GRAPHS OF TOTALLY GEODESIC FOLIATIONS ON PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. We study totally geodesic foliations (M, F) of arbitrary codimension q on n-dimensional pseudo-Riemannian manifolds, for which the induced metrics on leaves is non-degenerate. We assume that the q-dimensional orthogonal distribution $\mathfrak{M}$ to (M, F) is an Ehresmann connection for this foliation. Since the usual graph $G(F)$ is not Hausdorff manifold in general, we study the graph $G_{\mathfrak{M}}(F)$ of the foliation with an Ehresmann connection $\mathfrak{M}$ introduced early by the author. This graph is always a Hausdorff manifold. We prove that on the graph $G_{\mathfrak{M}}(F)$, a pseudo-Riemannian metric is defined, with respect to which the induced foliation and the simple foliations formed by the leaves of the canonical projections are totally geodesic. We show that the leaves of the induced foliation on the graph are non-degenerately reducible pseudo-Riemannian manifolds and their structure is described. The application to parallel foliations on nondegenerately reducible pseudo-Riemannian manifolds is considered. We also show that each foliation defined by the suspension of a homomorphism of the fundamental group of a pseudo-Riemannian manifold belongs to the considered class of foliations.

Keywords: totally geodesic foliation, pseudo-Riemaniann manifold, graph of a foliation, Ehresmann connection for a foliation.

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1. Introduction and main results

Foliation golonomy gruppoid was introduced by Ch. Ehresmann and an equivalent construction was proposed by H.E. Winkelnkemper [1] and was called the graph of a foliation. The graph of a foliation contains all information about the foliation and on germ holonomy groups conventionally accepted in the foliation theory [2]. C^*-algebras of complex-valued functions for foliations introduced by Connes [3] are defined on the holonomy gruppoids of these foliations.

Let (M, F) be a smooth foliation of codimension q on an n-dimensional smooth manifold M. R. Blumenthal and J. Hebda defined an Ehresmann connection for (M, F) as a q-dimensional distribution $\mathfrak{M}$ transversal to the leaves whose integral curves admit a translation along each curve in a leaf of the foliation [4], [5]. For an arbitrary leaf L_α of the foliation (M, F), the notion of a $\mathfrak{M}$-holonomy group was introduced. We provide exact definitions in Section 2.

Let us recall the construction of the graph $G_{\mathfrak{M}}(F)$ of a foliation F of a codimension q with an Ehresmann connection $\mathfrak{M}$ on an n-dimensional manifold M introduced by us [6], see also [7], [8]. Let us consider two points x and y in the same leaf L_α in this foliation, We denote by $A(x, y)$ the set of all piecewise smooth paths in L_α connecting x and y. Paths h and f in $A(x, y)$ are called equivalent $h \sim f$ if the loop $h \cdot f^{-1}$ equalling to the product of the paths h and f^{-1} generates a trivial element of a $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L_\alpha, x)$ of a leaf L_α at a point x. The equivalence class containing a path h is denoted by $\{h\}$. The set $G_{\mathfrak{M}}(F)$ of triples of form $(x, \{h\}, y)$, where $x \in M$, $y \in L(x)$, $h \in A(x, y)$, is called the *graph of the foliation (M, F) with an Ehresmann connection $\mathfrak{M}$*, while the mappings $p_1 : G_{\mathfrak{M}}(F) \to M : (x, \{h\}, y) \mapsto x$, $p_2 : G_{\mathfrak{M}}(F) \to M : (x, \{h\}, y) \mapsto y$ are called

the *canonical projections*. We show that the graph $G_{\mathfrak{M}}(F)$ is naturally equipped with a structure of $(2n-q)$-dimensional smooth *Hausdorff* manifold [6], see also [7], [8].

Thus, the graph of a foliation with an Ehresmann connection $G_{\mathfrak{M}}(F)$ is defined similar to a classical graph of a foliation $G(F)$ [1] by replacing the germ holonomy group $\Gamma(L, x)$ of a leaf L, $x \in M$, by the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L, x)$. In distinction to $G_{\mathfrak{M}}(F)$, a topological space of the graph $G(F)$ is generally speaking not Hausdorff.

The mapping

$$\beta : G_{\mathfrak{M}}(F) \to G(F), \beta(x, \{h\}, y) = (x, <h>, y),$$

where $(x, <h>, y) \in G(F)$, is a local diffeomorphism. Both graphs $G(F)$ and $G_{\mathfrak{M}}(F)$ are equipped with the structure of gruppoids and β is a homomorphism of these gruppoids, that is, it maps the product of the elements in one gruppoid into that of corresponding elements in another grouppoid.

Let $p : M \to B$ be a submersion and $\mathfrak{M}$ be a distribution on B. We shall employ the notation $p^*\mathfrak{M} := \{\mathfrak{N}_x \,|\, x \in M\}$, where $\mathfrak{N}_x = \{Y \in T_x M \,|\, p_{*x} Y \in \mathfrak{M}_{p(x)}\}$

We proved the following properties of the two aforementioned graphs of an arbitrary foliation with an Ehresmann connection [6], [7].

Theorem 1. *Let (M, F) be a foliation of codimension q on a smooth n-dimensional manifold M admitting an Ehresmann connection $\mathfrak{M}$. Then*

1. A homomorphism $\beta : G_{\mathfrak{M}}(F) \to G(F)$ of gruppoids is an isomorphism if and only if the graph $G(F)$ is Hausdorff.

2. The canonical projections $p_i : G_{\mathfrak{M}}(F) \to M$, $i = 1, 2$, are locally trivial fibrations.

3. The distribution $\mathfrak{N} := p_1^*\mathfrak{M} \cap p_2^*\mathfrak{M}$ is an Ehresmann connection for the induced foliation

$$\mathbb{F} := \{p_1^{-1}(L) \,|\, L \in F\} = \{p_2^{-1}(L) \,|\, L \in F\},$$

on the graph $G_{\mathfrak{M}}(F)$, and the holonomy groups $H_{\mathfrak{N}}(\mathbb{L}, z)$ and $H_{\mathfrak{M}}(L, x)$, $x = p_1(z)$ of leaves $\mathbb{L}$ and $L = p_1(\mathbb{L})$ and their germ holonomy groups are canonically isomorphic.

Definition 1. *A pseudogroup $\mathcal{H}$ of locally holonomic diffeomorphisms of a manifold M is called quasianalytic if the fact that for some open connected subset V in M the identity $h|_V = id_V$ holds for some element $h \in \mathcal{H}$ implies that $h = id_{D(h)}$ on the entire connected domain $D(h)$ of the element h containing V.*

According [9, Prop. 2], the Winkelnkemper critetion on the Hausdorff property of the graph $G(F)$ can be reformulated as follows.

Proposition 1. *A topological space of the graph $G(F)$ of the foloation (M, F) is Hausdorff if and only the holonomy pseudogroup of this foliation is quasianalytic.*

According Theorem 1 and Proposition 1, for the foliations with an Ehresmann connection having a quasianalytic holonomy pseudogroup, we can identify the graphs $G_{\mathfrak{M}}(F)$ and $G(F)$ by the canonical isomorphism β and we denote this by $G_{\mathfrak{M}}(F) \cong G(F)$. Therefore, the graph $G_{\mathfrak{M}}(F)$ can be considered as a desingularization of a non-Hausdorff graph $G(F)$, where by the singularity we mean the absence of the Hausdorff property. Such essential difference between the properties of these graphs is explained by the fact that the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L_\alpha, x)$ is of a global nature, while germ holonomy group $\Gamma(L_\alpha, x)$ has a local-global nature; the global one is over the leaves and the local one is over the transversals.

The following three foliations are induced on the graph $G_{\mathfrak{M}}(F)$:

$$F^{(1)} = \{p_1^{-1}(x) \,|\, x \in M\}, \qquad F^{(2)} = \{p_2^{-1}(x) \,|\, x \in M\}, \qquad \mathbb{F} = \{p_1^{-1}(L) \,|\, L \in F\}.$$

We observe $\mathbb{F} = \{p_2^{-1}(L) \,|\, L \in F\}$.

We introduce the notation $\mathfrak{N} = p_1^*\mathfrak{M} \cap p_2^*\mathfrak{M}$ and $\mathfrak{M}^{(1)} = \mathfrak{N} \oplus TF^{(2)}$. We stress that each smooth vector field X on the graph $G_{\mathfrak{M}}(F)$ is uniquely represented as $X = X^{(1)} + X^{\mathfrak{N}} + X^{(2)}$, where $X^{(i)} \in \mathfrak{X}_{F^{(i)}}(G_{\mathfrak{M}}(F))$, $i = 1, 2$, $X^{\mathfrak{N}} \in \mathfrak{X}_{\mathfrak{N}}(G_{\mathfrak{M}}(F))$, as well as in the form:

$$X = X^{(1)} + X^{\mathfrak{M}^{(1)}}, \tag{1}$$

where $X^{\mathfrak{M}^{(1)}} \in \mathfrak{X}_{\mathfrak{M}^{(1)}}(G_{\mathfrak{M}}(F))$.

Definition 2. *Let (M, F) be a foliation of a codimension q on n-dimensional pseudo-Riemannian manifold (M, g), $0 < q < n$, and on the leaves pseudo-Riemannian metrics are induced. Then for all vector fields $X, Y \in \mathfrak{X}(G_\mathfrak{M}(F))$ given by (1) the identity*

$$d(X, Y) := (p_1^* g)(X^{\mathfrak{M}^{(1)}}, Y^{\mathfrak{M}^{(1)}}) + (p_2^* g)(X^{(1)}, Y^{(1)}) \tag{2}$$

defines a pseudo-Riemannian metrics d on the graph $G_\mathfrak{M}(F)$ called the induced metrics.

As (M, F) is a transversally analytic Riemannian foliation and $\mathfrak{M}$ is the q-dimensional orthogonal distribution on a complete Riemannian manifold and the graph $G(F)$ is identified with the graph $G_\mathfrak{M}(F)$, the induced metrics d on $G(F)$ was considered by R. Wolak in [10].

Definition 3. *A distribution $\mathfrak{M}$ on a pseudo-Riemannian manifold (M, g) is called geodesically invariant if each smooth geodesic curve of the Levi-Civita connection of the metrics g touching the distribution $\mathfrak{M}$ at a single point is an integral curve of this distribution.*
Foliations with a geodesically invariant tangential distribution are called the totally geodesic foliations.

A following theorem is one of the main results of the present work.

