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DIRICHLET BOUNDARY VALUE PROBLEM FOR ALLER–LYKOV MOISTURE TRANSFER EQUATION WITH FRACTIONAL DERIVATIVE IN TIME

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Abstract. The heat-moisture transfer in soils is a fundamental base in addressing many problems of hydrology, agrophysics, building physics and other fields of science. The researchers focus on possibility of reflecting specific features of the studied arrays in the equations as well as their structure, physical properties, the processes going on in them, etc. In view of this, there arises a new class of fractional differential equations of state and transport being the base for most mathematical models describing a wide class of physical and chemical processes in media with a fractal structure and memory.

This paper studies the Dirichlet boundary value problem for the Aller–Lykov moisture transfer equation with the Riemann–Liouville fractional derivative in time. The considered equation is a generalization of the Aller-Lykov equation obtained by means of introducing the concept of the fractal rate of humidity change, which accounts the presence of flows moving against the moisture potential.

The existence of the solution to the Dirichlet boundary value problem is proved by the Fourier method. By means of energy inequalities method, for the solution we obtain an apriori estimate in terms of fractional Riemann-Liouville derivative, which implies the uniqueness of the solution.

Keywords: Aller–Lykov moisture transfer equation, Riemann–Liouville fractional derivative, Fourier method, apriori estimate

Mathematics Subject Classification: 35E99

1. Introduction

The moisture transfer in soil can be described by the quasilinear equation [1]

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(D(w)\frac{\partial w}{\partial x}\right),$$

where $w(x,t)$ is the moisture of the soil at the depth x at the time moment t expressed as a fraction, $D(w)$ is the diffusion coefficient. This equation is obtained on the base of the analysis of the diffusion mechanism in a porous media, when one takes into consideration the emergence of the moisture flows under the action of the gradient of the capillary pressure. However, numerous and convincing experiments sometimes demonstrate the inverse flow from stratum with a low moisture content to the stratum with a high moisture content. A right explanation of the moisture motion in the direct and inverse direction is possible on the base of the modified Aller equation [1]:

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(D(w)\frac{\partial w}{\partial x} + A\frac{\partial^2 w}{\partial x \partial t}\right), \tag{1}$$

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where an additional term $A\frac{\partial^3 w}{\partial x^2 \partial t}$ is involved to explain the motion of the moisture against the moisture gradient, A is a variable Aller coefficient.

Aller equation (1) assumes an infinite speed of distribution of the perturbation in a soil, while Lykov equation

$$\frac{\partial w}{\partial t} + A_1 \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(D(w)\frac{\partial w}{\partial x}\right) \tag{2}$$

supposes a finite speed. In (2), an additional term $A_1\frac{\partial^2 w}{\partial t^2}$ is introduced and its role becomes appreciable in the processes with fast oscillations of the moisture on the boundary of the studied soil pattern. Lykov assumes that $A_1 = Cx^2$, where $C = const$ depends on the diffusion coefficient, the porosity of the media, its capillary properties and the fluid viscosity [2].

Since a colloidal capillary-porous media of a polycapillar structure is an example of a fractal media or can be regarded in such way, A.M. Nakhushev presented a "principally new moisture transfer equation" on the base of equation (2) [2]:

$$D_{0t}^{\alpha} w = \frac{\partial}{\partial x}\left(D(w)\frac{\partial w}{\partial x}\right) - A_1 D_{0t}^{\alpha+1} w, \tag{3}$$

where D_{0t}^{γ} is the Riemann-Liouville operator of the fractional differentiation [2], $0 < \alpha < 1$. As $\alpha = 1$, equation (3) coincides with Lykov moisture transfer equation (2). Under such approach, in the case of Aller equation (1) we obtain a so-called modified moisture transfer equations with a fractional derivative considered in works [3, 4].

To describe the processes of evaporation and infiltration, V.Ya. Kulik [5] suggested to employ a hybrid equation combining the approaches by Aller and Lykov. Equations of such kind were considered in works [6, 7].

In the present work we consider the equation

$$A_1 D_{0t}^{\alpha+1} w + D_{0t}^{\alpha} w = \frac{\partial}{\partial x}\left(D(w)\frac{\partial w}{\partial x} + A D_{0t}^{\alpha}\frac{\partial w}{\partial x}\right),$$

where A, $A_1 = const > 0$, $0 < \alpha < 1$, in the case $D(\omega) \equiv 1$, which is a generalization of Aller-Lykov equation in the classical formulation.

Our study is based on the Fourier method and the method of apriori estimates. Earlier, boundary value problems for equations with fractional derivatives were studied by the Fourier method in works by S.Kh. Gekkieva [8], O.P. Agrawal [9], V.A. Nakhusheva [10], O.Kh. Masaeva [11] and other authors. The method of apriori estimates was applied in works [12], [13].

2. Formulation of problem

In domain $Q = \{(x,t):\ 0 < x < 1,\ t > 0\}$ we consider the equation

$$A_1 D_{0t}^{\alpha+1} u + D_{0t}^{\alpha} u = u_{xx} + A D_{0t}^{\alpha} u_{xx}, \qquad 0 < x < 1, \qquad t > 0. \tag{4}$$

A regular solution of equation (4) in the domain Q is a function $u = u(x,t)$ in the class $D_{0t}^{\alpha-1}u(x,t),\ D_{0t}^{\alpha}u(x,t) \in\ C\left(\overline{Q}\right)$; $D_{0t}^{\alpha+1}u(x,t),\ u_{xx}(x,t),\ D_{0t}^{\alpha}u_{xx}(x,t) \in C\left(Q\right)$ satisfying equation (4) at all points $(x,t) \in Q$.

