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ORBITS OF DECOMPOSABLE 7-DIMENSIONAL LIE ALGEBRAS WITH $\mathfrak{sl}(2)$ SUBALGEBRA

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Abstract. The problem on complete classification of holomorphically homogeneous real hypersurfaces in two-dimensional complex spaces was resolved by E. Cartan in 1932. A similar description in the three-dimensional case was recently obtained by A. Loboda. In this work we discuss a part of classification of locally holomorphic homogeneous hypersurfaces in 4-dimensional complex space being orbits in $\mathbb{C}^4$ by one family of 7-dimensional Lie algebra. As it was shown in works by Beloshapka, Kossovskii, Loboda and other, the ideas by E. Cartan allow one to obtain rather simply the descriptions of the orbits for the algebras having Abelian ideals for rather large dimensions. In particular, the presence of a 4-dimensional Abelian ideal in 7-dimensional Lie algebra of holomorphic vector fields in $\mathbb{C}^4$ often gives rise to the tubularity property for all orbits of such algebra. The Lie algebras in the family we consider are direct sums of the algebra $\mathfrak{sl}(2)$ and several 4-dimensional Lie algebras and they have at most 3-dimensional Abelian subalgebras. By means of a technique of the simultaneous «flattening» of vector field we obtain a complete description of all Levi non-degenerate holomorphically homogeneous hypersurfaces being the orbits of the considered algebras in $\mathbb{C}^4$. Many of the obtained homogeneous hypersurfaces turn out to be tubular manifolds. At the same time, the issue on possible reduction of other hypersurfaces to tubes requires further studying. As an effective tool for such study, as well as for a detailed investigation of issues on holomorphic equivalent of the obtained orbits, the technique of Moser normal forms can serve. By means of this technique, we study the issue on the sphericity for representatives of one of the obtained family of hypersurfaces. However, the application of the method of normal forms for the hypersurfaces in complex spaces of dimension 4 and higher requires a further developing of this technique.

Keywords: homogeneous hypersurface, holomorphic transformation, decomposable Lie algebra.

Mathematics Subject Classification: 17B66, 53B25, 32V40

1. Introduction

The problem of local description of real hypersurfaces in complex spaces homogeneous with respect to holomorphic transformation was completely solved in $\mathbb{C}^2$ by E. Cartan, see [1]. A similar classification in $\mathbb{C}^3$ consists of two big fragments, one of which provides the description of all Levi degenerate homogeneous hypersurfaces in $\mathbb{C}^3$, while the other contains the description of non-degenerate hypersurfaces, see [3]–[8].

Since the classification of holomorphically homogeneous hypersurfaces in $\mathbb{C}^3$ is complete, there arises a natural interest to obtaining similar descriptions in the spaces of higher dimensions, in particular, in $\mathbb{C}^4$. Apart of obvious tubes over affine homogeneous hypersurfaces in $\mathbb{R}^4$, see, for instance, [9]–[11], only particular examples of holomorphically homogeneous hypersurfaces are known in $\mathbb{C}^4$, see [12], [13].

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Similar to the three-dimensional case, it is reasonable to split the problem on classification of holomorphically homogeneous hypersurfaces in $\mathbb{C}^4$ into two pieces, namely, the description of Levi degenerate and non-degenerate hypersurfaces. We recall that by the Levi degeneracy (at the point 0) of a smooth hypersurface $M \subset \mathbb{C}^4$ containing the origin and defined by the equation $\operatorname{Im} z_4 = F(z_1, z_2, z_3, \operatorname{Re} z_4)$, $dF(0) = 0$ we mean the vanishing at the point 0 of the determinant of the Hessian matrix $(\partial^2 F/\partial z_k \partial \bar{z}_l)$ $(k, l \in \{1, 2, 3\})$.

The aim of the present work is to construct a complete description of all Levi non-degenerate homogeneous hypersurfaces being the orbits in $\mathbb{C}^4$ of four decomposable 7-dimensional algebras

$$\mathfrak{r}_k = \mathfrak{h}_k \oplus \mathfrak{sl}(2),$$

where 3-dimensional algebra $\mathfrak{sl}(2)$ is determined by commutation relations

$$[e_1, e_2] = e_1, [e_1, e_3] = 2e_2, [e_2, e_3] = e_3,$$

while the commutation relations for 4-dimensional algebras $\mathfrak{h}_k$ $(k = 1, \ldots, 4)$ are given in the following table $(|h| \leqslant 1, p \geqslant 0)$ [14]:

Table 1.1

Algebras	$[e_1, e_3]$	$[e_1, e_4]$	$[e_2, e_3]$	$[e_2, e_4]$	$[e_3, e_4]$
$\mathfrak{h}_1$		$2e_1$	e_1	e_2	$e_2 + e_3$
$\mathfrak{h}_2$		$(h+1)e_1$	e_1	e_2	he_3
$\mathfrak{h}_3$		$2pe_1$	e_1	$pe_2 - e_3$	$e_2 + pe_3$
$\mathfrak{h}_4$	e_1	$-e_2$	e_2	e_1	

The choice of exactly such algebras for studying is explained by the fact that the maximal dimension of their Abelian subalgebras is equal to three. It follows from works [7], [12], [15] that the presence of an Abelian subalgebra of dimension n in the algebra of vector fields in $\mathbb{C}^n$ simplifies essentially the study of such algebras and often leads either to the Levi degeneracy of these orbits or to their tubular structure. And the most interesting Levi non-degenerate homogeneous surfaces arose while considering the algebras with Abelian subalgebras of small dimensions, see [7], [15].

We note that among decomposable seven-dimensional Lie algebras possessing no Abelian subalgebras of dimension 4, only eight types of such algebras are direct sums of four-dimensional and three-dimensional terms. Four of these eight types are the algebras $\mathfrak{r}_k = \mathfrak{h}_k \oplus \mathfrak{sl}(2)$, $(k = 1, \ldots, 4)$, considered in the text. Extra four types are the algebras $\mathfrak{s}_k = \mathfrak{h}_k \oplus \mathfrak{su}(2)$ $(k = 1, \ldots, 4)$ with the same four-dimensional terms $\mathfrak{h}_k$ as in the first case. The study of the orbits of the algebras $\mathfrak{s}_k$ is not completed by the author yet.

In order to describe all Levi non-degenerate homogeneous hypersurfaces being the orbits in $\mathbb{C}^4$ of 7-dimensional algebras $\mathfrak{r}_k$ $(k = 1, \ldots, 4)$, we employ the technique of realizations Lie algebras as the algebras of holomorphic vector fields on homogeneous manifolds, see [16], and this technique develops the ideas by E. Cartan in work [1].

The main result of the present paper is formulated in the following statement.

Theorem 1.1. *Let p be a center of the germ of a real-analytic hypersurface M in $\mathbb{C}^4$, and $g(M)$ be a 7-dimensional algebras of germs of holomorphic vector fields on M having a full rank in p. If $g(M)$ and $\mathfrak{r}_1$ are isomorphic as Lie algebras, then M is necessarily Levi degenerate. Levi non-degenerate 7-dimensional orbits of algebras $\mathfrak{r}_2$, $\mathfrak{r}_3$, $\mathfrak{r}_4$ are exactly the surfaces from the* *following families (up to a locally holomorphic equivalence):*

$$\mathfrak{r}_2: \quad y_4 = y_3(y_1 + \ln y_3 + A\ln y_2), \quad A \in \mathbb{R}\setminus\{0\}, \tag{1.1}$$

$$\mathfrak{r}_2, \mathfrak{r}_3: \quad y_4 = A\ln y_1 - \ln\left(y_2 \pm y_3^2\right), \quad A \in \mathbb{R}\setminus\{0\}, \tag{1.2}$$

$$(y_4 \pm y_3^2)y_2 + y_1 = A|z_1|, \quad A \in \mathbb{R}\setminus\{\pm 1\}, \tag{1.3}$$

$$\mathfrak{r}_4: \quad y_2 = |z_1||y_3 + iy_4|^A e^{B\arg(y_3+iy_4)}, \quad (A, B) \in \mathbb{R}^2\setminus\{(0,0)\}, \tag{1.4}$$

$$(x_1 - y_2y_4)^2 + (y_1 - y_3y_4)^2 = (1-A)|z_1|^2, \quad A < 1, \quad A \neq 0, \tag{1.5}$$

$$y_1y_4 + x_1y_3 = |z_1|^2(\ln y_2 + A\arg z_1), \quad A \in \mathbb{R}, \tag{1.6}$$

where z_1, z_2, z_3, z_4 are variables in $\mathbb{C}^4$, $x_k = \operatorname{Re} z_k$, $y_k = \operatorname{Im} z_k$, $k = 1, \ldots, 4$.