Theorem 2. *Let (M, F) be a totally geodesic foliation of an arbitrary codimension q on an n-dimensional pseudo-Riemannian manifold (M, g), and on the leaves, pseudo-Riemannian metrics are induced. Assume that a q-dimensional distribution $\mathfrak{M}$ orthogonal to the foliation (M, F) is an Ehresmann connection for (M, F). Then the above defined foliations $F^{(1)}$, $F^{(2)}$ and $\mathbb{F}$ on the graph $G_\mathfrak{M}(F)$ with the induced metrics d are totally geodesic and the q-dimensional distribution $\mathfrak{N}$ orthogonal to $\mathbb{F}$ is geodesically invariant.*

The first statement of Theorem 1, Proposition 1 and Theorem 2 imply the following statement.

Corollary 1. *Assume that a foliation (M, F) satisfies the assumptions of Theorem 2 and the holonomy pseudogroup of this foliation is quasi-analytic. Then the graph $G(F)$ is identified with the graph $G_\mathfrak{M}(F)$ equipped with the induced metrics, the induced foliations $(G(F), \mathbb{F})$ and a foliation formed by the leaves of the canonical projections $p_i : G(F) \to M$, $i = 1, 2$, are totally geodesic.*

Since a holonomy pseudogroup of a transversally analytic foliation (M, F) is quasianalytic, the graphs $G(F)$ and $G_\mathfrak{M}(F)$ are identified. Since the completeness of the Riemannian manifold implies the completeness of each totally geodesic foliation on this manifold, according Proposition 4, the foliation $\mathfrak{M}$ complementary to TF with respect to the orthogonality is an Ehresmann connection for (M, F). This is why Theorem 1 implies the following statement.

Corollary 2. *If (M, F) is a transversally analytic foliation on a complete Riemannian manifold (M, g) and $G(F)$ is the graph of this foliation equipped with the induced metrics, then the induced foliation $(G(F), \mathbb{F})$ and the foliations $(G(F), F^{(i)})$, $i = 1, 2$, formed by the fibres of canonical projections $p_i : G(F) \to M$ are totally geodesic foliations.*

Remark 1. *Under the assumptions of Corollary 2, R. Wolak proved a totally geodesic property for the foliation formed by the fibres of only one canonical projection $p_1 : G(F) \to M$ [10, Thm. 2].*

Remark 2. *The graphs of pseudo-Riemannian foliations on pseudo-Riemannian manifold with non-degenerate metrics on the leaves were studied by A.Yu. Dolgonosova and the author in [11].*

Applying Theorem 2, we prove the following properties for the graphs of the considered class of totally geodesic foliations.

Theorem 3. *Let (M, F) be a foliation satisfying conditions of Theorem 2, $F^{(1)}$, $F^{(2)}$, $\mathbb{F}$ be the aforementioned foliations on the graph $G_\mathfrak{M}(F)$ and $L_0 = p_1^{-1}(x)$, $x \in M$, be an arbitrary fibre of the canonical projection and the graph and the leaves of the corresponding foliations are considered with the induced metrics d. Then*
1. Each leaf $\mathbb{L} = p_1^{-1}(L)$ of the induced foliation $(G_\mathfrak{M}(F), \mathbb{F})$ is a non-degenerately reduced pseudo-Riemannian manifold with a pair of parallel foliations $F^{(1)}|_\mathbb{L}$ and $F^{(2)}|_\mathbb{L}$ mutually complementary with respect to the orthogonality.
2. For each leaf L in the foliation (M, F) there exists a regular pseudo-Riemannian covering $f_L : L_0 \to L$ with the group of covering transformations isomorphic to the $\mathfrak{M}$-holonomy group $H_\mathfrak{M}(L)$.

3. The group $H_{\mathfrak{M}}(L)$ acts freely diagonally and properly discontinuous on the pseudo-Riemmanian product $L_0 \times L_0$ by means of an isometry group Ψ and there exists an isometry

$$\eta : \mathbb{L} \to (L_0 \times L_0)/\Psi$$

of the leaf $\mathbb{L} = p_1^{-1}(L)$ onto the quotient-manifold $(L_0 \times L_0)/\Psi$ mapping the leaves of parallel foliations $F^{(1)}|_{\mathbb{L}}$ and $F^{(2)}|_{\mathbb{L}}$ onto the respected leaves of the foliations covered by the product $L_0 \times L_0$.

According the second statement in Theorem 3, each leaf (L_α, g) of the foliation (M, F) is locally isometric to (L_0, d) that yields the following corollary.

Corollary 3. *If there exists a leaf L of a totally geodesic foliation (M, F) having a constant curvature, then each leaf L_α of this foliation has the same constant curvature.*

Let (M, g) be a non-degenerately reducible pseudo-Riemmanian manifold considered with the Levi-Civita connection. This means that there exists a subspace $\mathfrak{M}_x$ in the tangential vector space $T_x M$ at some point $x \in M$, on which the restriction of the metrics g is non-degenerate and $\mathfrak{M}_x$ is invariant with respect to the parallel translations along piece-wise smooth loops at the point x. A parallel translation of the subspace $\mathfrak{M}_x$ into any other point in the manifold M defines a distribution $\mathfrak{M}$ on M called *parallel*. Since the parallel translation preserves the metric tensor, then the subspace $\mathfrak{M}_x^\perp$ complementary with respect to the orthogonality is invariant with respect to parallel translations along the loops at the point x and therefore, it also defines a parallel distribution $\mathfrak{M}^\perp$ on M. As it is known, each parallel distribution is integrable and is tangential to some foliation, which is called *parallel*.

Thus, on each non-degenerately reducible pseudo-Riemmanian manifold there exists a pair of parallel foliations $(F, F^\perp)$ mutually complementary with respect to the orthogonality.

Theorem 4. *Let (M, g) be a non-degenerately reducible pseudo-Riemannian manifold F and $F^\perp$ be the parallel foliations of an additional dimension and $\mathfrak{M} = TF^\perp$ is an Ehresmann connection for the foliation (M, F). Then in terms of the above notations we have:*

1. The graphs $G(F)$ and $G_{\mathfrak{M}}(F)$ are canonically isomorphic and are identified; $G(F)$ is equipped with the induced pseudo-Riemmanian metrics d.

2. For almost all points $z \in G(F)$ and $x = p_1(z) \in M$ the leaves $\mathbb{L} = \mathbb{L}(z)$ and $L = L(x)$ have trivial holonomy groups and are isometric to $L_0 \times L_0$ and L_0, respectively, where L_0 is an arbitrary fixed leaf of the canonical projection $p_1 : G(F) \to M$ with the induced metrics.

3. There defined the foliations $F^{\mathfrak{N}}$, $\mathcal{F}^{(i)}$, $i = 1, 2$, such that $TF^{\mathfrak{N}} = \mathfrak{N}$ and $T\mathcal{F}^{(i)} = \mathfrak{M}^{(i)}$ on the graph $G(F)$.

4. The graph $G(F)$ with the induced metrics d is a non-degenerately reducible pseudo-Riemmanian manifold with three pairs of parallel mutually complementary with respect to the orthogonality foliations $(F^{(1)}, \mathcal{F}^{(1)})$, $(F^{(2)}, \mathcal{F}^{(2)})$, and $(F^{\mathfrak{N}}, \mathbb{F})$.

5. Each of the above six foliations possesses an integrable Ehresmann connection and for its graph Theorems 2 and 3 hold as well as Statements 1–4 of the present theorem.

Theorem 4 and the following two statements show that the considered class of foliations is rather wide.

Proposition 2. *Let (M, F) be a totally geodesic foliation of a codimension q on n-dimensional pseudo-Riemmanian manifold (M, g), where $0 < q < n$. If complete pseudo-Riemmanian metrics are induced on its leaves, then a q-dimensional distribution $\mathfrak{M}$ orthogonal to the leaves is an Ehresmann connection for the foliation (M, F).*

Proposition 3. *Let (B, g^B) be an arbitrary pseudo-Riemmanian manifold. If (M, F) is a foliation obtained by the suspension of a homomorphism*

$$\rho : \pi_1(B, b) \to Diff(T),$$

then on M there exists a pseudo-Riemmanian metrics such that

1) (M, F) is a totally geodesic foliation with an induced pseudo-Riemmanian metrics on the leaves and the leaves of the foliation (M, F) are complete pseudo-Riemmanian manifolds if and only if (B, g^B) is complete;

2) the associated locally trivial fibration is formed by the fibres of the pseudo-Riemmanian submersion $p : M \to B$;

3) the distribution formed by the tangential spaces to the fibres of the submersion $p : M \to B$ *is an integrable Ehhresmann connection for* (M, F);

4) the graph $G(F)$ *is Hausdorff if and only if the group* $\Psi := \rho(\pi_1(B, b))$ *acts quasi-analytically on the manifold* T.

Notations. Following [15], we denote by $P(N, H)$ the principle H-bundle over a manifold N. By $\mathfrak{X}(M)$ we denote the module of smooth vector fields on a manifold M over the algebra $\mathfrak{F}(M)$ of smooth functions. A foliation F on a manifold M is denoted both by a single letter and by the pair (M, F). Let $\mathfrak{M}$ be a smooth distribution on the manifold M, then $\mathfrak{X}_{\mathfrak{M}}(M) := \{X \in \mathfrak{X}(M) \mid X_u \in \mathfrak{M}_u \ \forall u \in M\}$. If $\mathfrak{M}$ is integrable and $\mathfrak{M} = TF$, then $\mathfrak{X}_{\mathfrak{M}}(M)$ is also denoted by $\mathfrak{X}_F(M)$.

By $\mathfrak{Fol}$ we denote the category of foliations, in which the morphisms map the leaves of one foliation into those of another.

The restriction of a foliation (or metrics) on a submanifold is denoted by the symbol as the original foliation (or metrics).

By $\cong$ we denote the isomorphism in a corresponding category, while $\oplus$ stands for the direct sum of vector spaces and distributions.

2. FOLIATIONS WITH EHRESMANN CONNECTION

2.1. Ehresmann connectl for foliations. Assume we are given a foliation F of an arbitrary codimension $q \geqslant 1$ on a smooth n-dimensional manifold M.

We denote by $\mathfrak{M}$ a q-dimensional transversal to F distribution, then the tangential space $T_x M$ to the manifold M at each point $x \in M$ can be represented as $T_x M = T_x F \oplus \mathfrak{M}_x$. The distribution $\mathfrak{M}$ and piece-wise smooth integral curves of this distribution are called $\mathfrak{M}$-horizontal or simply horizontal. A tangent distribution TF to the leaves of the foliation F and each vector X in $T_x F$, $x \in M$, are called vertical. A curve in the manifold M belonging to a single leaf in the foliation F is called vertical.