Let us formulate the Dirichlet problem for equation (4).

Problem 1. *Find a regular solution $u(x,t)$ for equation (4) in the domain Q obeying the boundary conditions*

$$u(0,t) = u(1,t) = 0, \qquad t > 0, \tag{5}$$

and the initial conditions

$$\lim_{t\to 0} D_{0t}^{\alpha-1}u(x,t) = \tau(x), \qquad \lim_{t\to 0} D_{0t}^{\alpha}u(x,t) = \nu(x), \tag{6}$$

where $\tau(x)$, $\nu(x)$ are given functions.

3. Solution for homogeneous Aller-Lykov equation with fractional derivative in time

To solve Problem 1, we apply the separation of variables.

First we find a class of non-trivial solutions to equation (4) obeying homogeneous boundary conditions (5), which can be represented as

$$u(x,t) = \varphi(x)y(t). \tag{7}$$

Substituting this representation in equation (4), we obtain the following equations for the functions $\varphi(x)$, $y(t)$:

$$\varphi'' + \lambda\varphi = 0, \qquad \varphi(0) = 0, \qquad \varphi(1) = 0, \tag{8}$$

$$A_1 D_{0t}^{\alpha+1} y + (1 + A\lambda)\, D_{0t}^{\alpha} y + \lambda y = 0$$

or

$$D_{0t}^{\alpha+1} y + a D_{0t}^{\alpha} y + by = 0, \tag{9}$$

where $a = \frac{1+A\lambda}{A_1}$, $b = \frac{\lambda}{A_1}$, $\lambda = const.$

As it is known, the solution of spectral problem (8) reads as

$$\varphi_k(x) = \sin(\pi k x), \qquad \lambda = \lambda_k = (\pi k)^2, \qquad k = 1, 2, \ldots \tag{10}$$

Before we write out a solution for equation (9), we note that ordinary differential equations of fractional order with constant coefficients were studied quite intensively. Explicit representations for solutions of initial and boundary value problems were found in terms of the generalized Mittag-Leffler functions and Wright functions. A detailed exposition of these results and the references can be found in works [14, 15]. To avoid technically complicated theory of special functions, in our work we solve equation (9) by reducing it to the second kind integral Volterra equation with a power kernel and solving it by the method of successive approximations.

Let $y_k(t)$ be a solution of equation (9) corresponding to an eigenvalue λ_k. Applying the operator of fractional integration of order $\alpha + 1$ [16] to equation (9), we obtain:

$$y_k(t) + \int\limits_0^t y_k(\tau) \left[a_k + \frac{b_k}{\Gamma(\alpha+1)(t-\tau)^{-\alpha}}\right] d\tau = f(t), \tag{11}$$

where

$$a_k = \frac{1 + A\lambda_k}{A_1}, \qquad b_k = \frac{\lambda_k}{A_1}, \qquad f(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} (\nu_k + a_k \tau_k) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \tau_k,$$

$$D_{0t}^{\alpha-1} y_k(t)\bigg|_{t=0} = \tau_k, \qquad D_{0t}^{\alpha} y_k(t)\bigg|_{t=0} = \nu_k.$$

We apply the theory of Volterra integral equations to equation (11). We introduce the notation

$$K_1(t,\tau) = \left[a_k + \frac{b_k (t-\tau)^{\alpha}}{\Gamma(\alpha+1)}\right] H(t-\tau), \tag{12}$$

where

$$H(z) = \begin{cases} 1, & z \geqslant 0, \\ 0, & z < 0 \end{cases}$$

is the Heaviside function, and we define the sequence of the kernels $\{K_n(t,\ \tau)\}_1^{\infty}$ by means of the recurrent relations:

$$K_{n+1}(t,\tau) = \int\limits_{\tau}^{t} K_n(t, t_1)\, K_1(t_1, \tau)\, dt_1. \tag{13}$$

By induction, we are going to prove that

$$K_{n+1}(t,\tau)=\left[\sum_{s=0}^{n+1}\left(\begin{array}{c}n+1\\ s\end{array}\right)a_k^{n+1-s}b_k^s\frac{(t-\tau)^{n+s\alpha}}{\Gamma(n+1+s\alpha)}\right]H(t-\tau), \tag{14}$$

where

$$\binom{n+1}{s}=\frac{(n+1)!}{s!(n+1-s)!}.$$

Indeed, as $n=0$, identity (14) follows (12). Assume that (14) holds for $l\leqslant n$:

$$K_l(t,\tau)=\left[\sum_{s=0}^{l}\binom{l}{s}a_k^{l-s}b_k^s\frac{(t-\tau)^{l+s\alpha-1}}{\Gamma(l+s\alpha)}\right]H(t-\tau),\qquad l\leqslant n. \tag{15}$$

Substituting (15) into (13), after some simple transformations we find (14) holds for $l=n+1$. This proves (14) for all n.

