation (M, F).

5.3. Non-degenerately reducible pseudo-Riemannian manifolds. Let (V, g) be a non-degenerately reducible n-dimensional pseudo-Riemannian manifold, then there exists a p-dimensional subspace $\mathfrak{M}_{x_0}$ of the tangential space $T_{x_0}V$, the restriction of the metrics g on which is non-degenerate and moreover, $T_{x_0}V$ is invariant with respect to its holonomy group. This means that $\mathfrak{M}_{x_0}$ is invariant with respect to the parallel translations along all piece-wise smooth curve closed at x_0. Since the parallel translation along a curve preserves the angles between vectors, the orthogonal complement $\mathfrak{M}_{x_0}^{\perp}$ to $\mathfrak{M}_{x_0}$ in $T_{x_0}V$ is also invariant with respect to the holonomy group of the pseudo-Riemannian space (V, g). The parallel translation of $\mathfrak{M}_{x_0}$ to an arbitrary point $x \in V$ is independent on the choice of the path connecting x_0 with x and defines the subspace $\mathfrak{M}_x \subset T_xV$. Thus, obtained distributions $\mathfrak{M}$ and $\mathfrak{M}^{\perp}$ are called *parallel*. As it is known, they are completely geodesic and integrable and therefore, they define two additional by orthogonality foliations $(F, F^{\perp})$, where $\dim(F^{\perp}) = n - p = q$; we denote them by (F_1, F_2). These foliations are called *parallel*. We note that the parallel property of the distribution is a local property in the sense that if some distribution is parallel in some neighbourhood of each point of a connected pseudo-Riemannian manifold V, then it is parallel on V. This is why we need the completeness of the metrics g to describe its global structure.

Let $\kappa\colon \tilde{V} \to V$ be the universal covering mapping and $\tilde{F}_i := \kappa^* F_i$, $i = 1, 2$. Then $(\tilde{V}, \tilde{g})$, where $\tilde{g} := \kappa^* g$, is a non-degenerately reducible pseudo-Riemannian manifold with a pair of orthogonal parallel distributions $(\tilde{F}_1, \tilde{F}_2)$. The completeness of the metrics g implies the completeness of the pseudo-Riemannian metrics $\tilde{g}$. According to Wu theorem [15], there exists an isometry of a pseudo-Riemannian manifold $\tilde{V}$ onto the product of pseudo-Riemannian manifolds $\tilde{V}_1 \times \tilde{V}_2$ mapping the leaves of the foliations $(\tilde{F}_1, \tilde{F}_2)$ into the corresponding leaves of the bifoliation (F_1, F_2). Thus, (V, F_1, F_2) is a natural bifoliation of a complete non-degenerately reducible pseudo-Riemannian manifold covered by product. The pseudogroups of holonomy of foliations (V, F_i) are formed by local isometries and this is why they are quasianalytic. Therefore, Theorem 1.1 implies the following statement.

Theorem 5.1. *Let (V, g) be a complete non-degenerately reducible n-dimensional pseudo-Riemannian manifold and (F_1, F_2) be orthogonal foliations of dimensions $n - q$ and q, respectively, on the leaves of which a pseudo-Riemannian metrics is induced. Then:*

1) for almost each point $x \in V$ the inclusions $f_1 : X_1 \to V$ and $f_2 : X_2 \to V$ of the leaves $X_1 = X_1(x)$ and $X_2 = X_2(x)$ induce the monomorphisms of fundamental groups $f_{i*} : \pi_1(X_i, x) \to \pi_1(V, x)$ and $G_{ii} = f_{i*}(\pi_1(X_i, x))$ are normal subgroups of the group $\pi_1(V, x)$;

2) on pseudo-Riemannian productr $X_1 \times X_2$ of the leaves passing x, a free and properly discontinuous action of the group of isometries Ψ is well-defined and this group is isomorphic to the quotient group $G/(G_{11} \times G_{22})$; a complete canonical pseudo-Riemannian manifold $(X_1 \times X_2)/\Psi$ with a pair of canonical parallel foliations $(\mathcal{F}_1, \mathcal{F}_2)$ is well-defined;

3) an isometry $\Theta : V \to (X_1 \times X_2)/\Psi$ is well-defined and it is an isomorphism in the category $\mathfrak{BiF}$.

Remark 5.1. *Theorem 5.1 generalizes the results by Ya.L. Shapiro* [7] *for reducible Riemannian manifolds.*

Remark 5.2. *We note that each parallel foliation on a pseudo-Riemannian manifold is completely geodesic. This is why the completeness of the geodesics in the leaves of the foliation (V, F_2) ensures that the distribution TF_2 is an Ehresmann connection for (V, F_1); that is, it is sufficient to have a transversal completeness of the foliation (V, F_1). Sometimes such completeness is called a partial completeness of a non-degenerately reducible Riemannian manifold (V, g).*

5.4. G-foliations with integrable Ehresmann connection. We recall that the foliation (M, F) admitting a transversal G-structure are called G-foliations. We observe that the pseudogroup of holonomy of each G-foliation is quasianalytic. A specific nature of G-foliation (M, F) with an integrable Ehresmann connection is reflected in the following theorem implied by Theorems 1.1 and 1.2.

Theorem 5.2. *Let (M, F) be a G-foliation with an integrable Ehresmann connection $\mathfrak{M}$ on an n-dimensional manifold M and the foliation (M, F^t) be such that $TF^t = \mathfrak{M}$. Let $\kappa : \widetilde{M} \to M$ be an universal covering mapping. Then*

1) the manifold $\widetilde{M}$ is diffeomorphic to the product of the manifolds $X_1 \times X_2$ and the G-structure is induced on X_2, with respect to which (M, F) is a G-foliation;

2) the canonical manifold $(X_1 \times X_2)/\Psi$ is well-defined and the group Ψ acts on X_2 by automorphisms of the mentioned G-structure;

3) germ groups of holonomy of the foliation (M, F) are isomorphic to the groups of $\mathfrak{M}$-holonomy and the manifold X_2 is a space of the holonomic covering for each leaf (M, F);

4) in the category $\mathfrak{BiF}$, the bifoliation (M, F, F^t) is isomorphic to the canonical bifoliation $((X_1 \times X_2)/\Psi, \mathcal{F}_1, \mathcal{F}_2)$ defined up to an isomorphism of the triple (X_1, X_2, Ψ) in the category $\mathfrak{T}$ preserving the G-structure on X_2.

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