A piece-wise smooth mapping $H : I \times I \to M$, where $I = [0, 1]$, is called the *vertical-horizontal homotopy* (VHH) if for each $(s, t) \in I \times I$, the curve $H|_{I \times \{t\}}$ is horizontal and $H|_{\{s\} \times I}$ is a vertical curve. The pair $(H|_{I \times \{0\}}, H|_{\{0\} \times I})$ is called the *base of VHH H*. Two curves (δ, τ) on M are called the *admissible pair of paths* if $\delta(0) = \tau(0)$ and the path δ is horizontal and τ is vertical.

If for each admissible pair of paths (δ, τ) there exists VHH with the base (δ, τ), then the distribution $\mathfrak{M}$ is called the *Ehresmann connection for F*. If at that the distribution $\mathfrak{M}$ is integrable, then the Ehresmann connection $\mathfrak{M}$ is called *integrable*.

We shall say that a curve $\widetilde{\delta}$ is obtained by the *translation of the curve δ along τ with respect to an Ehresmann connection* $\mathfrak{M}$ if $\widetilde{\delta} := H|_{I \times \{1\}}$. We denote this translation by $\delta \xrightarrow{\tau} > \widetilde{\delta}$.

2.2. $\mathfrak{M}$-holonomy groups. Let (M, F) be a foliation with an Ehresmann connection $\mathfrak{M}$. We denote by Ω_x, $x \in M$, the set of horizontal curves originating at the point x. The action of the fundamental group $\pi_1(L, x)$ of the leaf $L = L(x)$ on the set Ω_x is defined as follows: $\Phi_x : \pi_1(L, x) \times \Omega_x \to \Omega_x : ([h], \sigma) \mapsto \tilde{\sigma}$, where $[h] \in \pi_1(L, x)$ and $\tilde{\sigma}$ is the result of the translation of the curve $\sigma \in \Omega_x$ along h with respect to $\mathfrak{M}$. Let $K_{\mathfrak{M}}(L, x)$ be the kernel of the action Φ_x, that is, $K_{\mathfrak{M}}(L, x) = \{\alpha \in \pi_1(L, x) \mid \alpha(\sigma) = \sigma \ \forall \sigma \in \Omega_x\}$. The quotient group $H_{\mathfrak{M}}(L, x) = \pi_1(L, x)/K_{\mathfrak{M}}(L, x)$ is called the $\mathfrak{M}$-*holonomy group* of the leaf L [4]. Thanks to the connectivity of the leaves, the $\mathfrak{M}$-holonomy groups at different points of the same leaf are isomorphic. Let $\Gamma(L, x)$ be the germ holonomy group of a leaf L. There is an epimorphism of the groups $\chi : H_{\mathfrak{M}}(L, x) \to \Gamma(L, x)$ satisfying the identity

$$\chi \circ \mu = \nu, \tag{3}$$

where $\mu : \pi_1(L, x) \to H_{\mathfrak{M}}(L, x)$ is the quotient mapping and $\nu([h]) := < h >$ is the germ of a local holonomic diffeomorphism of a transversal q-dimensional disk along a loop h at the point x.

2.3. Local horizontal holonomic diffeomorphisms. We consider an arbitrary smooth foliation (M, F) of codimension q on a n-dimensional manifold M. Let $\mathfrak{M}$ be a smooth q-dimensional distribution on M transversal to this foliation, that is, $T_x M = T_x F \oplus \mathfrak{M}_x$ for all $x \in M$. In what follows we shall consider vertical-horizonatl homotopies with respect to the distributions TM and $\mathfrak{M}$. At each point $x \in M$ there exists a neighbourhood V_x such that for each admissible pair of paths (σ, h) in

V_x with a common origin at x, there exists a *VHH* in V_x with the base (σ, h). Let $\sigma : [0, 1] \to M$ be arbitrary smooth integral curve of the distribution $\mathfrak{M}$ with the origin $x_0 := \sigma(0)$ and the end $x_1 := \sigma(1)$. It is easy to make sure that there exist contractible neighbourhoods U_0 of the point x_0 in the leaf $L_0 \ni x_0$ and U_1 of the point x_1 in the leaf $L_1 \ni x_1$ such that for each point $x \in U_0$ and each piece-wise smooth path $h_x : [0, 1] \to U_0$ connecting $h(0) = x_0$ with $h(1) = x$, there exists *VHH* H_x with the base (σ, h_x). At that, $\mathfrak{M}$-horizontal curve $\sigma_x(s) := H_x(s, 1)$, $s \in [0, 1]$, is smooth and since U_0 is contractible, this curve is independent of the choice of the path h_x connecting x_0 with x in U_0. In what follows σ_x is called the *translation of the path* σ *to the point* $x \in U_0$. At that, the diffeomorphism is well-defined:

$$\Phi_\sigma : U_0 \to U_1 : x \mapsto \sigma_x(1), \qquad x \in U_0,$$

which is called the *locally horizontal holonomic diffeomorphism* along $\mathfrak{M}$-curve σ [5].

The definition of the Lie derivative $L_X g$ of 2-form g along a vector field X implies the following statement.

Lemma 1. *Assume that (M, F) is a foliation of codimension q on n-dimensional pseudo-Riemmanian manifold (M, g) and the induced metrics does not degenerate on its leaves, and that $\mathfrak{M}$ is the q-dimensional distribution orthogonal to TF. Let $\sigma : [0, 1] \to M$ be an arbitrary smooth $\mathfrak{M}$-horizontal curve in M, $\Phi_\sigma : U_0 \to U_1$ be a local holonomic diffeomorphism along σ and $W := \{\sigma_x(s) \,|\, x \in U_0, s \in (0, 1)\}$, where σ_x is the translation of σ to the point $x \in U_0$. Then the following two conditions are equivalent:*

1. *The diffeomorphism Φ_σ is an isometry of (U_0, g) and (U_1, g);*
2. *For a vector field $X \in \mathfrak{X}_\mathfrak{M}(W)$ such that $X|_{\sigma_x(s)} = \dot{\sigma}_x(s)$, where $x \in U_0$, $s \in [0, 1]$, the identity $(L_X g)(Y, Z) = 0$ holds for all $Y, Z \in \mathfrak{X}_F(W)$.*

3. PSEUDO-RIEMMANIAN SUBMERSIONS

The study of pseudo-Riemmanian submersions was initiated by B. O'Neill [12] and A. Gray [13]. A smooth surjective submersion $p : M \to B$ between two pseudo-Riemannian manifolds (M, g) and (B, g^B) is called the *pseudo-Riemmanian submersion* if the metrics induced on each fibre of the submersion $p^{-1}(b)$, where $b \in B$ is non-degenerate and p preserves the scalar product of the vectors orthogonal to the fibres of the submersion.

Numerous works by various authors are devoted to studying pseudo-Riemmanian submersions. Pseudo-Riemannian submersions with totally geodesic fibres are of a special interest, for some classes classification theorems were proved, see [14] and the references therein.

The following properties of pseudo-Riemannian submersions with totally geodesic fibres are essentially employed in the present work.

Proposition 4. *If (M, g) and (B, g^B) are pseudo-Riemannian manifolds of dimensions n and q, respectively, where $0 < q < n$, and $p : M \to B$ is a pseudo-Riemannian submersion with fibres being totally geodesic manifolds in (M, g), then*

(i) the projection $\sigma = p \circ \gamma$ of each geodesic curves γ in (M, g) is geodesic in (B, g^B);

(ii) the pre-image $p^{-1}(L)$ of each totally geodesic submanifold L from the base (B, g^B) is a totally geodesic submanifold in (M, g); it is not connected once the lfibres of the submersion p are not.

Proof. Let $p : M \to B$ be a pseudo-Riemannian submersion with totally geodesic fibres.

(i). The geodesic property of a curve in a pseudo-Riemannian manifold (M, g) is local. This is why it is sufficient to show that for each geodesic curve γ in an arbitrary coordinate neighbourhood U adapted to (M, F), its projection $p \circ \gamma$ is a geodesic curve in the neighbourhood $V := p(U)$ of the manifold (B, g^B). We observe that $p|_U : U \to V$ is a pseudo-Riemannian submersion on a contractible manifold with contractible fibres. This is why without loss of generality, in the present proof we let $M = U$, $B = V$, $p : M \to B$ be a pseudo-Riemannina submersion with totally geodesic fibres, and $F = \{p^{-1}(b) \,|\, b \in B\}$.

Assume that pseudo-Riemannian metrics g and g^B have the same signatures (k, s) and (k_1, s_1), respectively, where $k + s = n$, $k_1 + s_1 = q$. Let $H_1 = O(k_1, s_1)$, $H = O(k, s)$. By $\mathcal{L}(M, H) = M \times H$ and $P_1(B, H_1) = B \times H_1$ we denote the principle bundles of pseudo-orthogonal frames on M and B defined by the metrics g and g^B, respectively; they are trivial principal bundles with projections

$\pi : \mathcal{L} \to M$ and $f_1 : P_1 \to B$. Let $\mathfrak{M}$ be a q-dimensional distribution orthogonal to the fibres of the submersion p. We denote by $P = M \times H_1$ the bundle of $\mathfrak{M}$-transversal frames being the pre-image of the bundle $P_1(B, H_1)$ under the submersion p, that is, $P = \{(y, v) \in M \times P_1 \,|\, p(y) = f_1(v)\}$. At that, the projections $f : P \to M$, $f(y, v) := y$, $h : P \to P_1$, $h(y, v) = v$ for all $(y, v) \in P$, are well-defined and they satisfy the identity $f_1 \circ h = p \circ f$. The fibres of the submersion $h : P \to P_1$ form the foliation (P, F_P).