Thus, for the resolvent of equation (11) we have the formula:

$$R(t,\tau,\lambda)=\sum_{n=0}^{\infty}(-1)^nK_{n+1}(t,\tau)=\left[\sum_{n=0}^{\infty}(-1)^n\sum_{s=0}^{n+1}\binom{n+1}{s}a_k^{n+1-s}b_k^s\frac{(t-\tau)^{n+s\alpha}}{\Gamma(n+1+s\alpha)}\right]H(t-\tau).$$

Thus, integral equation (11) possesses the unique solution represented as

$$\begin{aligned}y_k(t)=&f(t)-\int_0^t R(t,\tau,\lambda)f(t)d\tau\\ =&(\nu_k+a_k\tau_k)\left[\frac{t^\alpha}{\Gamma(\alpha+1)}+\sum_{n=0}^{\infty}(-1)^{n+1}\sum_{s=0}^{n+1}\binom{n+1}{s}\frac{a_k^{n+1-s}b_k^st^{n+s\alpha+\alpha+1}}{\Gamma(n+s\alpha+\alpha+2)}\right]\\ &+\tau_k\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\sum_{n=0}^{\infty}(-1)^{n+1}\sum_{s=0}^{n+1}\binom{n+1}{s}\frac{a_k^{n+1-s}b_k^st^{n+s\alpha+\alpha}}{\Gamma(n+s\alpha+\alpha+1)}\right].\end{aligned}$$

Hence, solutions of equation (9) associated with eigenvalues λ_k are of the form:

$$\begin{aligned}y_k(t)=&(\nu_k+a\tau_k)\sum_{n=0}^{\infty}\sum_{s=0}^{n}(-1)^n\binom{n}{s}\frac{a_k^{n-s}b_k^st^{n+s\alpha+\alpha}}{\Gamma(n+s\alpha+\alpha+1)}\\ &+\tau_k\sum_{n=0}^{\infty}\sum_{s=0}^{n}(-1)^n\binom{n}{s}\frac{a_k^{n-s}b_k^st^{n+s\alpha+\alpha-1}}{\Gamma(n+s\alpha+\alpha)}.\end{aligned}$$

Returning back to problem (4)–(6), we conclude that the functions

$$\begin{aligned}u_k(x,t)=&\varphi_k(x)y_k(t)\\ =&\left((\nu_k+a_k\tau_k)\,t^\alpha\sum_{n=0}^{\infty}\sum_{s=0}^{n}(-1)^n\binom{n}{s}\frac{a_k^{n-s}b_k^st^{n+s\alpha}}{\Gamma(n+s\alpha+\alpha+1)}\right.\\ &\left.+\tau_kt^{\alpha-1}\sum_{n=0}^{\infty}\sum_{s=0}^{n}(-1)^n\binom{n}{s}\frac{a_k^{n-s}b_k^st^{n+s\alpha}}{\Gamma(n+s\alpha+\alpha)}\right)\sin(\pi kx)\end{aligned}$$

are particular solutions of equation (4) satisfying boundary conditions (5) that can be checked straightforwardly.

We proceed to solving problem (4)–(6) in the general case. We introduce the series

$$u(x,t)=\sum_{k=1}^{\infty}y_k(t)\varphi_k(x)=\sum_{k=1}^{\infty}\left((\nu_k+a_k\tau_k)\,t^{\alpha}\sum_{n=0}^{\infty}\sum_{s=0}^{n}\binom{n}{s}\frac{(-1)^n\,a_k^{n-s}b_k^s t^{n+s\alpha}}{\Gamma(n+s\alpha+\alpha+1)}\right.$$
$$\left.+\,\tau_k t^{\alpha-1}\sum_{n=0}^{\infty}\sum_{s=0}^{n+1}\binom{n}{s}\frac{(-1)^n\,a_k^{n-s}b_k^s t^{n+s\alpha}}{\Gamma(n+s\alpha+\alpha)}\right)\sin(\pi kx). \tag{16}$$

The function $u(x,t)$ satisfies the boundary condition since the same holds for all terms in series (16). Claiming initial conditions (6), we obtain:

$$\lim_{t\to0}D_{0t}^{\alpha-1}u(x,t)=\lim_{t\to0}\sum_{k=1}^{\infty}y_k(t)\varphi_k\left(x\right)=\sum_{k=1}^{\infty}\varphi_k\left(x\right)\lim_{t\to0}D_{0t}^{\alpha-1}y_k(t)$$
$$=\sum_{k=1}^{\infty}\varphi_k\left(x\right)\tau_k=\sum_{k=1}^{\infty}\tau_k\sin\left(\pi kx\right)=\tau(x),$$
$$\lim_{t\to0}D_{0t}^{\alpha}u(x,t)=\sum_{k=1}^{\infty}\varphi_k\left(x\right)\lim_{t\to0}D_{0t}^{\alpha}y_k(t)=\sum_{k=1}^{\infty}\varphi_k\left(x\right)\nu_k=\sum_{k=1}^{\infty}\nu_k\sin\left(\pi kx\right)=\nu(x).$$

