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ON ZEROS OF POLYNOMIAL

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Abstract. For a given polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0$$

with real or complex coefficients, the Cauchy bound

$$|z| < 1 + A, \qquad A = \max_{0\leqslant j\leqslant n-1} |a_j|$$

does not reflect the fact that for A tending to zero, all the zeros of $P(z)$ approach the origin $z = 0$. Moreover, Guggenheimer (1964) generalized the Cauchy bound by using a lacunary type polynomial

$$p(z) = z^n + a_{n-p}z^{n-p} + a_{n-p-1}z^{n-p-1} + \cdots + a_1 z + a_0, \qquad 0 < p < n.$$

In this paper we obtain new results related with above facts. Our first result is the best possible. For the case as A tends to zero, it reflects the fact that all the zeros of P(z) approach the origin $z = 0$; it also sharpens the result obtained by Guggenheimer. The rest of the related results concern zero-free bounds giving some important corollaries. In many cases the new bounds are much better than other well-known bounds.

Keywords: zeroes, region, Cauchy bound, Lacunary type polynomials.

Mathematics Subject Classification: 30C15; 30C10; 26C10

1. Introduction and statement of results

Let

$$P(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0$$

be a polynomial of degree n. A classical result by Cauchy [6, Ch. VI, Sect. 27, Thm. 27.2] concerning the bounds for the moduli of the zeros of a polynomial can be stated as follows.

Theorem A. *All the zeros of $P(z)$ lie in the disc*

$$|z| < 1 + A, \tag{1.1}$$

where

$$A = \max_{0\leqslant j\leqslant n-1} |a_j|.$$

Joyal, Labelle and Rahman [1] improved Cauchy bound (1.1) and proved the following theorem.

Theorem B. *If $B = \max_{0\leqslant k\leqslant n-2} |a_k|$, then all the zeros of $P(z)$ lie in the disc*

$$|z| \leqslant \frac{1}{2}\left[1 + |a_{n-1}| + \sqrt{(1 - |a_{n-1}|)^2 + 4B}\right]$$

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Datt and Govil [3] improved Cauchy bound (1.1) and obtained the following result.

Theorem C. *All the zeros of* $P(z)$ *lie in the ring-shaped region*

$$\frac{|a_0|}{2(1+A)^{n-1}(1+nA)} \leqslant |z| \leqslant 1 + A\left(1 - \frac{1}{(1+A)^n}\right).$$

One more improvement of Cauchy bound (1.1) was made by Jain [8], who proved the following statement.

Theorem D. *All the zeros of* $P(z)$ *with* $a_{n-1} = a_{n-2} = 0$ *lie in the disc*

$$|z| < 2^{\frac{2}{9}}(1+B)^{\frac{1}{3}},$$

except for $B > 1, |a_j| = B$ *for some* j, $0 \leqslant j \leqslant n-3$ *and* $|a_i| \leqslant \alpha = 2^{\frac{1}{3}} - 1$ *for all* $i \neq j$, $i \in \{0, 1, 2, \ldots, n-3\}$. *In the latter case, all the zeros of* $P(z)$ *lie in the disc*

$$|z| < (1+B)^{\frac{1}{3}}.$$

Guggenheimer [5] generalized the Cauchy bound (1.1) by using a class of lacunary type polynomial

$$p(z) = z^n + a_{n-p}z^{n-p} + a_{n-p-1}z^{n-p-1} + \cdots + a_1 z + a_0, \qquad 0 < p < n,$$

and proved the following theorem.

Theorem E. *All the zeros of* $p(z)$ *lie in the disc*

$$|z| < \delta,$$

where $\delta > 1$ *is the only positive root of the equation*

$$t^p - t^{p-1} = Q^n$$

and

$$Q^n = \max_{0 \leqslant k \leqslant n-p} |a_k|.$$

In this paper, we obtain three bounds of Cauchy type. The bound in Theorem 1.1 is best possible and sharpen of the Theorem E. Also, in many cases, the bound in Theorem 1.1 is better than some other known bounds. The bounds in Theorem 1.2 and Theorem 1.3 are zero free. More precisely, we prove

Theorem 1.1. *All the zeros of* $p(z)$ *lie in the disc*

$$|z| \leqslant \delta_0 Q^{\frac{n}{p}},$$

where $\delta_0 \in (1, 2)$ *provided* $Q \geqslant 1$, *otherwise,* $\delta_0 \in (1, \infty)$ *is the greatest positive root of the equation*

$$q^{n+1}\left(Q^{\frac{n}{p}}\right)^{n+1} - q^n\left(Q^{\frac{n}{p}}\right)^n - q^{n-p+1}\left(Q^{\frac{n}{p}}\right)^{n+1} + Q^n = 0.$$

Remark 1.1. *As* $Q \to 0$, *all zeroes of* $p(z)$ *approach the origin* $z = 0$.