Remark 1.1. *Equations (1.2) and (1.3) in the above list are associated simultaneously with two families of algebras $\mathfrak{r}_2$, $\mathfrak{r}_3$. Such writing means that each family of hypersurfaces defined by equations (1.2) and (1.3) is an orbit of both $\mathfrak{r}_2$ and $\mathfrak{r}_3$. The coincidence of the orbits of two Lie algebras is a rather ordinary phenomenon, in the case when these algebras are subalgebras of a complete algebra $g(M)$ for the original orbit, see, for instance,* [7]. *Exactly this is the case for surfaces (1.2) and (1.3), for each of these surfaces the dimension $g(M)$ turns out to exceed 7, which is the dimension of each algebra of the families $\mathfrak{r}_2$, $\mathfrak{r}_3$.*

We note that the hypersurfaces defined by equations (1.1), (1.2) are tubular manifolds, while (1.4) is reduced to such manifold. The surfaces in the families (1.3), (1.5), (1.6), are likely to be reduced to the tubes only for particular values of the parameters. However, in general, the issue of such reduction is rather difficult and requires an independent study. An important property of these hypersurfaces is their sphericity, that is, their local equivalence to a sphere or its analogues. An example of studying the sphericity of the hypersurfaces in one of the obtained families is given in the last section.

Remark 1.2. *The issue on possible equivalence of hypersurfaces (1.1)–(1.6) for particular values of the parameters A and B requires a special study and in the present work we do not consider it.*

2. Holomorphic realization of Lie algebras

There are several known approaches to constructing classification of holomorphically homogeneous hypersurfaces. For instance, homogeneous hypersurfaces can be described by means of normal (canonical) Moser equations, see [17], that is, by means of finite sets of Taylor coefficients from the equations defining these hypersurfaces. For instance, this was the way for obtaining some of the aforementioned pieces of the classification in $\mathbb{C}^3$ ([3], [4]). Another approach for describing hypersurfaces is related with employing groups acting on them and associated Lie algebras. In work [16] there was demonstrated a technique for obtaining holomorphically homogeneous hypersurfaces in $\mathbb{C}^3$ on the base of constructing holomorphic realizations of abstract Lie algebras. By means of this approach, an essential part of the results on classification of holomorphically homogeneous hypersurfaces in the space $\mathbb{C}^3$, see [6], [7], [15]. It should be noted that sometimes it is useful to apply simultaneously both aforementioned approaches, see, for instance, [18].

Let us consider in more details the technique of constructing holomorphic realizations of Lie algebras, which will be used later for the proving Theorem 1.1.

Definition 2.1. *A real hypersurface $M \subset \mathbb{C}^4$ is called holomorphically homogeneous at a point $p \in M$ if there exists the Lie algebra of holomorphic vector fields tangential to M and having rank 7 in the vicinity of the point p.*

We choose some real seven-dimensional Lie algebra defined by its commutations relations.

With the chosen seven-dimensional Lie algebra, we associate the set of seven germs, centered at some point $p \in \mathbb{C}^4$, of holomorphic vector fields

$$\begin{aligned} e_k =& a_k(z_1, z_2, z_3, z_4)\frac{\partial}{\partial z_1} + b_k(z_1, z_2, z_3, z_4)\frac{\partial}{\partial z_2} \\ &+ c_k(z_1, z_2, z_3, z_4)\frac{\partial}{\partial z_3} + d_k(z_1, z_2, z_3, z_4)\frac{\partial}{\partial z_4}, \quad k = 1, \ldots, 7, \end{aligned} \tag{2.1}$$

linearly independent over $\mathbb{R}$. A brief writing is

$$e_k = (a_k, b_k, c_k, d_k), \quad k = 1, \ldots, 7,$$

where a_k, b_k, c_k, d_k are the germs of holomorphic (at the point p) functions of complex variables z_1, z_2, z_3, z_4. In what follows, the real and imaginary parts of the variables z_j will be denoted respectively by x_j and y_j, $j = 1, \ldots, 4$.

Definition 2.2. *A Lie algebra g of holomorphic vector fields in $\mathbb{C}^4$ is called a holomorphic realization of an abstract Lie algebra $\mathfrak{g}$ if the commutation relations of these algebras coincide.*

The commutator of two fields $[e_k, e_j]$ is calculated in the following well-known way:

$$[e_k, e_j] = \left(a_k \frac{\partial e_j}{\partial z_1} + b_k \frac{\partial e_j}{\partial z_2} + c_k \frac{\partial e_j}{\partial z_3} + d_k \frac{\partial e_j}{\partial z_4} \right) - \left(a_j \frac{\partial e_k}{\partial z_1} + b_j \frac{\partial e_k}{\partial z_2} + c_j \frac{\partial e_k}{\partial z_3} + d_j \frac{\partial e_k}{\partial z_4} \right).$$

Employing holomorphic transformations, the functional coefficients of fields (2.1) related with a Lie algebra can be changed and reduced to a simpler form. Here by a simpler field we can mean a field, one or more coefficients of which vanish or depend on less variables after a transformation. In particular, the following statement holds true, see, for instance, [12].

Lemma 2.1. *If on a Levi non-degenerate hypersurface $M \subset \mathbb{C}^4$ there is a pair of germs of commuting holomorphic vector fields e_j and e_k linearly independent over $\mathbb{R}$, then this pair can be flattened, that is, to be reduced to the form*

$$e_j = (0, 0, 0, 1), \quad e_k = (0, 0, 1, 0).$$

The simplification of even two fields to the form given in Lemma 2.1 leads to an essential simplification of other fields due to the commutation relations between the fields. Considering if necessary a series of subcases corresponding to vanishing or non-vanishing of several components of several vector fields, we finally reduce the basis of the considered Lie algebra to a form convenient for further integration, that is, for obtaining the equation of the surfaces via the algebra of holomorphic tangential vector fields. Here we should also take into consideration that the rank of the system of vector fields should be seven that reduces essentially the number of the subcases to be considered.

The algebras of vector fields possess also the following property being a generalization of the statements given in [16], [19] for $\mathbb{C}^4$.

Lemma 2.2. *Assume that the algebra of holomorphic vector fields in $\mathbb{C}^4$ possesses a quadruple of linearly independent vector fields, three of which read as*

$$e_1 = (0, 0, 0, 1), \quad e_2 = (0, 0, 1, 0), \quad e_3 = (0, 1, 0, 0).$$

If the components of the fourth field, up to the terms $\varphi_k(z_1)$, $(k = 1, 2, 3, 4)$, are linear functions of other variables, then a holomorphic change of variables removes all $\varphi_k(z_1)$ from this field. At the same time, the flattened form of the first triple of the fields and the linear components of the field e_4 are preserved.

We recall that in this work we are interesting in only Levi non-degenerate holomorphic homogeneous hypersurfaces. In view of this let us provide two statements, which in some cases allow us to conclude on the degeneracy of the corresponding hypersurfaces by the form of vector fields; these statements are easily extended to the case $\mathbb{C}^4$ from the case $\mathbb{C}^3$ and this is why we provide them without proving, see [7].

Lemma 2.3. *Assume that we are given a germ, centered at a point p, of a real analytic hypersurface $M \subset \mathbb{C}^4$ and a 7-dimensional algebra $g(M)$ of germs of holomorphic vector fields tangential to M and having rank 7 at the point p. If six basis holomorphic fields of the mentioned algebra have a zero coefficient at the same operator $\frac{\partial}{\partial z_k}$, then the hypersurface M is Levi degenerate.*

Lemma 2.4. *Assume that we are given a germ, centered at a point p, of a real analytic hypersurface $M \subset \mathbb{C}^4$ and a 7-dimensional algebra $g(M)$ of germs of holomorphic vector fields tangential to M and having rank 7 at the point p. If a quadruple of basis holomorphic fields of the mentioned algebra reads as*

$$e_j = (0, 0, c_j(z_1, z_2, z_3, z_4), d_j(z_1, z_2, z_3, z_4)), \quad j = 1, \ldots, 4,$$

up to re-denoting the variables and renumerating the fields, then the hypersurface M is Levi degenerate.

3. Holomorphic realizations of decomposable 7-dimensional Lie algebras with $\mathfrak{sl}(2)$-subalgebra

While employing the above described technique of holomorphic realizations for constructing the classification of holomorphically homogeneous hypersurfaces in $\mathbb{C}^4$, one has to consider in particular all 7-dimensional Lie algebras. However, opposite to the case $\mathbb{C}^3$, where 5-dimensional algebras considered, there is no complete list of algebras having dimension 7. At present, the descriptions of Lie algebras of dimension up to 6 are known [20], and while there are just several tens of 5-dimensional algebras, see [20], [21], the list of 6-dimensional algebras contains already hundreds of representatives. The classification of 7-dimensional Lie algebras is obviously much more bulky. In view of this, it is natural to study particular classes of 7-dimensional Lie algebras having in mind the experience of describing holomorphically homogeneous hypersurfaces in $\mathbb{C}^3$.

In order to prove Theorem 1.1, we shall construct holomorphic realizations of four 7-dimensional decomposable Lie algebras being direct sums of the 3-dimensional algebra $\mathfrak{sl}(2)$ and several 4-dimensional algebras defined by the following commutation relations:

Table 3.1

	$[e_1, e_2]$	$[e_1, e_3]$	$[e_2, e_3]$	$[e_4, e_6]$	$[e_4, e_7]$	$[e_5, e_6]$	$[e_5, e_7]$	$[e_6, e_7]$
$\mathfrak{r}_1$	e_1	$2e_2$	e_3		$2e_4$	e_4	e_5	$e_5 + e_6$
$\mathfrak{r}_2$	e_1	$2e_2$	e_3		$(h+1)e_4$	e_4	e_5	he_6
$\mathfrak{r}_3$	e_1	$2e_2$	e_3		$2pe_4$	e_4	$pe_5 - e_6$	$e_5 + pe_6$
$\mathfrak{r}_4$	e_1	$2e_2$	e_3	e_4	$-e_5$	e_5	e_4	

In this table $|h| \leqslant 1$, $p \geqslant 0$.