Thus, by (10) we obtain the following fact: the expansions of initial functions into the Fourier series over sines

$$\tau\left(x\right)=\sum_{k=1}^{\infty}\tau_k\sin\left(\pi kx\right),\qquad \nu\left(x\right)=\sum_{k=1}^{\infty}\nu_k\sin\left(\pi kx\right), \tag{17}$$

is the necessary solvability condition of Problem 1 in the class of functions, which can represented by series (16). Representations (17) hold if and only if

$$\tau(0)=\tau(1),\qquad \nu(0)=\nu(1),\qquad \tau_k=2\int_0^1\tau(x)\sin(\pi kx)dx,\qquad \nu_k=2\int_0^1\nu(x)\sin(\pi kx)dx.$$

As it is known from the theory of Fourier series [17], if the function $\tau(x)$ has continuous derivatives up to the third order and satisfies the conditions $\tau(0)=\tau(1)=\tau''(0)=\tau''(1)=0$, and $\nu(x)$ has continuous derivatives up to the second order and $\nu(0)=\nu(1)=0$, then the function $u(x,t)$ represented by formula (16) possesses all needed derivatives, which can be found by pointwise differentiation in the right hand side in (16).

To justify the Fourier method, we shall need a lemma [18] on asymptotic properties of functions of Mittag-Leffler type

$$E_\rho(z;\mu)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu+k\rho^{-1})}.$$

Lemma 1. *Let $\rho>\frac{1}{2}$, μ be a real constant and α_1 be a fixed number in the interval $\left(\frac{\pi}{2\rho},\min\left\{\pi,\frac{\pi}{\rho}\right\}\right)$. Then the following estimates hold:*

1. If $|\arg z|\leqslant\alpha_1$ and $|z|\geqslant0$, then

$$|E_\rho(z;\mu)|\leqslant M_1\left(1+|z|\right)^{\rho(1-\mu)}e^{\operatorname{Re}z^\rho}+\frac{M_2}{1+|z|}.$$

2. If $\alpha_1\leqslant|\arg z|\leqslant\pi$ and $|z|\geqslant0$, then

$$|E_\rho(z;\mu)|\leqslant\frac{M_2}{1+|z|},$$

where M_1 and M_2 are constants independent of z.

Let us show that series (16) and the series of the derivatives $D_{0t}^{\alpha+1}u$, $D_{0t}^{\alpha}u$, u_{xx}, $D_{0t}^{\alpha}u_{xx}$ obtained from this series converge uniformly.

To prove the uniform convergence of series (16), we employ known estimate for the Fourier coefficients [17], the properties of the Gamma-function and the following relations:

$$\begin{aligned}
|u_k| \leqslant & \left|(\nu_k + a_k\tau_k)\, t^\alpha \sum_{n=0}^{\infty}\sum_{s=0}^{n}(-1)^n \binom{n}{s} \frac{a_k^{n-s} b_k^s t^{n+s\alpha}}{\Gamma(n+s\alpha+\alpha+1)}\right| \\
& + \left|\tau_k t^{\alpha-1} \sum_{n=0}^{\infty}\sum_{s=0}^{n}(-1)^n \binom{n}{s} \frac{a_k^{n-s} b_k^s t^{n+s\alpha}}{\Gamma(n+s\alpha+\alpha)}\right| \\
\leqslant & ct^\alpha |\nu_k| \left|\sum_{n=0}^{\infty}(-a_k t)^n \sum_{s=0}^{n} \binom{n}{s} \frac{\left(a_k^{-1} b_k t^\alpha\right)^s}{\Gamma(n+\alpha+1)}\right| \\
& + ct^\alpha a_k |\tau_k| \left|\sum_{n=0}^{\infty}(-a_k t)^n \sum_{s=0}^{n} \binom{n}{s} \frac{\left(a_k^{-1} b_k t^\alpha\right)^s}{\Gamma(n+\alpha+1)}\right| \\
& + ct^{\alpha-1} |\tau_k| \left|\sum_{n=0}^{\infty}(-a_k t)^n \sum_{s=0}^{n} \binom{n}{s} \frac{\left(a_k^{-1} b_k t^\alpha\right)^s}{\Gamma(n+\alpha)}\right| \\
\leqslant & ct^\alpha \frac{M_3}{k^2} \left|\sum_{n=0}^{\infty}(-a_k t)^n \frac{\left(1 + a_k^{-1} b_k t^\alpha\right)^n}{\Gamma(n+\alpha+1)}\right| + ct^\alpha \frac{M_4}{k^2} \left|\sum_{n=0}^{\infty}(-a_k t)^n \frac{\left(1 + a_k^{-1} b_k t^\alpha\right)^n}{\Gamma(n+\alpha+1)}\right| \\
& + ct^{\alpha-1} \frac{M_5}{k^2} \left|\sum_{n=0}^{\infty}(-a_k t)^n \frac{\left(1 + a_k^{-1} b_k t^\alpha\right)^n}{\Gamma(n+\alpha)}\right| \\
= & t^\alpha \frac{M_6}{k^2} \left|\sum_{n=0}^{\infty} \frac{\left[-\left(a_k t + b_k t^{\alpha+1}\right)\right]^n}{\Gamma(n+\alpha+1)}\right| + t^{\alpha-1} \frac{M_7}{k^2} \left|\sum_{n=0}^{\infty} \frac{\left[-\left(a_k t + b_k t^{\alpha+1}\right)\right]^n}{\Gamma(n+\alpha)}\right| \\
= & t^\alpha \frac{M_6}{k^2} \left|E_1\left[-\left(a_k t + b_k t^{\alpha+1}\right); \alpha+1\right]\right| + t^{\alpha-1} \frac{M_7}{k^2} \left|E_1\left[-\left(a_k t + b_k t^{\alpha+1}\right); \alpha\right]\right|.
\end{aligned}$$

Here we have also employed the following property of the Gamma-function

$$\frac{1}{\Gamma(n+s\alpha+\mu)} \leqslant \frac{c}{\Gamma(n+\mu)},$$

where $\mu > 0$, c depends on α and μ and is independent of s and n.