Since $M = U$ is a coordinate neighbourhood adapted to the foliation (M, F), at each point $y \in M$, a coordinate frame $(\frac{\partial}{\partial x^a}|_y, \frac{\partial}{\partial x^\alpha}|_y)$, $a = 1, \ldots, n - q$, $\alpha = n - q + 1, \ldots, n$, is well-defined and $\{\frac{\partial}{\partial x^a}|_y\}$ is a basis of $T_y F$ in the tangent space to the leaf of the foliation (M, F) at the point y. Each point $(y, v) \in P$ is a transversal frame, that is, a basis $\{Z_\alpha|_y\}$ in the vector space $\mathfrak{M}_y$ at the point $y \in M$ defined by the identity $p_{*y}(Z_\alpha|_y) = X_\alpha|_x$, where $\{X_\alpha|_x\} = v$ is the basis in the tangential vector space $T_x B$ at the point $x = f_1(v) = p(y)$. Therefore, the mapping $J : P \to \mathcal{L}$, $J(y, v) = \{\frac{\partial}{\partial x^a}|_y, Z_\alpha|_y\}$ is well-defined and this is an embedding of the manifold P into $\mathcal{L}$ satisfying the identity $\pi \circ J = f$.

Let E_{n-q} be the unit $(n - q)$-dimensional matrix. By $j : H_1 \to H : A \mapsto \left(\begin{smallmatrix} E_{n-q} & 0 \\ 0 & A \end{smallmatrix}\right)$ we denote an embedding of the group H_1 into the group H. At that, the pair (J, j) defines a reduction $\mathcal{R}$ of H-foliation $\mathcal{L}$ to a closed subgroup $j(H_1)$ [15, Ch. 1, Sect. 5]. Since the mapping $J : P \to \mathcal{R} = J(P)$ is an isomorphism of principle foliated spaces can be identified with P with $\mathcal{R}$ by means with J. At that, we have a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{L} \supset \mathcal{R} \cong P & \xrightarrow{\ h\ } & P_1 \\
{\scriptstyle \pi_{\mathcal{R}} = f} \downarrow & & \downarrow {\scriptstyle f_1} \\
M & \xrightarrow{\ p\ } & B,
\end{array}
\tag{4}
$$

where $\pi_{\mathcal{R}} := \pi|_{\mathcal{R}}$. Hence, on $\mathcal{R}$, the foliation $\mathcal{F}$ is well-defined as the image of the foliation (P, F_P) under the mentioned identification and the restriction of the mapping $\pi_{\mathcal{R}}$ on each leaf $(\mathcal{R}, \mathcal{F})$ is a diffeomorphism on a corresponding leaf (M, F).

The Levi-Civita connection ∇ of the pseudo-Riemannian manifold (M, g) defines a H-connection Q on $\mathcal{L}$. Let $Q^{(1)}$ be a H_1-connection on P_1 defined by the Levi-Civita connection ∇^B of the pseudo-Riemannian manifold (B, g^B). We denote by ω the $\mathfrak{gl}(n, \mathbb{R})$-valued 1-form of the connection Q, while θ stands for the canonical 1-form of this connection on $\mathcal{L}$ with values in $\mathbb{R}^n$. We recall that $B_\xi \in \mathfrak{X}(\mathcal{L})$ is called the *standard horizontal vector field* if $\omega(B_\xi) = 0$, that is, $B_\xi \in \mathfrak{X}_Q(\mathcal{L})$, and $\theta(B_\xi) = \xi = const \in \mathbb{R}^n$. As it is known [15, Ch. Ⅲ, Prop. 6.3], a curve γ is geodesic in (M, ∇) if and only if γ is a projection of an integral curve of some standard horizontal vector field.

Since the lift in $\mathcal{L}$ of each geodesic curve in (M, ∇) is an integral curve of the distribution Q, the complete geodesicity of (M, F) yields the inclusion $T\mathcal{F} \subset Q|_{\mathcal{R}}$. Therefore, $Q|_{\mathcal{R}} = T\mathcal{F} \oplus \mathfrak{N}$, where $\mathfrak{N} = \pi^*\mathfrak{M} \cap Q|_{\mathcal{R}}$.

Since $p : M \to B$ is a pseudo-Riemannian submersion, according [11, Thm. 1], the distribution $\mathfrak{M}$ is geodesically invariant and this is why the Q-lift $\widetilde{\gamma}$ to a point $u \in \pi^{-1}(\gamma_0) \cap \mathcal{R}$ of each $\mathfrak{M}$-horizontal geodesic curve $\gamma : [0, 1] \to M$ is an integral curve of the distribution $\mathfrak{N}$. Moreover, for each vector $Y \in \mathfrak{N}_u$, $u \in \mathcal{R}$ such that $\theta(Y) = \xi \in \{0_{n-q}\} \times \mathbb{R}^q$, where 0_{n-q} is the zero in $\mathbb{R}^{n-q}$, there exists a unique $\mathfrak{M}$-horizontal geodesic curve γ on M, whose Q-lift at the point u is an integral curve of a standard horizontal vector field B_ξ such that $B_\xi|_u = Y$.

By properties of a pseudo-Riemannian submersion, each $\mathfrak{M}$-horizontal geodesic curve is projected to a geodesic curve of the base (B, ∇^B) by means of $p : M \to B$ [12]. Hence, in view of the identity $f_1 \circ h = p \circ \pi_{\mathcal{R}}$, we obtain $h_{*u}(\mathfrak{N}_u) = h_{*u}(Q_u) = Q^{(1)}_{h(u)}$ for each point $u \in \mathcal{R}$.

Let γ be an arbitrary geodesic curve in (M, g) passing through a point $x = \gamma(0)$ along a vector $X = \dot{\gamma}(0) \in T_x M$. We take a point $u_0 \in \mathcal{R}$ such that $\pi(u_0) = x$. We treat the frame u_0 as a mapping $u_0 : \mathbb{R}^n \to T_x M$, which maps a vector in $\mathbb{R}^n$ with coordinates $\lambda_1, \ldots, \lambda_n$ in the standard basis in $\mathbb{R}^n$ into the vector in $T_x M$ with the same coordinates in the base u_0. Assume that $\eta := u_0^{-1}(X) \in \mathbb{R}^n$. Let $pr : \mathbb{R}^n \cong \mathbb{R}^{n-q} \times \mathbb{R}^q \to \mathbb{R}^q$ be the canonical projection onto the factor and $\xi := pr(\eta) \in \mathbb{R}^q$. Since γ is a geodesic curve, there exists an integral curve $\widehat{\gamma}$ of the standard vector field B_η on $\mathcal{R}$ with the origin at the point $\widehat{\gamma}(0) = u_0$. At that $\gamma = \pi \circ \widehat{\gamma}$ and $\widehat{\sigma} := h \circ \widehat{\gamma}$ is an integral curve of the standard vector field B_ξ on P_1 passing through the point $v_0 = h(u_0) = \widehat{\sigma}(0)$. Since $p \circ \pi_{\mathcal{R}} = f_1 \circ h$, we have the chain of the identities $\sigma = p \circ \gamma = p \circ (\pi \circ \widehat{\gamma}) = (p \circ \pi) \circ \widehat{\gamma} = (f_1 \circ h) \circ \widehat{\gamma} = f_1 \circ (h \circ \widehat{\gamma}) = f_1 \circ \widehat{\sigma}$.

Thus, the curve $\sigma := f_1 \circ \hat{\sigma}$ is geodesic on (B, g^B) and it is a projection of the geodesic curve γ of the manifold (M, g), that is, $p \circ \gamma = \sigma$, and this proves Statement (i).

(*ii*). Let L be a totally geodesic submanifold in (B, g^B) and $N := p^{-1}(L)$ be a smooth embedded submanifold in M; it is not connected once the fibres of the submersion $p : M \to B$ are not. We take an arbitrary point $y \in N$ and a vector $Y \in T_y N$. Let $\gamma = \gamma_Y(s)$, $s \in [0, 1]$, be a geodesic curve in (M, g) passing through the point $y = \gamma_Y(0)$ along the vector Y, that is, $Y = \dot{\gamma}_Y(0)$. According the proven statement (*i*), the curve $\sigma := p \circ \gamma$ is geodesic (B, g^B) and it passes through the point $b = p(y) = \sigma(0)$ along the vector $X = p_{*y} Y \in T_b L$. Since L is a totally geodesic submanifold, then $\sigma(s) \in L$ for all $s \in [0, 1]$, and hence, $\gamma(s) \in N$ for all $s \in [0, 1]$. This means that the submanifold N in (M, g) is totally geodesic. The proof is complete.   □

4.   Proof of Theorem 2

4.1.   Criterion of a total geodesicity of foliation.

Definition 4. *A vector field X on a manifold M is called foliated with respect to a foliation (M, F) if $[X, Y] \in \mathfrak{X}_F(M)$ for each $Y \in \mathfrak{X}_F(M)$.*

Proposition 5. *Let (M, F) be a foliation on a pseudo-Riemannian manifold (M, g) with the Levi-Civita connection ∇ and on the leaves of the foliation, pseudo-Riemannian metrics are induced, and let $\mathfrak{M}$ be the distribution complementary to TF with respect to the orthogonality. Then the following statements are equivalent:*

(1) the foliation (M, F) is totally geodesic;

(2) $L_X g(Y, Z) = 0$ for arbitrary vector fields $X \in \mathfrak{X}_{\mathfrak{M}}(M)$ and $Y, Z \in \mathfrak{X}_F(M)$;

(3) $L_X g(Y, Z) = 0$ for each foliated vector field $X \in \mathfrak{X}_{\mathfrak{M}}(M)$ and for each vector fields $Y, Z \in \mathfrak{X}_F(M)$;

(4) for each $\mathfrak{M}$-horizontal curve $\sigma : [0, 1] \to M$, the local horizontal holonomic diffeomorphism $\Phi_\sigma : U_0 \to U_1$ is an isometry between (U_0, g) and (U_1, g).

Proof. We denote by $\alpha_{\mathfrak{M}}(Y, Z)$ the orthogonal projection of $\nabla_Y Z$ into $\mathfrak{X}_{\mathfrak{M}}(M)$ with respect to the decomposition $TM = TF \oplus \mathfrak{M}$. The total geodesicity of the foliation (M, F) is equivalent to $\alpha_{\mathfrak{M}}(Y, Z) = 0$ for all $Y, Z \in \mathfrak{X}_F(M)$.

The equivalence of (1) and (2) was proved in [16, Prop. 2.7] as follows. By employing the property of the Levi-Civita connection ∇ of the pseudo-Riemannian manifold (M, g), the identity was obtained:

$$L_X g(Y, Z) = g(X, \alpha_{\mathfrak{M}}(Y, Z)) \qquad \text{for all} \quad X \in \mathfrak{X}_{\mathfrak{M}}(M), \quad Y, Z \in \mathfrak{X}_F(M). \tag{5}$$

And since the induced metrics on the leaves is non-degenerate, this implies the equivalence

$$L_X g(Y, Z) = 0 \qquad \text{for all} \quad X \in \mathfrak{X}_{\mathfrak{M}}(M) \quad \Leftrightarrow \quad \alpha_{\mathfrak{M}}(Y, Z) = 0,$$

that is, $(1) \Leftrightarrow (2)$.