We note that the maximal dimension of the Abelian subalgebras for all algebras $\mathfrak{r}_1, \ldots, \mathfrak{r}_4$ is equal to three.

We construct the holomorphic realizations for each of the mentioned algebras assuming that their basis field are of the form

$$e_j = (a_j(z_1, z_2, z_3, z_4), b_j(z_1, z_2, z_3, z_4), c_j(z_1, z_2, z_3, z_4), d_j(z_1, z_2, z_3, z_4)), \quad j = 1, \ldots, 7.$$

The study of commutation relations for the algebras $\mathfrak{r}_1, \ldots, \mathfrak{r}_4$ allow us to make some preliminary simplification of vector fields similar for all algebras in the list.

Lemma 3.1. *Each holomorphic realization of the algebras $\mathfrak{r}_1, \dots, \mathfrak{r}_4$ possessing Levi non-degeberate integral hypersurfaces in $\mathbb{C}^4$ is reduced, up to a local holomorphic equivalence, to algebras with bases of the following forms:*

$$\begin{array}{rcl} e_1 & = & (0,1,0,0), \\ e_2 & = & (a_2(z_1), z_2 + b_2(z_1), c_2(z_1), d_2(z_1)), \\ e_3 & = & (2a_2(z_1)z_2 + a_3(z_1), z_2^2 + 2b_2(z_1)z_2 + b_3(z_1), \\ & & 2c_2(z_1)z_2 + c_3(z_1), 2d_2(z_1)z_2 + d_3(z_1)), \\ e_4 & = & (0,0,0,1), \\ e_5 & = & (0,0,1,0), \\ e_6 & = & (a_6(z_1,z_3,z_4), b_6(z_1,z_3,z_4), c_6(z_1,z_3,z_4), d_6(z_1,z_3,z_4)), \\ e_7 & = & (a_7(z_1,z_3,z_4), b_7(z_1,z_3,z_4), c_7(z_1,z_3,z_4), d_7(z_1,z_3,z_4)). \end{array} \tag{3.1}$$

Proof. According to Lemma 2.1, two commuting vector fields among seven fields can be flattened for each algebra. We suppose in what follows that

$$e_4 = (0,0,0,1), \quad e_5 = (0,0,1,0).$$

The relations $[e_1, e_4] = 0$, $[e_2, e_4] = 0$, $[e_3, e_4] = 0$ lead us to the identities

$$\left(-\frac{\partial a_j}{\partial z_4}, -\frac{\partial b_j}{\partial z_4}, -\frac{\partial c_j}{\partial z_4}, -\frac{\partial d_j}{\partial z_4}\right) = (0,0,0,0), \quad j = 1,2,3,$$

which imply that the functional coefficients of the fields e_1, e_2, e_3 are independent of the variable z_4.

Employing the identities $[e_1, e_5] = 0$, $[e_2, e_5] = 0$, $[e_3, e_5] = 0$, in the same way we get:

$$\left(-\frac{\partial a_j}{\partial z_3}, -\frac{\partial b_j}{\partial z_3}, -\frac{\partial c_j}{\partial z_3}, -\frac{\partial d_j}{\partial z_3}\right) = (0,0,0,0), \quad j = 1,2,3.$$

Therefore, the functional coefficients of the fields e_1, e_2, e_3 are also independent of the variable z_3.

According to Lemma 2.4, for Levi non-degeberate integral hypersurfaces, at some point at least one of the following two inequalities hold true:

$$(a_k(z_1,z_2), b_k(z_1,z_2)) \not\equiv (0,0) \quad (k = 1,2).$$

Without loss of generality we assume that $(a_1(z_1,z_2), b_1(z_1,z_2)) \not\equiv (0,0)$. Then the field e_1 can be reduced to the form:

$$e_1 = (0,1,0,0).$$

In view of the simplified form of the field e_1, the relations $[e_1, e_6] = 0$, $[e_1, e_7] = 0$ lead us to the identities:

$$\left(\frac{\partial a_6}{\partial z_2}, \frac{\partial b_6}{\partial z_2}, \frac{\partial c_6}{\partial z_2}, \frac{\partial d_6}{\partial z_2}\right) = (0,0,0,0),$$
$$\left(\frac{\partial a_7}{\partial z_2}, \frac{\partial b_7}{\partial z_2}, \frac{\partial c_7}{\partial z_2}, \frac{\partial d_7}{\partial z_2}\right) = (0,0,0,0).$$

This implies that the functional coefficients of the fields e_6, e_7 are independent of the variable z_2.

Moreover, since $[e_1, e_2] = e_1$, then

$$\left(\frac{\partial a_2}{\partial z_2}, \frac{\partial b_2}{\partial z_2}, \frac{\partial c_2}{\partial z_2}, \frac{\partial d_2}{\partial z_2}\right) = (0,1,0,0),$$

and this gives the following form of the field e_2:

$$e_2 = (a_2(z_1), z_2 + b_2(z_1), c_2(z_1), d_2(z_1)).$$

Taking into consideration the obtained simplifications, the relation $[e_1, e_3] = 2e_2$ gives the identity

$$\left(\frac{\partial a_3}{\partial z_2}, \frac{\partial b_3}{\partial z_2}, \frac{\partial c_3}{\partial z_2}, \frac{\partial d_3}{\partial z_2}\right) = (2a_2(z_1), 2z_2 + 2b_2(z_1), 2c_2(z_1), 2d_2(z_1)),$$

which allows us to transform the field e_3 to the form given in the formulation of the lemma.

Let us show now that the case $(a_1, b_1) \equiv (0, 0)$ gives rise to a contradiction.

Let $(a_1(z_1, z_2), b_1(z_1, z_2)) \equiv (0, 0)$ at some point of the surface. As it has been mentioned above, in this case the inequality $(a_2(z_1, z_2), b_2(z_1, z_2)) \not\equiv (0, 0)$ is necessarily satisfied. Therefore, the field e_2 can be reduced to the form

$$e_2 = (0, 1, 0, 0).$$

Thus, the fields e_1, e_2 and e_3 are written as follows:

$$\begin{aligned} e_1 &= (0, 0, c_1(z_1, z_2), d_1(z_1, z_2)), \\ e_2 &= (0, 1, 0, 0), \\ e_3 &= (a_3(z_1, z_2), b_3(z_1, z_2), c_3(z_1, z_2), d_3(z_1, z_2)). \end{aligned}$$

According to the commutation relations of the algebra, the identity $[e_1, e_3] = 2e_2$ should hold. However, the first two components in the commutation $[e_1, e_3]$ are zero, while the first two components of the field e_2 read as $(0, 2)$. Therefore, the case $(a_1, b_1) \equiv (0, 0)$ is impossible. The proof is complete. □

Thus, we suppose that the set of seven fields for each of the algebras $\mathfrak{r}_1, \ldots, \mathfrak{r}_4$ is originally of form (3.1).

Remark 3.1. *Hereafter, while writing the coefficients of the vectors fields, by the symbols A_k, B_k, C_k, D_k $(k = 1, \ldots, 7)$ we denote complex constants.*

3.1. Holomorphic realizations of algebra $\mathfrak{r}_1$. By commutation relations $[e_4, e_6] = 0$, $[e_4, e_7] = 2e_4$, $[e_5, e_6] = e_4$, $[e_5, e_7] = e_5$ we get:

$$\begin{aligned} \left(\frac{\partial a_6}{\partial z_4}, \frac{\partial b_6}{\partial z_4}, \frac{\partial c_6}{\partial z_4}, \frac{\partial d_6}{\partial z_4}\right) &= (0, 0, 0, 0), \\ \left(\frac{\partial a_7}{\partial z_4}, \frac{\partial b_7}{\partial z_4}, \frac{\partial c_7}{\partial z_4}, \frac{\partial d_7}{\partial z_4}\right) &= (0, 0, 0, 2), \\ \left(\frac{\partial a_6}{\partial z_3}, \frac{\partial b_6}{\partial z_3}, \frac{\partial c_6}{\partial z_3}, \frac{\partial d_6}{\partial z_3}\right) &= (0, 0, 0, 1), \\ \left(\frac{\partial a_7}{\partial z_3}, \frac{\partial b_7}{\partial z_3}, \frac{\partial c_7}{\partial z_3}, \frac{\partial d_7}{\partial z_3}\right) &= (0, 0, 1, 0). \end{aligned}$$

These identities allow us to reduce the fields e_6 and e_7 to the form

$$\begin{aligned} e_6 &= (a_6(z_1), b_6(z_1), c_6(z_1), z_3 + d_6(z_1)), \\ e_7 &= (a_7(z_1), b_7(z_1), z_3 + c_7(z_1), 2z_4 + d_7(z_1)). \end{aligned}$$

A further constructing of holomorphic realizations of the algebra $\mathfrak{r}_1$ requires a consideration of a series of cases.