We consider the series

$$\sum_{k=1}^{\infty}\left(t^\alpha \frac{M_6}{k^2} E_1\left[-\left(a_k t + b_k t^{\alpha+1}\right); \alpha+1\right] + t^{\alpha-1} \frac{M_7}{k^2} E_1\left[-\left(a_k t + b_k t^{\alpha+1}\right); \alpha\right]\right). \qquad (18)$$

Employing the second estimate in Lemma 1, we obtain:

$$|u_k| \leqslant \frac{M_6 M_2 t^\alpha}{k^2\left[1 + |a_k t + b_k t^{\alpha+1}|\right]} + \frac{M_7 M_2 t^{\alpha-1}}{k^2\left[1 + |a_k t + b_k t^{\alpha+1}|\right]},$$

and this implies the uniform convergence of series (18). The convergence of the majorizing series having the order $\frac{1}{k^4}$ implies the uniform convergence of series (18) and therefore, of series (16) as $t \geqslant t_0 > 0$, where t_0 is an arbitrary number.

The uniform convergence of the series

$$D_{0t}^{\alpha+1}u(x,t) \sim \sum_{k=1}^{\infty} D_{0t}^{\alpha}u_k, \qquad D_{0t}^{\alpha}u(x,t) \sim \sum_{k=1}^{\infty} D_{0t}^{\alpha}u_k,$$
$$u_{xx}(x,t) \sim \sum_{k=1}^{\infty} \frac{\partial^2 u_k}{\partial x^2}, \qquad D_{0t}^{\alpha}u_{xx}(x,t) \sim \sum_{k=1}^{\infty} D_{0t}^{\alpha}u_k$$

can be proved in the same way and this implies the possibility of term-wise differentiating of series (17) and of applying the generalized superposition principle, that is, the function $u(x,t)$ defined by series (17) solves equation (4).

Thus, the following theorem holds.

Theorem 1. *Let* $\tau \in C^3[0,1]$, $\nu \in C^2[0,1]$ *and the matching conditions hold* $\tau(0)=\tau(1)=\tau''(0)=\tau''(1)=0$, $\nu(0)=\nu(1)=0$ *hold. Then the function defined by series (16):*

$$u(x,t) = \sum_{k=1}^{\infty}\left((\nu_k + a_k\tau_k)\, t^{\alpha} \sum_{n=0}^{\infty}\sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^n\, a_k^{n-s} b_k^s t^{n+s\alpha}}{\Gamma(n+s\alpha+\alpha+1)} \right.$$
$$\left. +\tau_k t^{\alpha-1}\Gamma(\alpha) \sum_{n=0}^{\infty}\sum_{s=0}^{n} \binom{n}{s} \frac{(-1)^n\, a_k^{n-s} b_k^s t^{n+s\alpha}}{\Gamma(n+s\alpha+\alpha)} \right) \sin(\pi k x),$$

where

$$\tau_k = 2\int_0^1 \tau(x)\sin(\pi k x)\,dx, \quad \nu_k = 2\int_0^1 \nu(x)\sin(\pi k x)\,dx,$$
$$a_k = \frac{1+A(\pi k)^2}{A_1}, \qquad b_k = \frac{(\pi k)^2}{A_1}, \qquad 0<\alpha<1,$$

is a regular solution of Problem 1.

4. Uniqueness of solution for inhomogeneous Aller-Lykov equation with fractional in time derivative

Problem 2. *In domain Q, we consider a boundary value problem for an inhomogeneous equation:*

$$A_1 D_{0t}^{\alpha+1}u + D_{0t}^{\alpha}u = u_{xx} + A D_{0t}^{\alpha}u_{xx} + f(x,t), \qquad 0<x<1, \qquad 0<t\leqslant T, \tag{19}$$

subject to boundary conditions (5) and initial conditions (6).

We denote $Q_T = \{(x,t):\ 0<x<1,\ 0<t<T\}$. Assuming the existence of a regular solution to equation (19) in domain Q_T, we formulate the following theorem.

Theorem 2. *Let* $f(x,t) \in C(\bar{Q}_T)$; $\nu(x) \in C[0,1]$, $\tau(x) \in C^2[0,1]$ *everywhere on* $\bar{Q}$ *and the condition* $\tau(0)=\tau(1)=0$ *holds. Then the solution of problem (19), (5), (6) satisfies the apriori estimate*

$$\|D_{0t}^{\alpha}u\|_0^2 + \|D_{0t}^{\alpha}u_x\|_{2,Q_t}^2 + \|D_{0t}^{\alpha}u\|_{2,Q_t}^2 \leqslant M_1(t)\left(\|f\|_{2,Q_t}^2 + \|\tau''(x)\|_0^2 + \|\nu(x)\|_0^2\right), \tag{20}$$

where

$$\|u\|_0^2 = \int_0^1 u^2(x,t)dx, \qquad \|u\|_{2,Q_t}^2 = \int_0^t \|u(x,\tau)\|_0^2 d\tau.$$