The implication $(2) \Rightarrow (3)$ is obvious. Assume that (3) holds. We observe that an arbitrary vector field $X \in \mathfrak{X}_{\mathfrak{M}}(M)$ is a linear combination of foliated vector fields in $\mathfrak{X}_{\mathfrak{M}}(M)$, that is, $X = \beta^k X_k$, where $X_k \in \mathfrak{X}_{\mathfrak{M}}(M)$ are foliated vector fields, $\beta^k \in \mathfrak{F}(M)$. Applying (5), thanks to the bilinearity of g, we obtain

$$L_X g(Y, Z) = g(X, \alpha_{\mathfrak{M}}(Y, Z)) = g(\beta^k X_k, \alpha_{\mathfrak{M}}(Y, Z)) = \beta^k g(X_k, \alpha_{\mathfrak{M}}(Y, Z)) = \beta^k L_{X_k} g(Y, Z).$$

According the assumption, $L_{X_k} g(Y, Z) = 0$ and this is why $L_X g(Y, Z) = 0$. Therefore, $(3) \Rightarrow (2)$ and $(2) \Leftrightarrow (3)$.

If σ is a piece-wise smooth $\mathfrak{M}$-horizontal curve, then Φ_σ is a composition of local horizontal holonomic diffeomorphisms corresponding to smooth pieces of the curve σ. This is why, without loss of generality, in (4) we can assume that $\sigma : [0, 1] \to M$ is a smooth $\mathfrak{M}$-horizontal curve.

Assume that (3) is satisfied. Let $\sigma : [0, 1] \to M$ be a smooth $\mathfrak{M}$-horizontal curve, $\Phi_\sigma : U_0 \to U_1$ be a local horizontal holonomic diffeomorphism along σ and X be a vector field induced by Φ_σ in the way described in Lemma 1. Since X is a foliated vector field orthogonal to the foliation (M, F), it follows from (3) that $L_X g(Y, Z) = 0$ for all $Y, Z \in \mathfrak{X}_F(M)$. Therefore, by Lemma 1, Φ_σ is an isometry between (U_0, g) and (U_1, g). Thus, $(3) \Rightarrow (4)$.

Now it is sufficient to show that $(4) \Rightarrow (3)$. Let $X \in \mathfrak{X}_{\mathfrak{M}}(M)$ be an arbitrary foliated vector field and $\sigma : [0, 1] \to M$ be its arbitrary integral curve. Since σ is a $\mathfrak{M}$-horizontal curve, then a local horizontal

holonomic diffeomorphism $\Phi_\sigma : U_0 \to U_1$ along σ is well-defined. Since X is a foliated vector field, the translation σ_x of the curve σ at the point $x \in U_0$ is also an integral curve of the field X. According Lemma 1, this implies that $L_X g(Y, Z) = 0$ for all $Y, Z \in \mathfrak{X}_F(M)$. Thus, $(4) \Rightarrow (3)$. $\qquad\square$

Remark 3. *Proposition 5 has been proved without assuming the existence of the Ehresmann connection for the foliation (M, F).*

4.2. Total geodesicity of foliation $F^{(1)}$. Let us prove that $F^{(1)}$ is a totally geodesic foliation on the graph $(G_\mathfrak{M}(F), d)$.

Let $\gamma : [0, 1] \to G_\mathfrak{M}(F)$ be a smooth $\mathfrak{N}$-curve in $G_\mathfrak{M}(F)$ with the origin at the point $\gamma(0) = z_0 = (x_0, \{h\}, y_0) \in G_\mathfrak{M}(F)$. Then $\sigma := p_1 \circ \gamma$ is a $\mathfrak{M}$-curve in M with the origin at the point $x_0 = \sigma(0) = p_1(z_0)$. Since (σ, h) is an admissible pair, there exists a *VHH H* with a base (σ, h). Let $\sigma \xrightarrow{h} \tilde{\sigma}$, at that $\tilde{\sigma} = p_2 \circ \gamma$. Without loss of generality we assume that V_0 is a neighbourhood of the point z_0 in the fibre $p_1^{-1}(x_0)$ such that the neighbourhood $U_0 = p_2(V_0)$ belonging to the leaf $L = L(x_0)$ is properly covered by the mapping $p_2|_{L^{(1)}} : L^{(1)} \to L$, where $L^{(1)} = L^{(1)}(z_0) = p_1^{-1}(x_0)$. If $\Phi_\gamma : V_0 \to V_1 : z \mapsto \Phi_\gamma(z)$ is a horizontal holonomic diffeomorphism along γ in $G_\mathfrak{M}(F)$, where V_1 is a neighbourhood of the point $\gamma(1)$ in the leaf $p_1^{-1}(\sigma(1))$, then for each point $z \in V_0$ and the $\mathfrak{N}$-lift γ_z of the curve σ at the point z, by the definition we have $\Phi_\gamma(z) = \gamma_z(1)$. At that, $z = (x_0, \{h_y\}, y)$, where $y = p_2(z)$, $h_y = h \cdot t_y$, t_y is a path in U_0 connecting y_0 with y in U_0. We observe that $\sigma_y = p_2 \circ \gamma_z$ is $\mathfrak{M}$-curve in M obtained by translating σ along the path h_y and $\Phi_{\tilde{\sigma}} := p_2 \circ \Phi_\gamma : U_0 = p_2(V_0) \to U_1 = p_2(V_1)$ is a local $\mathfrak{M}$-horizontal holonomic diffeomorphism along $\tilde{\sigma}$ satisfying a commutative diagram:

$$
\begin{array}{ccc}
V_0 & \xrightarrow{\Phi_\gamma} & V_1 \\
{\scriptstyle p_2|_{V_0}}\downarrow & & \downarrow{\scriptstyle p_2|_{V_1}} \\
U_0 & \xrightarrow{\Phi_{\tilde{\sigma}}} & U_1.
\end{array}
\tag{6}
$$

According Proposition 5, the total geodesicity of the foliation F implies that $\Phi_{\tilde{\sigma}} : U_0 \to U_1$ is an isometry between the pseudo-Riemannian manifolds (U_0, g) and (U_1, g). Hence, taking into consideration that $p_1|_{V_0} : V_0 \to U_0$, $p_2|_{V_1} : V_1 \to U_1$ are isometries, by the commutativity of diagram (6) we obtain that $\Phi_\gamma : V_0 \to V_1$ is an isometry between (V_1, d) and (V_2, d). Hence, similar to Proposition 5, we obtain

$$
(L_X d)(Y, Z) = 0 \qquad \text{for all} \quad X \in \mathfrak{X}_\mathfrak{N}(G_\mathfrak{M}(F)), \quad Y, Z \in \mathfrak{X}_{F^{(1)}}(G_\mathfrak{M}(F)).
\tag{7}
$$

We are going to show that $(L_X d)(Y, Z) = 0$ for all $Y, Z \in \mathfrak{X}_{F^{(1)}}(G_\mathfrak{M}(F))$ and for each $X \in \mathfrak{X}_{F^{(2)}}(G_\mathfrak{M}(F))$. We denote by $L^{(i)} = L^{(i)}(z_0)$, $i = 1, 2$, and $\mathbb{L} = \mathbb{L}(z_0)$ the leaves in the foliations $F^{(i)}$ and $\mathbb{F}$ respectively passing through z_0. Let γ be an arbitrary $TF^{(2)}$-curve with the origin at $z_0 = \gamma(0)$, that is, $\gamma(s) \in L^{(2)}$, $s \in [0, 1]$. A local horizontal diffeomorphism $\Phi_\gamma : W_0 \to W_1$ along γ is well-defined, where W_0 is a neighbourhood of the point z_0 in the leaf $L^{(1)}$, and W_1 is a neighbourhood of the point $z_1 = \gamma(1)$ in the leaf $L^{(1)}(z_1)$ in the foliation $F^{(1)}$ passing through z_1.

We recall that a subset of a foliated manifold (N, F_N) is F_N-saturated if it can be represented as the union of some leaves in the foliation (N, F_N).

Each leaf $\mathbb{L}$ of the induced foliation $(G_\mathfrak{M}(F), \mathbb{F})$ is both $F^{(1)}$-saturated and $F^{(2)}$-saturated submanifold of the graph $G_\mathfrak{M}(F)$ and according the definition of the metrics d, the leaf $\mathbb{L}$ with the induced pseudo-Riemannian metrics $(\mathbb{L}, d)$ is locally a pseudo-Riemannian product of pseudo-Riemanninan manifolds $(L^{(1)}, d)$ and $(L^{(2)}, d)$. This implies that $F^{(1)}|_\mathbb{L}$ and $F^{(2)}|_\mathbb{L}$ are parallel foliations on an non-degenerate reducible pseudo-Riemannina manifold $(\mathbb{L}, d)$, see, for instance, [19]. Therefore, $F^{(1)}|_\mathbb{L}$ and $F^{(2)}|_\mathbb{L}$ are totally geodesic foliations on $(\mathbb{L}, d)$ and according Proposition 5, $\Phi_\gamma : W_0 \to W_1$ is an isometry and the identity holds

$$
(L_X d)(Y, Z) = 0 \qquad \text{for all} \quad X \in \mathfrak{X}_{F^{(2)}}(G_\mathfrak{M}(F)), \quad Y, Z \in \mathfrak{X}_{F^{(1)}}(G_\mathfrak{M}(F)).
\tag{8}
$$

We observe that at each point of the graph $G_\mathfrak{M}(F)$ there exists a neighbourhood $\mathcal{W}$ adapted to the foliations $\mathbb{F}$, $F^{(1)}$ and $F^{(2)}$ simultaneously, at which both identities (7) and (8) hold. Since $\mathfrak{M}^{(1)} = \mathfrak{N} \oplus TF^{(2)}$, each vector field $X \in \mathfrak{X}_{\mathfrak{M}^{(1)}}(W)$ in the neighbourhood $\mathcal{W}$ of an arbitrary point

$z \in G_{\mathfrak{M}}(F)$ can be represented as $X = \alpha X^{\mathfrak{N}} + \beta X^{(2)}$, where $X^{\mathfrak{N}} \in \mathfrak{X}_{\mathfrak{N}}(G_{\mathfrak{M}}(F))$, $X^{(2)} \in \mathfrak{X}_{F^{(2)}}(G_{\mathfrak{M}}(F))$, $\alpha, \beta \in \mathfrak{F}(W)$. As in the proof of Proposition 5, we show that

$$(L_{\alpha X^{\mathfrak{N}} + \beta X^{(2)}} d)(Y, Z) = \alpha(L_{X^{\mathfrak{N}}} d)(Y, Z) + \beta(L_{X^{(2)}} d)(Y, Z) \quad \text{for all} \quad Y, Z \in \mathfrak{X}_{F^{(1)}}(G_{\mathfrak{M}}(F)).$$

Identities (7) and (8) imply that $(L_{X^{\mathfrak{N}}} d)(Y, Z) = 0$ and $(L_{X^{(2)}} d)(Y, Z) = 0$. Thus, we obtain:

$$(L_X d)(Y, Z) = 0 \qquad \text{for all} \quad X \in \mathfrak{X}_{\mathfrak{M}^{(1)}}(G_{\mathfrak{M}}(F)), \quad Y, Z \in \mathfrak{X}_{F^{(1)}}(G_{\mathfrak{M}}(F)).$$

According Proposition 5, this means that $F^{(1)}$ is a totally geodesic foliation.