Remark 1.2. *The bound* $\delta_0 Q^{\frac{n}{p}}$ *in Theorem 1.1 is the best possible and it is attained at the polynomial*

$$p(z) = z^n - Q^n\left(z^{n-p} + z^{n-p-1} + \cdots + z + 1\right).$$

Remark 1.3. *Theorem 1.1 is an improvement of Theorem E, which can be seen by observing that*

$$\begin{aligned}&\left(\delta Q^{-\frac{n}{p}}\right)^{n+1}\left(Q^{\frac{n}{p}}\right)^{n+1} - \left(\delta Q^{-\frac{n}{p}}\right)^n\left(Q^{\frac{n}{p}}\right)^n - \left(\delta Q^{-\frac{n}{p}}\right)^{n-p+1}\left(Q^{\frac{n}{p}}\right)^{n+1} + Q^n \\ &= \delta^{n+1} - \delta^n - \delta^{n-p+1}\left(Q^{\frac{n}{p}}\right)^p + Q^n = \delta^{n-p+1}\left(\delta^p - \delta^{p-1} - Q^n\right) + Q^n = Q^n > 0,\end{aligned}$$

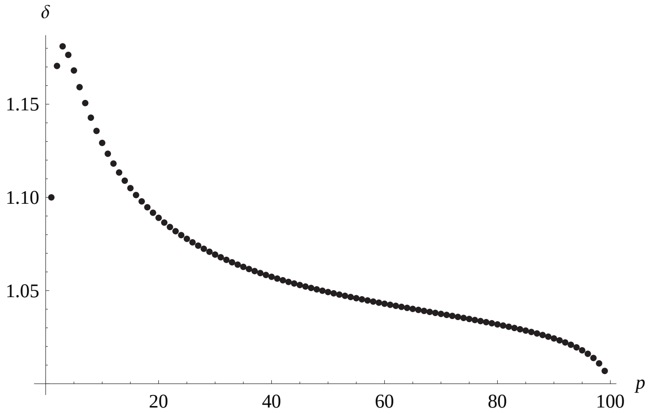

FIGURE 1. Variation of δ, as p varies from 1 to 99.

which implies

$$\delta_0 < \delta Q^{-\frac{n}{p}}, \quad i.e. \quad \delta_0 Q^{\frac{n}{p}} < \delta.$$

Remark 1.4. *In many cases, the Theorem 1.1 gives a better bound than those in previous results. In order to demonstrate this, we consider the polynomial*

$$p(z) = z^5 + a_2 z^2 + a_1 z + a_0,$$

with $|a_0| = 8, |a_1| = 2, |a_2| = 6$. *Here n=5, p=3 and* $Q^n = 8$. *By Theorem 1.1, we obtain* $\delta_0 = 1.17174$ *and all the zeros of* $p(z)$ *lie in* $|z| \leqslant 2.34349$, *whereas all the zeros of* $p(z)$ *lie in the regions*

$$\begin{aligned} |z| &< 9, && \text{by Theorem A} \\ |z| &< 3.37, && \text{by Theorem B} \\ |z| &< 8.989, && \text{by Theorem C} \\ |z| &< 2.426, && \text{by Theorem D} \\ |z| &< 2.3948, && \text{by Theorem E} \\ |z| &< 8.999, && \text{by [4, Thm. 1]} \\ |z| &< 6.75, && \text{by [9, Thm. 1].} \end{aligned}$$

Example 1. *Theorem 1.1 gives an idea for finding the bound of zeros for a class of lacunary type of polynomials*

$$\Gamma_n^p(Q^n) = \left\{ p(z) = z^n + \sum_{j=0}^{n-p} a_j z^j : \max_{0 \leqslant k \leqslant n-p} |a_k| \leqslant Q^n \right\}, \qquad 0 < p < n.$$

By Theorem 1.1, all the zeros of each polynomial of the class $\Gamma_n^p(Q^n)$ *always lie in the region* $|z| \leqslant \delta_0 Q^{\frac{n}{p}}$. *It is very difficult to find the value of* δ_0 *for each non-negative real value of* Q^n, *as* n *is a fixed natural number and* p *varies from* $0 < p < n$. *Here we consider an example by choosing* $Q^n = 10$ *and* $n = 100$ *and we draw a picture for the variation of* δ, *when* p *varies from* 1 *to* 99, *see Figure 1. Once we determine the value of* δ_0 *for a particular value of* $p\,(0 < p < 100)$, *we can easily obtain the bound of each polynomial from the class of polynomials* $\Gamma_{100}^p(10)$.

In particular, for $p = 17$, *the value of* δ_0 *is 1.09786. Using Theorem 1.1, we see that all zeros of each polynomial in the class* $\Gamma_{100}^{17}(10)$ *lie in the region*

$$|z| \leqslant 1.25709.$$

Theorem 1.2. *All the zeros of $p(z)$ with $|a_0| \neq 0$ never vanish in the region*

$$|z| \leqslant \frac{1}{t_0},$$

where $t_0\,(> 1)$ is the greatest positive root of the equation

$$|a_0|\, t^{n+1} - (|a_0| + Q^n)\, t^n - t + 1 = 0.$$

With the help of Theorem 1.1 and Theorem 1.2, we can easily obtain the following Corollary.