Proof. Following [19], we introduce a new unknown function $v(x,t)$ by letting

$$u(x,t) = v(x,t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\tau(x).$$

The function $v(x,t)$ is the deviation of the function $u(x,t)$ from the known function $\frac{t^{\alpha-1}}{\Gamma(\alpha)}\tau(x)$. Since $D_{0t}^{\alpha+1}t^{\alpha-1} = 0$, $D_{0t}^{\alpha}t^{\alpha-1} = 0$ [16], we get:

$$\begin{aligned} A_1 D_{0t}^{\alpha+1}v + D_{0t}^{\alpha}v - v_{xx} - AD_{0t}^{\alpha}v_{xx} =& f(x,t) - \left(A_1 D_{0t}^{\alpha+1} + D_{0t}^{\alpha} - \frac{\partial^2}{\partial x^2} - AD_{0t}^{\alpha}\frac{\partial^2}{\partial x^2}\right)\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\tau(x)\right) \\ =& f(x,t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\tau''(x). \end{aligned}$$

The function $v(x,t)$ solves the equation

$$A_1 D_{0t}^{\alpha+1}v + D_{0t}^{\alpha}v - v_{xx} - AD_{0t}^{\alpha}v_{xx} = F(x,t), \qquad 0 < x < 1, \qquad 0 < t \leqslant T, \tag{21}$$

subject to the initial conditions

$$\begin{aligned} &\lim_{t\to 0} D_{0t}^{\alpha-1}v(x,t) = \lim_{t\to 0} D_{0t}^{\alpha-1}\left(u(x,t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\tau(x)\right) = \tau(x) - \frac{\tau(x)}{\Gamma(\alpha)}\lim_{t\to 0} D_{0t}^{\alpha-1}t^{\alpha-1} = 0, \\ &\lim_{t\to 0} D_{0t}^{\alpha}v(x,t) = \lim_{t\to 0} D_{0t}^{\alpha}\left(u(x,t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\tau(x)\right) = \nu(x) - \frac{\tau(x)}{\Gamma(\alpha)}\lim_{t\to 0} D_{0t}^{\alpha}t^{\alpha-1} = \nu(x) \end{aligned} \tag{22}$$

and boundary conditions

$$v(0,t) = v(1,t) = 0, \qquad 0 \leqslant t \leqslant T, \tag{23}$$

where

$$F(x,t) = f(x,t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\tau''(x).$$

We are going to obtain an apriori estimate in terms of the fractional Riemann-Liouville derivative. In order to do this, we calculate the scalar product of equation (21) with $D_{0t}^{\alpha}v$:

$$A_1\left(D_{0t}^{\alpha+1}v, D_{0t}^{\alpha}v\right) + \left(D_{0t}^{\alpha}v, D_{0t}^{\alpha}v\right) - \left(v_{xx}, D_{0t}^{\alpha}v\right) - A\left(D_{0t}^{\alpha}v_{xx}, D_{0t}^{\alpha}v\right) = \left(F, D_{0t}^{\alpha}v\right), \tag{24}$$

where

$$(u,v) = \int_0^1 uv\,dx, \qquad (u,u) = \|u\|_0^2.$$

We transform the terms in (24) by means of (22), (23):

$$\begin{aligned} A_1\left(D_{0t}^{\alpha+1}v, D_{0t}^{\alpha}v\right) =& A_1\int_0^1 \frac{1}{\Gamma(1-\alpha)}\frac{\partial^2}{\partial t^2}\int_0^t \frac{v(x,\tau)d\tau}{(t-\tau)^{\alpha}}\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{v(x,\tau)\,d\tau}{(t-\tau)^{\alpha}}dx \\ =& \frac{A_1}{2}\int_0^1 \frac{\partial}{\partial t}\left(D_{0t}^{\alpha}v\right)^2 dx = \frac{A_1}{2}\frac{\partial}{\partial t}\left\|D_{0t}^{\alpha}v\right\|_0^2, \end{aligned}$$

$$\left(D_{0t}^{\alpha}v, D_{0t}^{\alpha}v\right) = \left\|D_{0t}^{\alpha}v\right\|_0^2,$$

$$\begin{aligned} \left(v_{xx}, D_{0t}^{\alpha}v\right) =& \frac{1}{\Gamma(1-\alpha)}\int_0^1 v_{xx}(x,t)\frac{\partial}{\partial t}\int_0^t \frac{v(x,\tau)d\tau}{(t-\tau)^{\alpha}}dx \\ =& \frac{1}{\Gamma(1-\alpha)}\left\{ v_x(x,t)\frac{\partial}{\partial t}\int_0^t \frac{v(x,\tau)d\tau}{(t-\tau)^{\alpha}}\Bigg|_0^1 - \int_0^1 v_x(x,t)\frac{\partial}{\partial t}\int_0^t \frac{v_x(x,\tau)d\tau}{(t-\tau)^{\alpha}}dx \right\}. \end{aligned}$$