4.3. Geodesic invariance of distributions $\mathfrak{M}^{(1)}$, $T\mathbb{F}$ and $TF^{(2)}$.

The definition of the metrics d implies that $p_1 : G_{\mathfrak{M}} \to M$ is a pseudo-Riemannian submersion and $\mathfrak{M}^{(1)}$ is a distribution complementary to the leaves of this submersion with respect to the orthogonality. This is why according [11, Thm. 1], the distribution $\mathfrak{M}^{(1)}$ is geodesically invariant.

As it has been proved above, the foliation $F^{(1)}$ is totally geodesic and hence, $p_1 : G_{\mathfrak{M}}(F) \to M$ is a pseudo-Riemannian submerions with totally geodesic fibres. Applying Statement (ii) of Proposition 4, we obtain that the total geodesicity of (M, F) implies the same for the induced foliation $(G_{\mathfrak{M}}(F), \mathbb{F})$, since each its leaf $\mathbb{L} = p_1^{-1}(L)$ is the pre-image of some leaf L of the foliation (M, F) being a totally geodesic submanifold in (M, g).

Since $TF^{(2)} = \mathfrak{M}^{(1)} \cap T\mathbb{F}$, then $TF^{(2)}$ is a geodesically invariant distribution as the intersection of geodesically invariant distributions $\mathfrak{M}^{(1)}$ and $T\mathbb{F}$. Therefore, $(G_{\mathfrak{M}}(F), F^{(2)})$ is a totally geodesic foliation on $(G_{\mathfrak{M}}(F), d)$.

5. Proof of Theorem 3

1. The first statement of Theorem 3 was proved in the proof of Theorem 2, see Subsection 4.2).

2. Let x_1 and x_2 be two points on the pseudo-Riemannina manifold (M, g). According [4, Lm. 1.1], the leaves $L_1 \ni x_1$ and $L_2 \ni x_2$ can be connected by a $\mathfrak{M}$-horizontal curve $\sigma : [0, 1] \to M$, where $y_1 = \sigma(0) \in L_1$, $y_2 = \sigma(1) \in L_2$. We connect x_i with y_i by the curve σ_i in the leaf L_i, $i = 1, 2$. Then the product of the paths $\gamma := \sigma_1 \cdot \sigma \cdot \sigma_2^{-1}$ connects a point x_1 with a point x_2. As it is known [7, Lm. 1], each $\mathfrak{M}$-horizontal curve in M possesess $\mathfrak{N}$-horizontal lifts in $G_{\mathfrak{M}}(F)$. The curves σ_1 and σ_2 possess $TF^{(2)}$-horizontal lifts in $G_{\mathfrak{M}}(F)$, since for each leaf $\mathbb{L}$ in the induced foliation $(G_{\mathfrak{M}}(F), \mathbb{F})$, the distribution $TF^{(2)}|_{\mathbb{L}}$ is an Ehresmann connection for the submersion $p_1|_{\mathbb{L}}$. Therefore, for each point z in $p_1^{-1}(x_1)$, there exists a $\mathfrak{M}^{(1)}$-horizontal lift $\widehat{\gamma}$ of the curve γ with the origin at $\widehat{\gamma}(0) = z$ and the end $\widehat{\gamma}(1) \in p_1^{-1}(x_1)$. Since

$$\Phi_{\widehat{\gamma}} : p_1^{-1}(x_0) \to p_1^{-1}(x_1) : z \mapsto \widehat{\gamma}(1)$$

is a horizontal holonomic diffeomorphism with respect to the foliation $(G_{\mathfrak{M}}, F^{(1)})$ along a $\mathfrak{M}^{(1)}$-horizontal path $\widehat{\gamma}$, then by the complete geodesicity of the mentioned foliation, it follows from Proposition 5 that the mapping $\Phi_{\widehat{\gamma}}$ is an isometry. Therefore, there exists a pseudo-Riemannian manifold $L_0^{(1)}$ isometric to each fibre of the submersion p_1. In the same way, there exists a pseudo-Riemannian manifold $L_0^{(2)}$ isometric to each fibre in the submersion p_2. We fix a point $z_0 = (x_0, \{1_{x_0}\}, x_0) \in G_{\mathfrak{M}}(F)$, where 1_{x_0} is the constant path at the point x_0. We note that the restriction of the inversion $i : G_{\mathfrak{M}}(F) \to G_{\mathfrak{M}}(F)$, $i(x, \{h\}, y) = (y, \{h^{-1}\}, x)$ on the leaf $p_1^{-1}(x_0)$ is an isometry $p_1^{-1}(x_0)$ onto the leaf $p_2^{-1}(x_0)$. This implies that $L_0^{(1)}$ and $L_0^{(2)}$ are isometric. We denote by L_0 a pseudo-Riemannian manifold isometric to $L_0^{(i)}$, $i = 1, 2$.

Let $L = L(x)$, $x \in M$, be an arbitrary leaf in the foliation (M, F). The definition of the graph $G_{\mathfrak{M}}(F)$ implies that the restriction of the canonical projection $p_1|_{p_2^{-1}(x)} : p_2^{-1}(x) \to L$ is a regular covering mapping with a group of covering mappings isomorphic to the $\mathfrak{M}$-holonomy group $H_{\mathfrak{M}}(L, x)$. According the definition of the pseudo-Riemannian metrics d on $G_{\mathfrak{M}}(F)$, this mapping is a local isometry and therefore, is a pseudo-Riemannian covering. The above proven isometricity of $p_2^{-1}(x)$ and L_0 implies Statement (i) of the theorem.

We consider an arbitrary leaf $\mathbb{L} = p_1^{-1}(L)$ of the induced foliation $\mathbb{F}$ on the graph $G_{\mathfrak{M}}(F)$ with the metrics $d|_{\mathbb{L}}$. As it has been shown in Subsection 4.2, a pseudo-Riemannian manifold $(\mathbb{L}, d)$ is non-degenerately reducible and $F^{(1)}|_{\mathbb{L}}$ and $F^{(2)}|_{\mathbb{L}}$ its orthogonal parallel leaves. We stress that $TF^{(1)}|_{\mathbb{L}}$ is

an integrable Ehresmann connection for $F^{(2)}|_{\mathbb{L}}$, and $TF^{(2)}|_{\mathbb{L}}$ is an integrable Ehresmann connection for $F^{(1)}|_{\mathbb{L}}$. Thus, $(\mathbb{L}, p_1|_{\mathbb{L}}, p_1|_{\mathbb{L}}, L, L)$ is a symmetric simple transversal bifibration in the sense [8].

We fix a point $z = (x, \{1_x\}, x) \in \mathbb{L}$. According (i), the group $\Psi \cong H_{\mathfrak{M}}(L, x)$ acts by isometries on a pseudo-Riemannian covering manifold L_0 as a group of covering transformations and this is why, a diagonal action Φ of the group Ψ on a pseudo-Riemannian product $L_0 \times L_0$ is well-defined by the rule $\psi(z_1, z_2) = (\psi(z_1), \psi(z_2))$, $(z_1, z_2) \in L_0 \times L_0$. The action of Φ is free, properly discontinuous and preserves the structure of the product. The quotient manifold $(L_0 \times L_0)/\Psi$ is well-defined with a pair of foliations F_1, F_2 covered by the product $L_0 \times L_0$. Since Φ preserves the metrics of the pseudo-Riemannian product on $L_0 \times L_0$, then on $(L_0 \times L_0)/\Psi$, a pseudo-Riemannian metrics is induced, and with respect to this metrics, the quotient mapping $L_0 \times L_0 \to (L_0 \times L_0)/\Psi$ is a pseudo-Riemannian covering. At that, on $(L_0 \times L_0)/\Psi$ a pair of parallel foliations (F_1, F_2) is well-defined, whose leaves are covered by the leaves of trivial foliations of the product $L_0 \times L_0$.

As it is known [8, Props. 5, 6], there exists a diffeomorphism

$$\Theta : \mathbb{L} \to (L_0 \times L_0)/\Psi$$

being an isomorphism of both pairs of the foliations $F^{(i)}|_{\mathbb{L}}$ and F_i, $i = 1, 2$ in the category of foliations $\mathfrak{Fol}$. It is easy to see that $L_0 \times L_0$ is a common pseudo-Riemannian covering space for $\mathbb{L}$ and for $(L_0 \times L_0)/\Psi$, therefore, Θ is a diffeomorphism being a local isometry, that is, Θ is an isometry and this completes the proof of Statement (ii) of Theorem 3.

6. GRAPHS OF PARALLEL FOLIATIONS

6.1. Criterion of existence of integrable Ehresmann connection.

Definition 5. *A pair of transversal foliations of additional dimensions (F_1, F_2) on a manifold M is called the a bifibration.*

If (F_1, F_2) is a bifibration on M, then at each point $x \in M$, the identity $T_x M = T_x F_1 \oplus T_x F_2$ holds true.

Definition 6. *Let (F_1, F_2) be a bifibration on M and $\kappa : \widehat{M} \to M$ be an universal covering mapping. If the conditions hold:*
1) $\widehat{M} = M_1 \times M_2$ is a product of simply-connected manifolds M_1 and M_2,
2) $\kappa^* F_1 = \{M_1 \times \{x_2\} \,|\, x_2 \in M_2\}$, $\kappa^* F_2 = \{x_1 \times M_2 \,|\, x_1 \in M_1\}$,
then one says that the bifibration (F_1, F_2) is covered by a product.