Case 1. Let $a_2(z_1) \not\equiv 0$. Then the field e_2 in set (3.1) can be reduced to

$$e_2 = (1, z_2, 0, 0).$$

By identities $[e_2, e_6] = 0$, $[e_2, e_7] = 0$ we get the relations:

$$\begin{aligned} (a_6'(z_1), b_6'(z_1) - b_6(z_1), c_6'(z_1), d_6'(z_1)) &= (0, 0, 0, 0), \\ (a_7'(z_1), b_7'(z_1) - b_7(z_1), c_7'(z_1), d_7'(z_1)) &= (0, 0, 0, 0). \end{aligned}$$

Hence,

$$a_6(z_1) = A_6, \quad b_6(z_1) = B_6 e^{z_1}, \quad c_6(z_1) = C_6, \quad d_6(z_1) = D_6,$$
$$a_7(z_1) = A_7, \quad b_7(z_1) = B_7 e^{z_1}, \quad c_7(z_1) = C_7, \quad d_7(z_1) = D_7.$$

By identity $[e_6, e_7] = e_5 + e_6$ we get

$$(0, e^{z_1}(A_6B_7 - A_7B_6), C_6, z_3 - C_7 + 2D_6) = (A_6, B_6e^{z_1}, 1 + C_6, z_3 + D_6).$$

Comparing the third components in the left hand side and the right hand side, we arrive at the identity

$$C_6 = 1 + C_6.$$

Thus, case 1 leads to a contradiction.

Case 2. Let $a_2(z_1) \equiv 0$, $a_6(z_1) \not\equiv 0$. Then, in view of simplified fields e_6, e_7, in the set (3.1) we can get:

$$e_2 = (0, z_2 + b_2(z_1), c_2(z_1), d_2(z_1)),$$
$$e_6 = (1, 0, 0, z_3).$$

By identity $[e_2, e_6] = 0$ we obtain

$$(0, -b_2'(z_1), -c_2'(z_1), -d_2'(z_1) + c_2(z_1)) = (0, 0, 0, 0).$$

This yields

$$b_2(z_1) = B_2, \quad c_2(z_1) = C_2, \quad d_2(z_1) = C_2z_1 + D_2.$$

The relation $[e_2, e_7] = 0$ leads us to the following identity:

$$(0, -b_7(z_1), C_2, -a_7(z_1)C_2 + 2C_2z_1 + 2D_2) = (0, 0, 0, 0).$$

We then get $C_2 = 0$, $D_2 = 0$. Thus, the fields e_1 and e_2 become

$$e_1 = (0, 1, 0, 0),$$
$$e_2 = (0, z_2 + B_2, 0, 0),$$

which is impossible for a non-degenerate hypersurface.

Case 3. Let $a_2(z_1) \equiv 0$, $a_6(z_1) \equiv 0$, $a_7(z_1) \not\equiv 0$. Then the field e_7 can be reduced to the form

$$e_7 = (1, 0, z_3, 2z_4).$$

The relation $[e_2, e_3] = e_3$ allows to conclude immediately on the degeneracy of the hypersurfaces in this case. Indeed, the first component of the commutator $[e_2, e_3]$ vanishes, while the first component of the field e_3 equals to $a_3(z_1)$. Thus, $a_3(z_1) \equiv 0$, and all first components of the fields $e_1, \dots, e_6$ vanish, which gives the degeneracy according to Lemma 2.3.

Thus, we conclude that the algebra $\mathfrak{r}_1$ has no non-degenerate holomorphic realizations.

3.2. Holomorphic realization of algebra $\mathfrak{r}_2$. Here the commutation relations are $[e_4, e_6] = 0$, $[e_4, e_7] = (h+1)e_4$, $[e_5, e_6] = e_4$, $[e_5, e_7] = e_5$. Similarly to the previous case, they give rise to a simplified form of the fields e_6 and e_7:

$$e_6 = (a_6(z_1), b_6(z_1), c_6(z_1), z_3 + d_6(z_1)),$$
$$e_7 = (a_7(z_1), b_7(z_1), z_3 + c_7(z_1), (h+1)z_4 + d_7(z_1)).$$

Case 1. Let $a_2(z_1) \not\equiv 0$. Then the field e_2 can be reduced to the form

$$e_2 = (1, z_2, 0, 0).$$

By identities $[e_2, e_6] = 0$, $[e_2, e_7] = 0$ we obtain:

$$(a_6'(z_1), b_6'(z_1) - b_6(z_1), c_6'(z_1), d_6'(z_1)) = (0, 0, 0, 0),$$
$$(a_7'(z_1), b_7'(z_1) - b_7(z_1), c_7'(z_1), d_7'(z_1)) = (0, 0, 0, 0),$$

and this yields that

$$a_6(z_1)=A_6, b_6(z_1)=B_6e^{z_1}, c_6(z_1)=C_6, d_6(z_1)=D_6,$$
$$a_7(z_1)=A_7, b_7(z_1)=B_7e^{z_1}, c_7(z_1)=C_7, d_7(z_1)=D_7.$$

The relation $[e_2,e_3]=e_3$ gives rise to the identity

$$\big(a_3'(z_1)+2z_2, z_2^2-b_3(z_1)+b_3'(z_1), c_3'(z_1), d_3'(z_1)\big)=(2z_2+a_3(z_1), z_2^2+b_3(z_1), c_3(z_1), d_3(z_1)),$$

which yields a simplified form of the field e_3:

$$a_3(z_1)=A_3e^{z_1},\quad b_3(z_1)=B_3e^{2z_1},\quad c_3(z_1)=C_3e^{z_1},\quad d_3(z_1)=D_3e^{z_1}.$$

The consideration of remaining relations $[e_3,e_6]=0$, $[e_3,e_7]=0$, $[e_6,e_7]=he_6$ allow us to obtain the following system of equations relating the coefficients of the fields and the parameter h of the algebra:

$$\begin{gathered}A_3A_6+2B_6=0,\quad A_3B_6-2A_6B_3=0,\quad A_6C_3=0,\quad A_6D_3-C_3=0,\\ A_3A_7+2B_7=0,\quad A_3B_7-2A_7B_3=0,\quad C_3(A_7-1)=0,\\ D_3(A_7-h-1)=0,\quad hA_6=0,\quad A_6B_7-A_7B_6-B_6h=0,\\ C_6(h-1)=0,\quad D_6-C_7=0.\end{gathered}\tag{3.2}$$

This system has eight solutions but only three of them give the bases of the algebra of the holomorphic vector fields corresponding to non-degenerate hypersurfaces. These solutions are as follows.

a) A solution of system (3.2) is

$$\begin{gathered}B_3=-\frac{1}{4}A_3^2,\quad B_6=-\frac{1}{2}A_3A_6,\quad B_7=-\frac{1}{2}A_3A_7,\\ C_3=0,\quad C_6=0,\quad C_7=D_6,\quad D_3=0,\quad h=0.\end{gathered}$$

The bases of the algebras of holomorphic vectors fields are

$$\begin{aligned}e_1&=(0,1,0,0),\\ e_2&=(1,z_2,0,0),\\ e_3&=\left(2z_2+A_3e^{z_1}, z_2^2-\frac{1}{4}A_3^2e^{2z_1},0,0\right),\\ e_4&=(0,0,0,1),\\ e_5&=(0,0,1,0),\\ e_6&=\left(A_6,-\frac{1}{2}A_3A_6e^{z_1},0,z_3+D_6\right),\\ e_7&=\left(A_7,-\frac{1}{2}A_3A_7e^{z_1},z_3+D_6,z_4+D_7\right).\end{aligned}\tag{3.3}$$

b) A solution of system (3.2) is

$$A_6=0,\quad A_7=2,\quad B_3=-\frac{1}{4}A_3^2,\quad B_6=0,\quad B_7=-A_3,\quad C_3=0,\quad C_7=D_6,\quad h=1.$$

The bases of the algebras of holomorphic vectors fields are

$$
\begin{aligned}
e_1 &= (0,1,0,0)\,,\\
e_2 &= (1,z_2,0,0)\,,\\
e_3 &= \left(2z_2 + A_3e^{z_1}, z_2^2 - \frac{1}{4}A_3^2e^{2z_1}, 0, D_3e^{z_1}\right),\\
e_4 &= (0,0,0,1)\,,\\
e_5 &= (0,0,1,0)\,,\\
e_6 &= (0,0,C_6,z_3+D_6)\,,\\
e_7 &= (2,-A_3e^{z_1}, z_3+D_6, 2z_4+D_7)\,.
\end{aligned}
\tag{3.4}
$$

c) A solution of system (3.2) is

$$A_6 = 0, \quad B_3 = -\frac{1}{4}A_3^2, \quad B_6 = 0, \quad B_7 = -\frac{1}{2}A_3A_7, \quad C_3 = 0, \quad C_7 = D_6, \quad D_3 = 0, \quad h = 1.$$

The bases of the algebras of holomorphic vectors fields are

$$
\begin{aligned}
e_1 &= (0,1,0,0)\,,\\
e_2 &= (1,z_2,0,0)\,,\\
e_3 &= \left(2z_2 + A_3e^{z_1}, z_2^2 - \frac{1}{4}A_3^2e^{2z_1}, 0, 0\right),\\
e_4 &= (0,0,0,1)\,,\\
e_5 &= (0,0,1,0)\,,\\
e_6 &= (0,0,C_6,z_3+D_6)\,,\\
e_7 &= \left(A_7, -\frac{1}{2}A_3A_7e^{z_1}, z_3+D_6, 2z_4+D_7\right).
\end{aligned}
\tag{3.5}
$$