By (23),

$$v_x(x,t)\frac{\partial}{\partial t}\int\limits_0^t \frac{v(x,\tau)d\tau}{(t-\tau)^\alpha}\Bigg|_0^1 = 0,$$

then

$$(v_{xx}, D_{0t}^\alpha v) = -\frac{1}{\Gamma(1-\alpha)}\int\limits_0^1 v_x(x,t)\frac{\partial}{\partial t}\int\limits_0^t \frac{v_x(x,\tau)d\tau}{(t-\tau)^\alpha}dx.$$

In the same way we obtain:

$$\begin{aligned}
A\left(D_{ot}^\alpha v_{xx}, D_{ot}^\alpha v\right) =& A\int\limits_0^1 \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int\limits_0^t \frac{v_{xx}(x,\tau)d\tau}{(t-\tau)^\alpha}\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int\limits_0^t \frac{v(x,\tau)d\tau}{(t-\tau)^\alpha}dx \\
=& A\frac{1}{\Gamma^2(1-\alpha)}\int\limits_0^1 \frac{\partial}{\partial t}\int\limits_0^t \frac{v_{xx}(x,\tau)d\tau}{(t-\tau)^\alpha}\frac{\partial}{\partial t}\int\limits_0^t \frac{v(x,\tau)d\tau}{(t-\tau)^\alpha}dx \\
=& A\frac{1}{\Gamma^2(1-\alpha)}\left\{\frac{\partial}{\partial t}\int\limits_0^t \frac{v_x(x,\tau)d\tau}{(t-\tau)^\alpha}\frac{\partial}{\partial t}\int\limits_0^t \frac{v(x,\tau)d\tau}{(t-\tau)^\alpha}\Bigg|_0^1\right. \\
&\left. - A\int\limits_0^1 \frac{\partial}{\partial t}\int\limits_0^t \frac{v_x(x,\tau)d\tau}{(t-\tau)^\alpha}\frac{\partial}{\partial t}\int\limits_0^t \frac{v_x(x,\tau)d\tau}{(t-\tau)^\alpha}dx\right\} \\
=& - A\int\limits_0^1 \left(\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int\limits_0^t \frac{v_x(x,\tau)d\tau}{(t-\tau)^\alpha}\right)^2 dx = -A\left\|D_{0t}^\alpha v_x\right\|_0^2.
\end{aligned}$$

We employ the Cauchy-Schwarz inequality and ε-inequality [20] with arbitrary $\varepsilon > 0$:

$$(F, D_{0t}^\alpha v) \leqslant \frac{1}{4\varepsilon}\left\|F\right\|_0^2 + \varepsilon\left\|D_{0t}^\alpha v\right\|_0^2.$$

In view of the obtained inequalities, by (24) we get:

$$\begin{aligned}
&\frac{A_1}{2}\frac{\partial}{\partial t}\left\|D_{0t}^\alpha v\right\|_0^2 + \left\|D_{0t}^\alpha v\right\|_0^2 + \frac{1}{\Gamma(1-\alpha)}\int\limits_0^1 v_x(x,t)\frac{\partial}{\partial t}\int\limits_0^t \frac{v_x(x,\tau)d\tau}{(t-\tau)^\alpha}dx + A\left\|D_{0t}^\alpha v_x\right\|_0^2 \\
&\qquad\leqslant \frac{1}{4\varepsilon}\left\|F\right\|_0^2 + \varepsilon\left\|D_{0t}^\alpha v\right\|_0^2.
\end{aligned} \tag{25}$$

We integrate (25) over τ from 0 to t:

$$\begin{aligned}
&\frac{A_1}{2}\left\|D_{0t}^\alpha v\right\|_0^2 + \int\limits_0^t \left\|D_{0t}^\alpha v(x,\tau)\right\|_0^2 d\tau + \frac{1}{\Gamma(1-\alpha)}\int\limits_0^t d\tau\int\limits_0^1 v_x(x,\tau)\frac{\partial}{\partial\tau}\int\limits_0^\tau \frac{v_x\left(x,\tau_1\right)d\tau_1}{\left(\tau-\tau_1\right)^\alpha}dx \\
&\quad + A\int\limits_0^t \left\|D_{0t}^\alpha v_x(x,\tau)\right\|_0^2 d\tau \leqslant \frac{1}{4\varepsilon}\left\|F\right\|_{2,Q_t}^2 + \varepsilon\int\limits_0^t \left\|D_{0t}^\alpha v(x,\tau)\right\|_0^2 d\tau + \frac{A_1}{2}\left\|D_{0t}^\alpha v(x,0)\right\|_0^2.
\end{aligned}$$

Since the integral in the left hand side of this inequality is non-negative [2], we find:

$$A_1\left\|D_{0t}^\alpha v\right\|_0^2 + 2A\left\|D_{0t}^\alpha v_x\right\|_{2,Q_t}^2 + 2\varepsilon_1\left\|D_{0t}^\alpha v\right\|_{2,Q_t}^2 \leqslant \frac{1}{2\varepsilon}\left\|f\right\|_{2,Q_t}^2 + A_1\left\|\nu(x)\right\|_0^2,$$

where

$$\|D_{0t}^{\alpha}v\|_{2,Q_t}^2 = \int_0^t \|D_{0t}^{\alpha}v(x,t)\|_0^2\, d\tau, \qquad \varepsilon_1 = 1-\varepsilon.$$