Employing the theorem by S. Kashiwabara [17, Thm. 2], it is easy to obtain the criterion of the existence of an integrable Ehresmann connection for a smooth foliation. We formulate this criterion in the following form convenient for us.

Theorem 5. *A bifibration (F_1, F_2) on a manifold M is covered by a product if and only if the distribution TF_2 is an integrable Ehresmann connection for the foliation (M, F_1).*

Let (F_1, F_2) be a bifibration on the manifold M. The definition of Ehresmann connection implies that TF_2 is an Ehresmann connection for the foliation (M, F_1) if and only if TF_1 is an Ehresmann connection for (M, F_2).

6.2. Lemma. We shall employ the following statement, which is essentially of a local character.

Lemma 2. *Let $(F, F^{\perp})$ be mutually orthogonal foliations of additional dimensions on a pseudo-Riemannian manifold (M, g) and the induced metrics on the leaves of the foliations (M, F) is non-degenerate. Then the following four statements are equivalent:*
(1) the foliation (M, F) is simultaneously pseudo-Riemannian and totally geodesic;
(2) both foliations (M, F) and $(M, F^{\perp})$ are totally geodesic;
(3) both foliations (M, F) and $(M, F^{\perp})$ are pseudo-Riemannian;
(4) both foliations (M, F) and $(M, F^{\perp})$ are parallel.

Proof. Assume that under the assumptions of the lemma $dim(M) = n$ and $dim(F) = q$, where $0 < q < n$; at that, $dim(F^{\perp}) = n - q$.

Let the foliation (M, F) be simultaneously pseudo-Riemannian and totally geodesic. Since (M, F) is a pseudo-Riemannian foliation, according [11, Thm. 1], the orthogonal $(n-q)$-dimensional distribution

$\mathfrak{M}^\perp$ is totally geodesic. Since $\mathfrak{M}^\perp = TF^\perp$, this means that the foliation $(M, F^\perp)$ is totally geodesic. Hence, $(1) \Rightarrow (2)$.

Assume that both foliations (M, F) and $(M, F^\perp)$ are totally geodesic. According [11, Thm. 1], both these foliations are pseudo-Riemannian, that is, $(2) \Rightarrow (3)$.

Let both foliations (M, F) and $(M, F^\perp)$ are pseudo-Riemannian. As it is known, for each bifibration $(F, F^\perp)$, at each point $x \in M$, there exists a chart (U, φ) such that $U = U_1 \times U_2$, where $F|_U = \{U_1 \times \{x_2\} \mid x_2 \in U_2\}$ and $F^\perp|_U = \{\{x_1\} \times U_2 \mid x_1 \in U_1\}$. Since the foliations (M, F) and $(M, F^\perp)$ are pseudo-Riemannian, this implies the existence of pseudo-Riemannian metrics g_1 and g_2 on U_1 and U_2, respectively, such that the projections on the factors $U_1 \times U_2 \to U_i$, $i = 1, 2$, are pseudo-Riemannian submersions $(U_1 \times U_2, g)$ on (U_i, g_i). This means that $(U_1 \times U_2, g)$ is a pseudo-Riemannian product of pseudo-Riemannian manifolds (U_1, g_1) and (U_2, g_2). Since x is an arbitrary point in the manifold M, this yields that (M, g) is a non-degenerate pseudo-Riemannian manifold with parallel foliations (M, F) and $(M, F^\perp)$, see, for instance, [19]. Thus, we have proved that $(3) \Rightarrow (4)$.

Assume that (M, F) and $(M, F^\perp)$ are parallel foliations on (M, g). Since the tangential vectors to a geodesic curve forms a field of parallel translation, the definition of the parallel foliations imply that they are totally geodesic and this is why it follows from the proven implication $(2) \Rightarrow (3)$ that both foliations are pseudo-Riemannian. Thus, $(4) \Rightarrow (1)$. $\square$

6.3. Holonomy groups of parallel foliations.

Proposition 6. *Let F be a parallel foliation on a pseudo-Riemannian manifold (M, g), the induced metrics be non-degenerate on the leaves and the distribution complementary with respect to the orthogonality is an Ehresmann connection for F. Then almost each leaf in the foliation F has a trivial holonomy group.*

Proof. The assumptions imply the existence a parallel foliation $F^\perp$ complementary to F with respect to the orthogonality. Since $TF^\perp$ is an Ehresmann connection for F, it follows from Theorem 5 that the bifibration $(F, F^\perp)$ is covered by a product, that is, the universal covering for M reads as $\kappa : M_1 \times M_2 \to M$ and $\kappa^* F = \{M_1 \times \{z\} \mid z \in M_2\}$. We fix $x_0 \in M$ and $(y_0, z_0) \in M_1 \times M_2$, $p_1(y_0, z_0) = x_0$. By L and $L^\perp$ we denote the leaves of the foliations F and $F^\perp$ passing through x_0. We define $M_1 \cong M_1 \times \{z_0\}$ and $M_2 \cong \{y_0\} \times M_2$. Let $g_1 = g|_L$ and $g_2 = g|_{L^\perp}$. Since $\kappa|_{M_1} : M_1 \to L$ and $\kappa|_{M_2} : M_2 \to L^\perp$ are universal covering mappings, the pseudo-Riemannina manifolds $(M_1, \kappa^* g_1)$ and $(M_2, \kappa^* g_2)$ are well-defined. Since locally (M, g) is a product of pseudo-Riemannian manifolds induced on local leaves of foliations F and $F^\perp$, the pseudo-Riemannian manifold $(M_1 \times M_2, \kappa^* g)$ is a product of pseudo-Riemannian manifolds $(M_1, \kappa^* g_1)$ and $(M_2, \kappa^* g_2)$.

The fundamental group $\pi_1(M, x_0)$ acts on $M_1 \times M_2$ as a group of covering transformations G of the covering κ preserving the structure of the product and a pseudo-Riemannian metrics $\kappa^* g$. This is why on M_2, a group of isometries Ψ is induced and an epimorphism of the groups $\chi : G \to \Psi$ is well-defined. Since the action of the group Ψ on M_2 is quasi-analytic, the holonomy group $\Gamma(L, x)$ of an arbitrary leaf $L = L(x)$ of the foliation F is isomorphic to the stationary subgroup Ψ_z of the group Ψ at a point $z \in pr(\kappa^{-1}(x))$, where $pr : M_1 \times M_2 \to M_2$ is the canonical projection onto the second factor. Therefore, the leaf $L = L(x)$ possesses a trivial holonomy group if and only if the group Ψ_z is trivial as $z \in pr(\kappa^{-1}(x))$.

Let $fix(\psi)$ be the set of fixed points of an isometry $\psi \in \Psi$. Let us prove that union of all leaves in the foliation(M, F) with non-trivial holonomy groups has a zero measure in M. This is equivalent to the fact that the set $K = \bigcup_{\psi \in \Psi} fix(\psi)$ has a zero measure in M_2.

We recall that a subset N of a m-dimensional manifold has a zero measure if at each point there exists a chart (U, f) of this manifold such that the subset $f(U \cap N) \subset \mathbb{R}^m$ has a zero measure in $\mathbb{R}^m$.

Let ψ be an arbitrary element in Ψ and z be an arbitrary point in $fix(\psi)$. Since a pseudo-Riemannian metrics defines a G-structure of the first order, there exists an isomorphism $\mu : \Psi_z \to D\Psi_z$, $\mu(\{\psi\}_z) = \psi_{*z}$, of the stationary subgroup Ψ_z onto the linear group $D\Psi_z$ mapping each isometry $\psi \in \Psi_z$ into the differential ψ_{*z} at the point z.

There exists a neighbourhood W_0 of the zero in $T_z M_2$ such that the exponential mapping $Exp|_{W_0} : W_0 \to M_2$ is a diffeomorphism onto an open neighbourhood W of the point z in M_2. By the continuity of ψ_{*z}, there exists a neighbourhood W_0' of the zero in $T_z M_2$ obeying $\psi_{*z}(W_0') \subset W_0$. Let $W' =$

$Exp(W_0')$. Since ψ is an isometry, it satisfies the identity

$$Exp \circ \psi_{*z}|_{W_0'} = \psi \circ Exp|_{W_0'},$$

and therefore, $Exp^{-1}(fix(\psi) \cap W') = fix(\psi_{*z}) \cap W_0'$. Since ψ_{*z} is a non-trivial linear mapping of the vector space $T_z M_2$, the set $fix(\psi_{*z})$ is a proper subspace in $T_z M_2$ and this is why set $fix(\psi_{*z}) \cap W_0'$ has a zero measure in $T_z M_2 \cong \mathbb{R}^q$. We observe that $(W', Exp^{-1}|_{W'})$ can be treated as a chart at the point z in the manifold M_2. Therefore, the set $fix(\psi)$ has a zero measure in M_2. Since the group Ψ is at most countable, this implies that the set K has also a zero measure in M_2. $\qquad\square$

6.4. Proof of Theorem 4. Assume that (M, g) is a non-degenerate reducible pseudo-Riemannian manifold, while F and $F^\perp$ are its parallel foliations of additional dimension and $\mathfrak{M} = TF^\perp$ is an Ehresmann connection for the foliation (M, F). By Lemma 2, the foliations F and $F^\perp$ are simultaneously totally geodesic and pseudo-Riemannian. This is why the holonomy pseudo-group of the foliation (M, F) is formed by local isometries and is quasi-analytic. Hence, Proposition 1 implies that the graph $G(F)$ is Hausdorff. According Theorem 1, the graphs $G(F)$ and $G_{\mathfrak{M}}(F)$ are canonically isomorphic and are identified and this proves Statement 1.

According [11, Thm. 2], the induced foliation $(G(F), \mathbb{F})$ is also pseudo-Riemannian and Statement 2 follows Proposition 6 and Statements 2, 3 of Theorem 3.