Case 2. Let $a_2(z_1) \equiv 0$, $a_6(z_1) \not\equiv 0$. Then the field e_6 can be transformed to the form

$$e_6 = (1,0,0,z_3)\,.$$

By the relations $[e_2,e_6] = 0$, $[e_3,e_6] = 0$, $[e_6,e_7] = he_6$ we obtain the identities

$$
\begin{aligned}
&(0, -b_2'(z_1), -c_2'(z_1), -d_2'(z_1) + c_2(z_1)) = (0,0,0,0),\\
&(-a_3'(z_1), -b_3'(z_1), -c_3'(z_1), -d_3'(z_1) + c_3(z_1)) = (0,0,0,0),\\
&(a_7'(z_1), b_7'(z_1), c_7'(z_1), z_3h - c_7(z_1) + d_7'(z_1)) = (h,0,0,hz_3).
\end{aligned}
$$

The solutions of these equations allow us to simplify the form of the fields e_2, e_3 and e_7:

$$
\begin{aligned}
e_2 &= (0, z_2 + B_2, C_2, C_2z_1 + D_2)\,,\\
e_3 &= \left(A_3, z_2^2 + 2B2z_2 + B_3, 2C_2z_2 + C_3, 2(C_2z_1 + D_2)z_2 + C_3z_1 + D_3\right),\\
e_7 &= (hz_1 + A_7, B_7, z_3 + C_7, (h+1)z_4 + C_7z_1 + D_7)\,.
\end{aligned}
$$

The commutation relation $[e_2,e_7] = 0$ for the transformed fields gives the identity

$$(0, -B_7, C_2, C_2z_1 + D_2h + D_2 - A_7C_2) = (0,0,0,0),$$

which implies that

$$B_7 = 0, \quad C_2 = 0, \quad D_2(h+1) = 0.$$

As $C_2 = 0$, the field e_2 becomes $e_2 = (0, z_2 + B_2, 0, D_2)$ and we assume that $D_2 = 0$, we get a field of form $e_2 = (0, z_2 + B_2, 0, 0)$, which under the presence $e_1 = (0,1,0,0)$ gives a degeneracy of the hypersurface. This is why the case $D_2 = 0$ is impossible and the identity $h = -1$ should hold true.

We write the relation $[e_3, e_7] = 0$ in an expanded form:

$$(-A_3, 0, C_3, A_3C_7 - C_3A_7 + C_3z_1) = (0, 0, 0, 0).$$

We then get that $A_3 = 0$ and $C_3 = 0$. Thus, all first and third components in the fields $e_1, \dots, e_4$ vanish and according to Lemma 2.4 this means a degeneracy of the hypersurface.

Case 3. Let $a_2(z_1) \equiv 0$, $a_6(z_1) \equiv 0$ and $a_7(z_1) \not\equiv 0$. Then the field e_7 can be reduced to the form

$$e_7 = (1, 0, z_3, (h+1)z_4).$$

The first component in the commutation $[e_2, e_3]$ is obviously zero and at the same time, the identity $[e_2, e_3] = e_3$ should be satisfied. Since the first component of the field e_3 is equal $a_3(z_1)$, then we necessarily have $a_3(z_1) = 0$. However in this case all first components of the fields $e_1, \dots, e_6$ vanish and this means the degeneracy of the hypersurface by Lemma 2.3.

3.3. Holomorphic realization of algebra $\mathfrak{r}_3$. Here by the commutation relations $[e_4, e_6] = 0$, $[e_4, e_7] = 2pe_4$, $[e_5, e_6] = e_4$ we get:

$$\begin{aligned} e_6 &= (a_6(z_1), b_6(z_1), c_6(z_1), z_3 + d_6(z_1))\,, \\ e_7 &= (a_7(z_1, z_3), b_7(z_1, z_3), c_7(z_1, z_3), 2pz_4 + d_7(z_1, z_3))\,. \end{aligned}$$

We write in an expanded form the relation $[e_5, e_7] = pe_5 - e_6$:

$$\left(\frac{\partial a_7}{\partial z_3}, \frac{\partial b_7}{\partial z_3}, \frac{\partial c_7}{\partial z_3}, \frac{\partial d_7}{\partial z_3}\right) = (-a_6(z_1), -b_6(z_1), p - c_6(z_1), -z_3 - d_6(z_1)).$$

We get the following form of the field e_7:

$$\begin{aligned} e_7 = \Big(&- a_6(z_1)z_3 + a_7(z_1), -b_6(z_1)z_3 + b_7(z_1)\,, \\ & pz_3 - c_6(z_1)z_3 + c_7(z_1), 2pz_4 - \frac{1}{2}z_3^2 - d_6(z_1)z_3 + d_7(z_1)\Big)\,. \end{aligned}$$

Case 1. Let $a_2(z_1) \not\equiv 0$, then the field e_2 can be reduced to the form

$$e_2 = (1, z_2, 0, 0)\,.$$

By relation $[e_2, e_6] = 0$ we get the identity

$$(a_6'(z_1), b_6'(z_1) - b_6(z_1), c_6'(z_1), d_6'(z_1)) = (0, 0, 0, 0),$$

which allows us to simplify the form of the fields e_6 and e_7:

$$\begin{aligned} e_6 &= (A_6, B_6e^{z_1}, C_6, z_3 + D_6)\,, \\ e_7 &= \Big(- A_6z_3 + a_7(z_1), -B_6e^{z_1}z_3 + b_7(z_1)\,, \\ & pz_3 - C_6z_3 + c_7(z_1), 2pz_4 - \frac{1}{2}z_3^2 - D_6z_3 + d_7(z_1)\Big)\,. \end{aligned}$$

As a result, the relation $[e_2, e_7] = 0$ gives rise to the identity

$$(a_7'(z_1), b_7'(z_1) - b_7(z_1), c_7'(z_1), d_7'(z_1)) = (0, 0, 0, 0),$$

by which we find the coefficients of the field e_7. We finally obtain:

$$\begin{aligned} e_7 = \Big(&- A_6z_3 + A_7, -B_6e^{z_1}z_3 + B_7e^{z_1}\,, \\ & pz_3 - C_6z_3 + C_7, 2pz_4 - \frac{1}{2}z_3^2 - D_6z_3 + D_7\Big)\,. \end{aligned}$$

We now employ relation $[e_2, e_3] = e_3$:

$$\left(a_3'(z_1) + 2z_2, z_2^2 - b_3(z_1) + b_3'(z_1), c_3'(z_1), d_3'(z_1)\right) = \left(2z_2 + a_3(z_1), z_2^2 + b_3(z_1), c_3(z_1), d_3(z_1)\right).$$

This gives:

$$e_3 = \left(2z_2 + A_3 e^{z_1}, z_2^2 + B_3 e^{2z_1}, C_3 e^{z_1}, D_3 e^{z_1}\right).$$

The remaining identities $[e_3, e_6] = 0$, $[e_3, e_7] = 0$, $[e_6, e_7] = e_5 + pe_6$ lead us to a system of equations similar to (3.2). Some of the solutions to this system, as in case (3.2), are associated to the algebras having only degenerate orbits. Here we provide only solutions which generate only more interesting for us algebras admitting Levi non-degenerate orbits.

a) $A_6 = 0$, $B_3 = -\frac{1}{4}A_3^2$, $B_6 = 0$, $B_7 = -\frac{1}{2}A_3 A_7$, $C_3 = 0$, $C_6 = \pm i$, $C_7 = D_6(p \mp i)$, $D_3 = 0$. The bases of the algebras of holomorphic vectors fields are

$$\begin{aligned}
e_1 &= (0, 1, 0, 0), \\
e_2 &= (1, z_2, 0, 0), \\
e_3 &= \left(2z_2 + A_3 e^{z_1}, z_2^2 - \frac{1}{4}A_3^2 e^{2z_1}, 0, 0\right), \\
e_4 &= (0, 0, 0, 1), \\
e_5 &= (0, 0, 1, 0), \\
e_6 &= (0, 0, \pm i, z_3 + D_6), \\
e_7 &= \left(A_7, -\frac{1}{2}A_3 A_7 e^{z_1}, (z_3 + D_6)(p \mp i), 2pz_4 - \frac{1}{2}z_3^2 - D_6 z_3 + D_7\right).
\end{aligned} \tag{3.6}$$

b) $A_6 = 0$, $A_7 = 2p$, $B_3 = -\frac{1}{4}A_3^2$, $B_6 = 0$, $B_7 = -pA_3$, $C_3 = 0$, $C_6 = \pm i$, $C_7 = D_6(p \mp i)$. The bases of the algebras of holomorphic vectors fields are

$$\begin{aligned}
e_1 &= (0, 1, 0, 0), \\
e_2 &= (1, z_2, 0, 0), \\
e_3 &= \left(2z_2 + A_3 e^{z_1}, z_2^2 - \frac{1}{4}A_3^2 e^{2z_1}, 0, D_3 e^{z_1}\right), \\
e_4 &= (0, 0, 0, 1), \\
e_5 &= (0, 0, 1, 0), \\
e_6 &= (0, 0, \pm i, z_3 + D_6), \\
e_7 &= \left(2p, -pA_3 e^{z_1}, (z_3 + D_6)(p \mp i), 2pz_4 - \frac{1}{2}z_3^2 - D_6 z_3 + D_7\right).
\end{aligned} \tag{3.7}$$