This implies the estimate

$$\|D_{0t}^{\alpha}v\|_0^2 + \|D_{0t}^{\alpha}v_x\|_{2,Q_t}^2 + \|D_{0t}^{\alpha}v\|_{2,Q_t}^2 \leqslant M(t)\left(\|F\|_{2,Q_t}^2 + \|\nu(x)\|_0^2\right)$$

or, returning back to $u\,(x,t)$, we obtain (20). The proof is complete. □

Inequality (20) implies the uniqueness of problem (19), (5), (6). Indeed, let u be a solution of homogeneous problem, that is, $f = \tau = \nu = 0$, then by (20) we have

$$\|D_{0t}^{\alpha}u\|_0^2 + \|D_{0t}^{\alpha}u_x\|_{2,Q_t}^2 + \|D_{0t}^{\alpha}u\|_{2,Q_t}^2 = 0.$$

Applying the generalized Newton-Leibnitz formula [16],

$$D_{0t}^{-\alpha}D_{0t}^{\alpha}u(x,t) = u(x,t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\lim_{t\to 0} D_{0t}^{\alpha-1}u(x,t),$$

we obtain

$$u(x,t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\lim_{t\to 0} D_{0t}^{\alpha-1}u(x,t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\tau(x) = 0 \qquad \text{in} \qquad Q_T.$$

Since T is arbitrary, we conclude that $u(x,t) \equiv 0$ at all points $(x,t) \in Q$.

BIBLIOGRAPHY

1. A.F. Chudnovsky. *Thermal physics of soils.* Nauka, Moscow (1976). (in Russian).
2. A.M. Nakhushev. *Fractional calculus and its application.* Fizmatgiz, Moscow (2003). (in Russian).
3. M.A. Keferov. *On one boundary value problem for a modified moisture transfer equation with a fractional derivative in time* // Dokl. Adyg. Mezhd. Akad. Nauk. **4**:1, 12–14 (1999). (in Russian).
4. M.A. Keferov, S.Kh. Gekkieva. *Boundary value problems for a modified for a modified moisture transfer equation with a fractional derivative in time in a multi-dimensional domain* // Nauchnye Vedomosti Belgorod. Gosud. Univ. Ser. Matem. Fiz. **41**:2 (220), 17–23 (2015). (in Russian).
5. V.Ya. Kulik. *Study of motion of soil water from the point of view of invariance of relatively continuous transformations groups* // in "Study of processes of energy and matter transfer in the system soil-plan-air". Nauka, Leningrad (1972). (in Russian).
6. M.M. Lafisheva, M.A. Kerefov, R.V. Dyshekova. *Difference schemes for the Aller-Lykov moisture transfer equations with a nonlocal condition* // Vladikavkaz. Matem. Zhurn. **19**:1, 50–58 (2017). (in Russian).
7. S.Kh. Gekkieva. *Boundary value problem for Aller-Lykov moisture transfer equation with a fractional derivative in time* // Proc. of All-Russian conference with international participation "Stable development: problems, concepts, models". Nalchik, 99–102 (2017). (in Russian).
8. S.Kh. Gekkieva. *Boundary value problem for a generalized transfer equation with a fractional derivative in time* // Dokl. Adyg. Mezhd. Akad. Nauk. **1**:1, 17–18 (1994). (in Russian).
9. O.P. Agrawal. *Solution for a fractional diffusion-wave equation defined in a bounded domain* // Nonl. Dynam. **29**:1–4, 145–155 (2002).
10. V.A. Nakhusheva. *Differential equations of mathematical models of nonlocal processes.* Nauka, Moscow (2006). (in Russian).
11. O.Kh. Masaeva. *Uniqueness of solutions to Dirichlet problems for generalized Lavrent'ev – Bitsadze equations with a fractional derivative* // Electron. J. Diff. Equat. **2017**, 1–8 (2017).
12. V.Kh. Shogenov, S.K. Kumykova, M.Kh. Shkhanukov-Lafishev. *The generalized transport equation and fractional derivatives* // Dopov. Akad. Nauk Ukr. 12, 47–54 (1997). (in Russian).
13. M.A. Kerefov. *Boundary value problems for modified moisture transfer equation with a fractional derivative in time.* PhD thesis. Nalchik (2000). (in Russian).
14. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. *Theory and applications of fractional differential equations.* North-Holland Math. Stud. **204**. Elsevier, Amsterdam, (2006).

15. A.V. Pskhu. *Initial-value problem for a linear ordinary differential equation of noninteger order* // Matem. Sborn. **202**:4, 111–122 (2011). [Sb. Math., **202**:4, 571–582 (2011).]
16. A.V. Pskhu. *Partial differential equations of fractional order*. Nauka, Moscow (2005). (in Russian).
17. V.I. Smirnov. *Course of higher mathematics. V. 2*. BKhV-St.-Petersburg, St.-Petersburg (2008). (in Russian).
18. M.M. Dzhrbashyan. *Integral transformations and representations of functions in complex domain.* Nauka, Moscow (1966). (in Russian).
19. M.A. Kerefov, S.Kh. Gekkieva. *Dirichlet boundary value problem for inhomogeneous nonlocal wave equation* // Vestnik Buryat. Gosud. Univ. Matem. Inform. 4, 76–86 (2016). (in Russian).
20. A.A. Samarski. *The theory of difference schemes.* Nauka, Moscow, 1989. [Marcel Dekker, New York (2001).]

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