Let us show that the distribution $\mathfrak{N}$ is integrable. For each $X \in \mathfrak{X}_{\mathfrak{N}}(G(F))$, the properties of the differential implies $p_{i*}(X) \in \mathfrak{X}_{\mathfrak{M}}(M)$ as $i = 1, 2$. Taking this into consideration, by the integrability of $\mathfrak{M}$ we have

$$p_{1*}([X, Y]) = [p_{1*}(X), p_{1*}(Y)] \in \mathfrak{X}_{\mathfrak{M}}(M)$$

and

$$p_{2*}([X, Y]) = [p_{1*}(X), p_{2*}(Y)] \in \mathfrak{X}_{\mathfrak{M}}(M).$$

Hence, in view of the identity $\mathfrak{N} = p_1^*\mathfrak{M} \cap p_2^*\mathfrak{M}$, we obtain $[X, Y] \in \mathfrak{X}_{\mathfrak{N}}(G(F))$. By the Frobenius theorem, the distribution $\mathfrak{N}$ is integrable and defines a foliation, which we denote by $F^{\mathfrak{N}}$. It follows from Theorem 2 that the foliations $F^{\mathfrak{N}}$ and $\mathbb{F}$ are totally geodesic. According Lemma 2 this is equivalent to the fact that $(F^{\mathfrak{N}}, \mathbb{F})$ is a pair of parallel foliations complementary with respect to the orthogonality.

Let us show that the distribution $\mathfrak{M}^{(2)}$ is also integrable. We take arbitrary vector fields X, Y tangential to $\mathfrak{M}^{(2)}$. Let $Z := [X, Y]$. Since $\mathfrak{M}^{(2)} = TF^{(1)} \oplus \mathfrak{N} = p_1^*\mathfrak{M}$, thanks to the integrability of the distribution $\mathfrak{M} = TF^\perp$, the chain of identities holds:

$$p_{1*}(Z) = p_{1*}([X, Y]) = [p_{1*}(X), \qquad p_{1*}(Y)] \in \mathfrak{X}_{\mathfrak{M}}(M),$$

and this is why it is necessary that $Z \in \mathfrak{X}_{\mathfrak{M}^{(2)}}(G(M))$. According Frobenius theorem, the distribution $\mathfrak{M}^{(2)}$ is integrable. We denote by $\mathcal{F}^{(2)}$ the foliation, for which $\mathfrak{M}^{(2)} = T\mathcal{F}^{(2)}$. By the assumptions, the foliations (M, F) and $(M, F^\perp)$ are parallel and this is why it follows from Lemma 2 that the foliation (M, F) is totally geodesic. By Theorem 2, the total geodesicity of the foliation (M, F) implies the same for the foliation $(G(F), \mathcal{F}^{(2)})$, and according [11, Thm. 2], the pseudo-Riemannian property of (M, F) implies the same for the foliation $(G(F), \mathcal{F}^{(2)})$. This is why, thanks to Lemma 2, the foliations $F^{(2)}$ and $\mathcal{F}^{(2)}$ on the graph $(G(F), d)$ complementary with respect to the orthogonality are parallel. In the same way we prove that the pair of complementary with respect to the orthogonality foliations $(F^{(1)}, \mathcal{F}^{(1)})$ are parallel. This completes the proof of Statements 2 and 3.

We consider the universal covering mapping $f : \widetilde{G(F)} \to G(F)$. Since $\mathfrak{N} = TF^{\mathfrak{N}}$ is an integrable Ehresmann connection for the foliation $(G(M), \mathbb{F})$, by Theorem 5, $\widetilde{G(F)} = \widetilde{L^{\mathfrak{N}}} \times \widetilde{\mathbb{L}}$ is a product of simply-connected manifolds and $f^*F^{\mathfrak{N}} = \{\widetilde{L^{\mathfrak{N}}} \times \{v\} \,|\, v \in \widetilde{\mathbb{L}}\}$ and $f^*\mathbb{F} = \{\{u\} \times \widetilde{\mathbb{L}} \,|\, u \in \widetilde{L^{\mathfrak{N}}}\}$. It follows from Theorem 2 that $\widetilde{\mathbb{L}} \cong \widetilde{L_0} \times \widetilde{L_0}$, where $f|_{\widetilde{L_0}} : \widetilde{L_0} \to L_0$ is an universal covering mapping for L_0. Thus, $\widetilde{G(F)} \cong \widetilde{L^{\mathfrak{N}}} \times \widetilde{L_0} \times \widetilde{L_0}$. Since the bifibration $(\mathcal{F}^{(2)}, F^{(2)})$ is covered by a product $(\widetilde{L^{\mathfrak{N}}} \times \widetilde{L_0}) \times \widetilde{L_0}$, according Theorem 5, $T\mathcal{F}^{(2)}$ is an integrable Ehresmann connection for the foliation $(G(F), \mathcal{F}^{(2)})$. And vice versa, $T\mathcal{F}^{(2)}$ is an integrable Ehresmann connection for $(G(F), F^{(2)})$. In the same way, $TF^{(1)}$ is an integrable Ehresmann connection for $(G(F), \mathcal{F}^{(1)})$ and vice versa. This completes the proof of Statement 3 of the theorem.

7.   Two classes of studied foliations

7.1.   Proof of Proposition 4. The family of piece-wise smooth geodesic curves of each pseudo-Riemannian manifold form a system of paths in the sense of [18]. Therefore, on each leaf (L_α, g) of the foliation (M, F) a system of paths is well-defined. By the completeness of the induced metrics on the leaves, the affine parameter of each geodesic curve lying in a leaf varies on the entire real line. This means that the mentioned system of paths is complete. According Lemma 1 and [16, Prop. 2.7], thanks to the total geodesicity of the foliation (M, F), for each integral curve σ of the distribution $\mathfrak{M}$, a local horizontal holonomic diffeomorphism $\Phi_\sigma : U_0 \to U_1$ is an isometry. Therefore, Φ_σ maps a geodesic curve into a geodesic curve keeping the parameter. This means that the foliation (M, F) is $\mathfrak{M}$-conformed with the systems of paths on the leaves. Hence, it follows from [18, Thm. 6.1] that $\mathfrak{M}$ is an Ehresmann connection for the foliation (M, F).

7.2.   Suspended foliations over pseudo-Riemannian manifolds.

Suspended foliations. Let (B, g^B) be an arbitrary m-dimensional pseudo-Riemannian manifold and T be an arbitrary smooth q-dimensional manifold. Suppose that we are given a homomorphism $\rho : G \to Diff(T)$ of the group $G = \pi_1(B, b)$ into the group $Diff(T)$ of diffeomorphisms of the manifold T. Let the group G act from the right as a group of covering transformations on the universal covering space $\widehat{B}$. Then the identity $f(x, t, g) = (x \cdot g, \rho(g^{-1})(t))$, where $(x, t, g) \in \widehat{B} \times T \times G$, defines a right action of the group G on the product of the manifolds $\widehat{B} \times T$. A quotient mapping $f : \widehat{B} \times T \to M$ onto the quotient manifold $M := (\widehat{B} \times T)/G$ induces a smooth foliation $F = \{f(\widehat{B} \times \{v\}) \,|\, v \in T\}$ on M, which is called *suspended* and is denoted by $(M, F) = Sus(T, B, \rho)$. The projection $p : M = (\widehat{B} \times T)/G \to B = \widehat{B}/G$ forms a locally trivial fibration, which is called *associated*. The group $\Psi = \rho(\pi_1(B, b))$ is called the *structural group* of the suspended foliation (M, F).

7.3.   Proof of Proposition 3. Let $p : M \to B$ be the associated fibration and $\mathfrak{M}$ be a distribution formed by tangential spaces to its fibres. Each vector field X on M is uniquely represented as $X = X^F + X^{\mathfrak{M}}$, where $X^F \in \mathfrak{X}_F(M)$, $X^{\mathfrak{M}} \in \mathfrak{X}_{\mathfrak{M}}(M)$. Let g^M be a pseudo-Riemannian metrics M non-degenerate on the leaves of the foliation (M, F). A mentioned metrics g^M exists since as g^M we can take an arbitrary Riemannian metrics. Then the identity

$$g(X, Y) := (p^* g^B)(X^F, Y^F) + g^M(X^{\mathfrak{M}}, Y^{\mathfrak{M}}) \qquad \text{for all} \quad X, Y \in \mathfrak{X}(M)$$

defines a pseudo-Riemannian metrics g on the manifold M. The definition of the metrics g implies that $p : M \to B$ is a pseudo-Riemannian submersion (M, g) on (B, g^B). Therefore, local horizontal holonomic diffeomorphisms of the foliation F are isometries and this is why, according Proposition 5, F is a totally geodesics foliation on a pseudo-Riemannian manifold (M, g). By the properties of the suspended manifold, the restriction $p|_{L_\alpha}$ of the projection p on an arbitrary leaf L_α of the foliation F is a covering mapping onto the base B and therefore, $p|_{L_\alpha} : L_\alpha \to B$ is a pseudo-Riemannian covering mapping. This implies that the leaves L_α equipped with the induced pseudo-Riemannian metrics are complete pseudo-Riemannian manifolds if and only if (B, g^B) is a complete pseudo-Riemannian manifold.

The completeness of the metrics g^B is not assumed. Thus, we have constructed a totally geodesic foliation (M, F) on the pseudo-Riemannian manifold (M, g) with induced pseudo-Riemannian metrics on the leaves, whose orthogonal q-dimensional distribution $\mathfrak{M}$ is an integrable Ehresmann connection for this foliation.

The proof of Statements 1)–3) is complete.

We stress that the graph of the foliation $G(F)$ is generally speaking not Hausdorff. Since the holonomy pseudo-group $\mathcal{H}(F)$ of the foliation F is determined by the transformations in the group $\Psi := \rho(\pi_1(B, b)) \subset Diff(T)$, by applying Proposition 1 we conclude that the graph $G(F)$ is Hausdorff if and only if the group Ψ acts quasianalytically on T. This completes the proof of Statement 4).

Thus, the foliations obtained by a suspension of a homomorphism of the fundamental group of a pseudo-Riemannian manifold belong to the studied class of the foliations.

Remark 4. *In terms of the above introduced notations, the graph $G_{\mathfrak{M}}(F)$ of the suspended foliation (M, F) is a Hausdorff smooth $(2m + q)$-dimensional manifold with a pseudo-Riemannian metrics d. The fibres of the canonical projections p_1 and p_2 are totally geodesic m-dimensional submanifolds in*

$(G_{\mathfrak{M}}(F), d)$ *isometric to arbitrary leaf* (L_0, d) *with the trivial* $\mathfrak{M}$*-holonomy group of the foliation* (M, F) *once it exists. A q-dimensional distributioin* $\mathfrak{N}$ *orthogonal to the induced totally geodesic foliation* $\mathbb{F}$ *is integrable and is tangent to some pseudo-Riemannian foliation.*

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