Case 2. Let $a_2(z_1) \equiv 0$, $a_6(z_1) \not\equiv 0$. Then the field e_6 can be reduced to the form

$$e_6 = (1, 0, 0, z_3).$$

We write in an expanded form the relations $[e_2, e_6] = 0$, $[e_6, e_7] = e_5 + pe_6$:

$$(0, -b_2'(z_1), -c_2'(z_1), -d_2'(z_1) + c_2(z_1)) = (0, 0, 0, 0),$$
$$(a_7'(z_1), b_7'(z_1), c_7'(z_1), d_7'(z_1) - c_7(z_1) + pz_3) = (p, 0, 1, pz_3).$$

Solving the written equations, we obtain a simplified form of the fields e_2 and e_7:

$$\begin{aligned}
e_2 &= (0, z_2 + B_2, C_2, C_2 z_1 + D_2), \\
e_7 &= \left(pz_1 - z_3 + A_7, B_7, pz_3 + z_1 + C_7, 2pz_4 - \frac{1}{2}z_3^2 + \frac{1}{2}z_1^2 + C_7 z_1 + D_7\right).
\end{aligned}$$

Employing the relation $[e_2, e_7] = 0$, we get

$$(-C_2, -B_7, pC_2, pC_2 z_1 - A_7 C_2 + 2pD_2) = (0, 0, 0, 0),$$

and this implies
$$C_2 = 0, \quad B_7 = 0, \quad pD_2 = 0.$$

If we assume that $D_2 = 0$, then we get $e_2 = (0, z_2 + B_2, 0, 0)$, and under the presence of the field $e_1 = (0, 1, 0, 0)$ this is possible only if the hypersurface is degenerate. Hence, we can assume that $D_2 \neq 0$ and therefore, $p = 0$.

By relation $[e_3, e_6] = 0$ we obtain the identity
$$(-a_3'(z_1), -b_3'(z_1), -c_3'(z_1), -d_3'(z_1) + c_3(z_1)) = (0,0,0,0),$$
which allows us to get a modified form of the field e_3:
$$e_3 = \left(A_3, 2B_2 z_2 + z_2^2 + B_3, C_3, C_3 z_1 + 2D_2 z_2 + D_3\right).$$

Employing the identity $[e_3, e_7] = 0$ written in an expanded form as
$$(-C_3, 0, A_3, A_3 C_7 + A_3 z_1 - C_3 A_7) = (0,0,0,0),$$
we get that $A_3 = C_3 = 0$, that is,
$$e_3 = \left(0, z_2^2 + 2B_2 z_2 + B_3, 0, 2D_2 z_2 + D_3\right).$$

Thus, in the field $e_1, \ldots, e_4$ all first and third components turn out to be zero and by Lemma 2.4 this means the degeneracy of the hypersurface.

Case 3. Let $a_2(z_1) \equiv 0$, $a_6(z_1) \equiv 0$ and $a_7(z_1) \not\equiv 0$. Then the field e_7 can be reduced to the form
$$e_7 = (1, -b_6(z_1) z_3, p z_3 - c_6(z_1) z_3, 2p z_4 - \frac{1}{2} z_3^2 - d_6(z_1) z_3).$$

To prove that in this case only degenerate hypersurfaces are possible, we note that the first component in the commutator $[e_2, e_3]$ vanishes and the first component in the field e_3 is equal to $a_3(z_1)$. Since the identity $[e_2, e_3] = e_3$, we necessarily have $a_3(z_1) = 0$ but in this case all components of the fields $e_1, \ldots, e_6$ vanish and this gives the degeneracy by Lemma 2.3.

3.4. Holomorphic realizations of algebra $\mathfrak{r}_4$. Expanding the commutation relations $[e_4, e_6] = e_4$, $[e_4, e_7] = -e_5$, $[e_5, e_6] = e_5$, $[e_5, e_7] = e_4$, we get a simplified form of the fields e_6 and e_7:
$$e_6 = (a_6(z_1), b_6(z_1), z_3 + c_6(z_1), z_4 + d_6(z_1)),$$
$$e_7 = (a_7(z_1), b_7(z_1), -z_4 + c_7(z_1), z_3 + d_7(z_1)).$$

Case 1. Let $a_2(z_1) \not\equiv 0$. Then, employing holomorphic change of variables, the field e_2 can be reduced to the form
$$e_2 = (1, z_2, 0, 0).$$
The relations $[e_2, e_6] = 0$, $[e_2, e_7] = 0$ give rise to the identities
$$(a_6'(z_1), b_6'(z_1) - b_6(z_1), c_6'(z_1), d_6'(z_1)) = (0,0,0,0),$$
$$(a_7'(z_1), b_7'(z_1) - b_7(z_1), c_7'(z_1), d_7'(z_1)) = (0,0,0,0),$$
by which we get
$$e_6 = (A_6, B_6 e^{z_1}, z_3 + C_6, z_4 + D_6),$$
$$e_7 = (A_7, B_7 e^{z_1}, -z_4 + C_7, z_3 + D_7).$$
One more relation $[e_2, e_3] = e_3$, implying the identity
$$\left(a_3'(z_1) + 2z_2, b_3'(z_1) - b_3(z_1) + z_2^2, c_3'(z_1), d_3'(z_1)\right) = \left(2z_2 + a_3(z_1), z_2^2 + b_3(z_1), c_3(z_1), d_3(z_1)\right),$$
give the following form for the field e_3:
$$\left(2z_2 + A_3 e^{z_1}, z_2^2 + B_3 e^{2z_1}, C_3 e^{z_1}, D_3 e^{z_1}\right).$$

The remaining relations $[e_3, e_6] = 0$, $[e_3, e_7] = 0$, $[e_6, e_7] = 0$ here also lead to a rather bulky system of equations for the coefficients of the fields e_3, e_6, e_7. This system has four solutions and three of them give the bases of the algebras of holomorphic vector fields corresponding to non-degenerate hypersurfaces.

a) $B_3 = -\frac{1}{4}A_3^2$, $B_6 = -\frac{1}{2}A_3A_6$, $B_7 = -\frac{1}{2}A_3A_7$, $C_3 = 0$, $C_6 = D_7$, $C_7 = -D_6$, $D_3 = 0$. The bases of the algebras of holomorphic vectors fields are

$$\begin{aligned}
e_1 &= (0,1,0,0)\,,\\
e_2 &= (1,z_2,0,0)\,,\\
e_3 &= \left(2z_2 + A_3e^{z_1}, z_2^2 - \frac{1}{4}A_3^2e^{2z_1}, 0, 0\right),\\
e_4 &= (0,0,0,1)\,,\\
e_5 &= (0,0,1,0)\,,\\
e_6 &= \left(A_6, -\frac{1}{2}A_3A_6e^{z_1}, z_3 + D_7, z_4 + D_6\right),\\
e_7 &= \left(A_7, -\frac{1}{2}A_3A_7e^{z_1}, -(z_4 + D_6), z_3 + D_7\right).
\end{aligned} \tag{3.8}$$

b) Extra two solutions unified via the sign «±»:

$$A_6 = 1, \quad A_7 = \pm i, \quad B_3 = -\frac{1}{4}A_3^2, \quad B_6 = -\frac{1}{2}A_3,$$
$$B_7 = \mp\frac{i}{2}A_3, \quad C_3 = \pm iD_3, \quad C_6 = D_7, \quad C_7 = -D_6.$$

The bases of the algebras of holomorphic vectors fields are

$$\begin{aligned}
e_1 &= (0,1,0,0)\,,\\
e_2 &= (1,z_2,0,0)\,,\\
e_3 &= \left(2z_2 + A_3e^{z_1}, z_2^2 - \frac{1}{4}A_3^2e^{2z_1}, \pm iD_3e^{z_1}, D_3e^{z_1}\right),\\
e_4 &= (0,0,0,1)\,,\\
e_5 &= (0,0,1,0)\,,\\
e_6 &= \left(1, -\frac{1}{2}A_3e^{z_1}, z_3 + D_7, z_4 + D_6\right),\\
e_7 &= \left(\pm i, \mp\frac{i}{2}A_3e^{z_1}, -(z_4 + D_6), z_3 + D_7\right).
\end{aligned} \tag{3.9}$$

Case 2. Let $a_2(z_1) \equiv 0$, $a_6(z_1) \not\equiv 0$. Then the field e_6 can be reduced to the form

$$e_6 = (1, 0, z_3, z_4)\,.$$

Employing the relations $[e_2, e_6] = 0$, $[e_6, e_7] = 0$, we obtain

$$(0, -b_2'(z_1), -c_2'(z_1) + c_2(z_1), -d_2'(z_1) + d_2(z_1)) = (0,0,0,0),$$
$$(a_7'(z_1), b_7'(z_1), c_7'(z_1) - c_7(z_1), d_7'(z_1) - d_7(z_1)) = (0,0,0,0).$$

Hence,

$$\begin{aligned}
e_2 &= (0, z_2 + B_2, C_2e^{z_1}, D_2e^{z_1})\,,\\
e_7 &= (A_7, B_7, -z_4 + C_7e^{z_1}, z_3 + D_7e^{z_1})\,.
\end{aligned}$$

The commutation relation $[e_3, e_6] = 0$ yields the identity

$$(-a_3'(z_1), -b_3'(z_1), -c_3'(z_1) + c_3(z_1), -d_3'(z_1) + d_3(z_1)) = (0,0,0,0),$$

which implies the following form for the field e_3:

$$e_3 = \left(A_3, z_2^2 + 2B_2 z_2 + B_3, (2C_2 z_2 + C_3)e^{z_1}, (2D_2 z_2 + DC_3)e^{z_1}\right).$$

Considering the final identities $[e_2, e_3] = e_3$, $[e_2, e_7] = 0$, $[e_3, e_7] = 0$ lead us to three admissible sets of the coefficients of the fields e_3, e_6, e_7. And only two of them, unified via the sign «$\pm$», give the bases of algebras of holomorphic vector fields admitting non-degenerate integral hypersurfaces.

The values of the coefficients

$$A_3 = 0, \quad A_7 = \pm i, \quad B_3 = B_2^2, \quad B_7 = 0, \quad C_2 = \pm i D_2, \quad C_3 = \pm 2iB_2D_2, \quad D_3 = 2B_2D_2.$$

The bases of the algebras of holomorphic vectors fields are

$$\begin{aligned}
e_1 &= (0,1,0,0)\,,\\
e_2 &= (0, z_2 + B_2, \pm i D_2 e^{z_1}, D_2 e^{z_1})\,,\\
e_3 &= \left(0, (z_2+B_2)^2, (\pm 2iD_2 z_2 \pm 2iB_2D_2)e^{z_1}, 2D_2 e^{z_1}(z_2 + B_2)\right),\\
e_4 &= (0,0,0,1)\,,\\
e_5 &= (0,0,1,0)\,,\\
e_6 &= (1, 0, z_3, z_4)\,,\\
e_7 &= (\pm i, 0, C_7 e^{z_1} - z_4, z_3 + D_7 e^{z_1})\,.
\end{aligned} \tag{3.10}$$

Case 3. Let $a_2(z_1) \equiv 0$, $a_6(z_1) \equiv 0$ and $a_7(z_1) \not\equiv 0$. Then the field e_7 can be reduced to the form

$$e_7 = (1, 0, -z_4, z_3).$$

By relation $[e_2, e_6] = 0$ we get

$$(0, -b_6(z_1), c_2(z_1), d_2(z_1)) = (0,0,0,0),$$

and this yields that

$$e_2 = (0, z_2 + b_2(z_1), 0, 0).$$

The field e_2 of such under the presence of the field $e_1 = (0,1,0,0)$ is possible only for Levi degenerate hypersurfaces.

4. Equations of hypersurfaces

The next step after finding the holomophic realizations of the Lie algebras is the obtaining of their orbits. A necessary condition for a real hypersurface M defined by an equation $\Phi = 0$ to be an orbit of a holomorphic realization of an algebra $\mathfrak{g}$ is the identity

$$\mathrm{Re}\,\left(e_k\,(\Phi)|_M\right) \equiv 0, \tag{4.1}$$

which should be satisfied for each basis field e_k of this algebra.

Thus, finding the orbits of holomorphic realizations of algebras $\mathfrak{r}_2$, $\mathfrak{r}_3$, $\mathfrak{r}_4$ is reduced to solving a system of partial differential equations. For instance, for one of realizations (3.6) we need to solve the system of the following form:

$$
\begin{aligned}
&\operatorname{Re}\left(\frac{\partial \Phi}{\partial z_2}\right)=0, \quad \operatorname{Re}\left(\frac{\partial \Phi}{\partial z_3}\right)=0, \quad \operatorname{Re}\left(\frac{\partial \Phi}{\partial z_4}\right)=0,\\
&\operatorname{Re}\left(\frac{\partial \Phi}{\partial z_1}+z_2\frac{\partial \Phi}{\partial z_2}\right)=0,\\
&\operatorname{Re}\left(i\frac{\partial \Phi}{\partial z_3}+\left(z_3+D_6\right)\frac{\partial \Phi}{\partial z_4}\right)=0,\\
&\operatorname{Re}\left(\left(2z_2+A_3e^{z_1}\right)\frac{\partial \Phi}{\partial z_1}+\left(z_2^2-\frac{A_3^2}{4}e^{2z_1}\right)\frac{\partial \Phi}{\partial z_2}\right)=0,\\
&\operatorname{Re}\left(A_7\frac{\partial \Phi}{\partial z_1}+\left(-\frac{A_3A_7}{2}e^{z_1}\right)\frac{\partial \Phi}{\partial z_2}+\left(z_3+D_6\right)(p-i)\frac{\partial \Phi}{\partial z_3}\right.\\
&\qquad\left.+\left(2pz_4-D_6z_3-\frac{1}{2}z_3^2-D_6z_3+D_7\right)\frac{\partial \Phi}{\partial z_4}\right)=0.
\end{aligned}
\tag{4.2}
$$

We note that it is often convenient to make some elementary changes in the coefficients of the fields before writing the system of equations. For instance, in this case we can replace e^{z_1} by z_1^*, which allows us to work with completely polynomial components of the fields, and at that, in the first components of the fields an additional factor z_1^* appears, while other components remain unchanged. We specify that here and in what follows after each step of multi-level change of variables the sign «$*$» is omitted.

Under the passing to real coordinates, the first three simplest identities of system (4.2) allow us to conclude that the defining function of the hypersurface is independent of the variables x_2, x_3, x_4. Solving other equations by standard methods, after some simple final holomorphic transformations we obtain an equation for the hypersurface:

$$
y_4=A\ln y_1-\ln\left(y_2-y_3^2\right). \tag{4.3}
$$

Writing and solving systems similar to (4.2), we obtain all equations in Theorem 1.1. At the same time, there can arise Levi degenerate hypersurfaces, which we do not consider.

Now we are going to discuss briefly the issues on studying certain properties of holomorphically homogeneous hypersurfaces using equation (4.3) as an example. Here we employ the method of Moser normal forms [17].

Employing the expansion into the Taylor series, we represent the equation of a Levi nondegenerate real-analytic hypersurface $M\subset\mathbb{C}^4$ as

$$
y_4=H(z,\bar z)+\sum_{k,l\geqslant 2,m\geqslant 0}N_{klm}(z,\bar z)x_4^m, \tag{4.4}
$$

where $H(z,\bar z)$ is the Levi form of the hypersurface containing Hermitian terms, which are linear in the variables z and $\bar z$; here $N_{klm}(z,\bar z,x_4)$ are homogeneous polynomials of total powers k and l of the variables z and $\bar z$, respectively, $z=(z_1,z_2,z_3)$. The polynomials N_{22k}, N_{32k}, N_{33k} obey additional restrictions called tr-conditions, see [3], [4].

In many cases the study of lower terms in normal equation (4.4) allows one to justify or disprove conjectures on holomorphic equivalence of various hypersurfaces. For instance, it is known that a homogeneous real-analytic hypersurface in the space $\mathbb{C}^n$ is spherical if and only if the term $N_{220}(z,\bar z)$ in its normal Moser equation vanishes.

Let us demonstrate a calculation procedure for checking the sphericity on the example of equation (4.3).

We shift to the point $(i,i,0,0)$ and write an expansion for the right hand side of the equation

$$
y_4=A\ln(y_1+1)-\ln\left(y_2+1-y_3^2\right)
$$

into the Taylor series up to the fourth powers; according to the normalization procedure, the terms of zero and first power can be removed:

$$y_4 = -\frac{1}{2}Ay_1^2 + \frac{1}{2}y_2^2 + y_3^2 + \frac{1}{3}Ay_1^3 - \frac{1}{3}y_2^3 - y_2y_3^2 - \frac{1}{4}Ay_1^4 + \frac{1}{4}y_2^4 + y_2^2y_3^2 + \frac{1}{2}y_3^4 + \dots. \tag{4.5}$$

We pass to the complex coordinates and write the Levi form for this equation:

$$H(z,\bar{z}) = -\frac{1}{4}A|z_1|^2 + \frac{1}{4}|z_2|^2 + \frac{1}{2}|z_3|^2.$$

We see that as $A < 0$ this form is positive definite, that this, the hypersurface is strictly pseudoconvex, and as $A > 0$ we get a sign-indefinite non-degenerate form. As $A = 0$ we get a degenerate hypersurface.

Let consider the case $A < 0$. The change of variables

$$z_1 = \frac{2}{\sqrt{-A}}z_1^*, \quad z_2 = 2z_2^*, \quad z_3 = \sqrt{2}z_3^*$$

reduces the Levi form to the canonical form

$$|z_1|^2 + |z_2|^2 + |z_3|^2.$$

After passing to complex variables and change of variables, expansion (4.5) becomes (we group terms by the total powers of the polynomials involved in this expression):

$$y_4 = \sum_{k+l\geqslant 2} F_{kl}(z,\bar{z}) = (F_{20} + F_{11} + F_{02}) + (F_{30} + F_{21} + F_{12} + F_{03}) + \dots, \tag{4.6}$$

where k, l are the powers of the corresponding terms in the variables z and $\bar{z}$ respectively.

According to normalization procedure [17], by means of holomorphic change of variables, we can remove all terms of form F_{k0}, F_{k1} from equation (4.6) and also, by symmetry, all terms F_{0k}, F_{1k}. After the mentioned changes, equation (4.6) becomes:

$$y_4 = |z_1|^2 + |z_2|^2 + |z_3|^2 + H_{22} + H_{32} + H_{23} + \dots. \tag{4.7}$$

In order to transform the terms while passing from equation (4.6) to (4.7), we can use the generalization of the formulae given in works [22]. In particular,

$$H_{22} = F_{22} - \langle f_2, f_2 \rangle, \tag{4.8}$$

where F_{22} is the term in equation (4.6), f_2 is a vector function, the components of which are homogeneous polynomials of second order with respect to the variable z, and this function is calculated by the formula $F_{21} = \langle f_2, z \rangle$. Here $\langle f, g \rangle = f^T H \bar{g}$, where f and g are the vector functions and H is the matrix of the Hermitian Levi form.

For the considered equation we have:

$$F_{22} = -\frac{3}{2A}z_1^2\bar{z}_1^2 + \frac{3}{2}z_2^2\bar{z}_2^2 + \frac{1}{2}z_2^2\bar{z}_3^2 + 2z_2\bar{z}_2z_3\bar{z}_3 + \frac{1}{2}z_3^2\bar{z}_2^2 + \frac{3}{4}z_3^2\bar{z}_3^2,$$

$$f_2 = \begin{pmatrix} -\frac{i\sqrt{-A}}{A}z_1^2 \\ \frac{i}{2}(2z_2^2 + z_3^2) \\ iz_2z_3 \end{pmatrix},$$

$$\langle f_2, f_2 \rangle = -\frac{1}{A}z_1^2\bar{z}_1^2 + z_2^2\bar{z}_2^2 + \frac{1}{2}z_3^2\bar{z}_2^2 + \frac{1}{2}z_2^2\bar{z}_3^2 + \frac{1}{4}z_3^2\bar{z}_3^2 + z_2z_3\bar{z}_2\bar{z}_3.$$

By formula (4.8) we obtain:

$$H_{22} = -\frac{1}{2A}|z_1|^4 + \frac{1}{2}|z_2|^4 + \frac{1}{2}|z_3|^4 + |z_2|^2|z_3|^2. \tag{4.9}$$

The polynomial H_{22} belongs to a 36-dimensional space of polynomials $\mathcal{F}_{22}$, which is expanded into the direct sum of 27-dimensional space $\mathcal{N}_{22}$ and 9-dimensional space $\mathcal{R}_{22}$, the entries of which are divisible by the form $|z_1|^2+|z_2|^2+|z_3|^2$. At that, the projection of H_{22} into the space $\mathcal{N}_{22}$ is exactly the polynomial N_{220} in equation (4.4).

The mentioned expansion for polynomial (4.9) can be written as

$$\begin{aligned}
H_{22} =& N_{220} + R_{220}\\
=&\frac{1}{40A}(A-1)(3(|z_1|^4-4|z_1|^2|z_2|^2+|z_2|^4)\\
&+3(|z_1|^4-4|z_1|^2|z_3|^2+|z_3|^4)-(|z_2|^4-4|z_2|^2|z_3|^2+|z_3|^4))\\
&+\frac{1}{20A}\left(-(3A+7)|z_1|^2+(9A+1)|z_2|^2+(9A+1)|z_3|^2\right)(|z_1|^2+|z_2|^2+|z_3|^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
N_{220} =&\frac{1}{40A}(A-1)(3(|z_1|^4-4|z_1|^2|z_2|^2+|z_2|^4)\\
&+3(|z_1|^4-4|z_1|^2|z_3|^2+|z_3|^4)-(|z_2|^4-4|z_2|^2|z_3|^2+|z_3|^4)).
\end{aligned}$$

As $A<0$, the polynomial N_{220} is non-zero and therefore, the hypersurface described by equation (4.3) is locally holomorphically non-equivalent to a sphere.

Remark 4.1. *We note that as $A=1$, equation (4.3) can be rewritten as*

$$y_1 = y_3^2 e^{y_4} + y_2 e^{y_4}.$$

This equation describes an indefinite spherical tube, see formula (7) in the main theorem in [23].

All equations written in Theorem 1.1 can be studied in the same way. However, such study is too bulky and goes beyond this paper.

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BIBLIOGRAPHY

1. E. Cartan. *Sur la géométrie pseudoconforme des hypersurfaces de l'espace de deux variables complexes* // Ann. Math. Pura Appl. **11**, 17–90 (1933).
2. G. Fels, W. Kaup. *Classification of Levi degenerate homogeneous CR-manifolds in dimension 5* // Acta Math. **201**:1, 1–82 (2008).
3. A.V. Loboda. *Homogeneous real hypersurfaces in ^{3} with two-dimensional isotropy groups* // Trudy Matem. Inst. Steklova RAN. **235**, 114–142 (2001). [Proc. Steklov Inst. Math. **235**, 107–135 (2001).]
4. A.V. Loboda. *Homogeneous strictly pseudoconvex hypersurfaces in $\mathbb{C}^3$ with two-dimensional isotropy groups* // Matem. Sborn. **192**:12, 3–24 (2001). [Sb. Math. **192**:12, 1741–1761 (2001).]
5. B. Doubrov, A. Medvedev, D. The. *Homogeneous Levi non-degenerate hypersurfaces in $\mathbb{C}^3$* // Math. Zeit. **297**: 1-2, 669–709 (2021).
6. I. Kossovskiy, A. Loboda. *Classification of homogeneous strictly pseudoconvex hypersurfaces in $\mathbb{C}^3$* // Preprint: arXiv:1906.11345 (2019).
7. A.V. Loboda *Holomorphically homogeneous real hypersurfaces in $\mathbb{C}^3$* // Trudy MMO. **81**:2, 61–136 (2020). [Trans. Moscow Math. Soc. **81**:2, 169–228 (2020).]
8. B. Doubrov, J. Merker, D. The. *The classification of simply-transitive Levi non-degenerate hypersurfaces in $\mathbb{C}^3$* // Int. Math. Res. Notic. rnab147 (2021).

9. M.G. Eastwood, V.V. Ezhov. *Homogeneous hypersurfaces with isotropy in affine four-space* // Trudy Matem. Inst. Steklova RAN. **235**, 57–70 (2001). [Proc. Steklov Inst. Math. **235**, 49–63 (2001).]
10. M.G. Eastwood, V.V. Ezhov. *A classification of non-degenerate homogeneous equiaffine hypersurfaces in four complex dimensions* // Asian J. Math. **5**:4, 721–740 (2001).
11. F. Dillen, L. Vrancken. *3-dimensional affine hypersurfaces in* $\mathbb{R}^4$ *with parallel cubic form* // Nagoya Math. J. **124**, 41–53 (1991).
12. A.V. Loboda, R.S. Akopyan, V.V. Krutskikh. *On the orbits of nilpotent 7-dimensional lie algebras in 4-dimensional complex space* // J. Siber. Feder. Univ. Math. Phys. **13**:3, 360–372 (2020). (in Russian).
13. R.S. Akopyan, A.V. Atanov. *Non-degenerate orbits in* $\mathbb{C}^4$ *of decomposable* 7*-dimensional Lie algebras* // in "Modern methods in theory of boundary value problems", Proc. Int. Conf. "Voronezh Spring Mathematical School. Pontryagin Readins – XXXI", 30–32 (2020). (in Russian).
14. G.M. Mubarakzyanov. *On solvable Lie algebras* // Izv. VUZov. Matem. **1**, 114–123 (1963). (in Russian).
15. A.V. Atanov, I.G. Kossovskiy, A.V. Loboda. *On orbits of action of 5-dimensional non-solvable Lie algebras in three-dimensional complex space* // Dokl. Math. **100**:1, 377–379 (2019).
16. V.K. Beloshapka, I.G. Kossovskiy. *Homogeneous hypersurfaces in* $\mathbb{C}^3$*, associated with a model CR-cubic* // J. Geom. Anal. **20**:3, 538–564 (2010).
17. S.S. Chern, J.K. Moser. *Real hypersurfaces in complex manifolds* // Acta Math. **133**, 219–271 (1974).
18. A.V. Atanov, A.V. Loboda. *On the orbits of one non-solvable 5-dimensional Lie algebra* // Matem. Fiz. Kompyut. Model. **22**:2, 5–20 (2019). (in Russian).
19. A.V. Atanov, A.V. Loboda. *Decomposable five-dimensional Lie algebras in the problem of holomorphic homogeneity in* $\mathbb{C}^3$ // Itogi Nauki Tekh. Ser. Sovrem. Mat. Pril. Temat. Obz. **173**, 86–115 (2019).
20. L. Šnobl, P. Winternitz. *Classification and Identification of Lie Algebras.* AMS, Providence, R.I. (2014).
21. G.M. Mubarakzyanov. *Classification of real structures of Lie algebras of fifth order* // Izv. VUZov. Matem. 3(34), 99–106 (1963). (in Russian).
22. A.V. Loboda. *Affine-homogeneous real hypersurfaces in* 3*-dimensional complex space* // Vestnik Voronezh. Gosud. Univ. Ser. Fiz. Matem. **2**, 71–91 (2009). (in Russian).
23. A.V. Isaev, M.A. Mishchenko. *Classification of spherical tube hypersurfaces having one minus in the signature of the levi form* // Izv. AN SSSR. Ser. Matem. **52**:6, 1123–1153 (1988). [Math. USSR-Izv. **33**:3, 441–472 (1989).]

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