Corollary 1.1. *All zeroes of the polynomial $p(z)$ with $|a_0| \neq 0$ lie in the ring-shaped region*

$$\frac{1}{t_0} < |z| \leqslant \delta_0 Q^{\frac{n}{p}},$$

where δ_0 and t_0 are the greatest positive roots of the equations

$$q^{n+1}\left(Q^{\frac{n}{p}}\right)^{n+1} - q^n\left(Q^{\frac{n}{p}}\right)^n - q^{n-p+1}\left(Q^{\frac{n}{p}}\right)^{n+1} + Q^n = 0$$

and

$$|a_0|\, t^{n+1} - (|a_0| + Q^n)\, t^n - t + 1 = 0,$$

respectively.

Theorem 1.3. *All the zeros of $p(z)$ with $|a_0| \neq 0$ never vanish in the disc*

$$|z| < \frac{1}{1 + \frac{Q}{|a_0|}}$$

provided

$$Q < \min\left\{1, 2^p\, |a_0|^2\right\}.$$

One can easily obtain the following corollaries by using Theorem 1.3.

Corollary 1.2. *If*

$$\left|a_{n-3} - a_{n-1}a_{n-2} + a_{n-1}\left(a_{n-1}^2 - a_{n-2}\right)\right| \neq 0, \left|a_0\left(a_{n-1}^2 - a_{n-2}\right)\right| \neq 0$$

and

$$\Gamma^{n+2} = \max_{3 \leqslant p \leqslant n+2} \left|a_{n-p} - a_{n-1}a_{n-p+1} + a_{n-p+2}\left(a_{n-1}^2 - a_{n-2}\right)\right|$$

with $a_{-1} = a_{-2} = 0$, then the polynomial $P(z)$ of degree $n > 3$ never vanish in the region

$$|z| < \frac{1}{1 + \frac{\Gamma}{\left|a_0\left(a_{n-1}^2 - a_{n-2}\right)\right|}}$$

provided

$$\Gamma < \min\left\{1, 2^3\left|a_0\left(a_{n-1}^2 - a_{n-2}\right)\right|\right\}.$$

The Corollary 1.2 can be easily obtained by using Theorem 1.3 on the polynomial

$$R(z) = \left(z^2 - a_{n-1}z + a_{n-1}^2 - a_{n-2}\right) P(z).$$

Corollary 1.3. *The polynomial $P(z)$ of degree $n > 4$ never vanish in the region*

$$|z| < \frac{1}{1 + \frac{\Upsilon}{\left|a_0\left(2a_{n-1}a_{n-2} - a_{n-3} - a_{n-1}^3\right)\right|}}$$

provided

$$\Upsilon < \min\left\{1, 2^4\left|a_0\left(2a_{n-1}a_{n-2} - a_{n-3} - a_{n-1}^3\right)\right|\right\},$$

where

$$\left|a_0\left(2a_{n-1}a_{n-2} - a_{n-3} - a_{n-1}^3\right)\right| \neq 0, \qquad |\zeta| \neq 0, \qquad \Upsilon^{n+3} = \max_{4 \leqslant p \leqslant n+3} |\eta_p|,$$

$$\zeta = a_{n-4} - a_{n-1}a_{n-3} + a_{n-2}\left(a_{n-1}^2 - a_{n-2}\right) + a_{n-1}\left(2a_{n-1}a_{n-2} - a_{n-3} - a_{n-1}^3\right),$$
$$\eta_p = a_{n-p} - a_{n-1}a_{n-p+1} + a_{n-p+2}\left(a_{n-1}^2 - a_{n-2}\right) + a_{n-p+3}\left(2a_{n-1}a_{n-2} - a_{n-3} - a_{n-1}^3\right)$$

with $a_{-1} = a_{-2} = a_{-3} = 0$.

The Corollary 1.3 obtained by using Theorem 1.3 on the polynomial
$$R(z) = \left(z^3 - a_{n-1}z^2 + \left(a_{n-1}^2 - a_{n-2}\right)z + 2a_{n-1}a_{n-2} - a_{n-3} - a_{n-1}^3\right)P(z).$$

2. Proof of theorems

Proof of Theorem 1.1. As $|z| > 1$, we have
$$|p(z)| \geqslant |z|^n - Q^n\frac{|z|^{n-p+1} - 1}{|z| - 1} = \frac{1}{|z| - 1}\left\{|z|^{n+1} - |z|^n - Q^n|z|^{n-p+1} + Q^n\right\}.$$
We introduce a function
$$f(t) = t^{n+1} - t^n - Q^n t^{n-p+1} + Q^n.$$
For $q > 0$ we get
$$\begin{aligned} f\left(qQ^{\frac{n}{p}}\right) &= \left(qQ^{\frac{n}{p}}\right)^{n+1} - \left(qQ^{\frac{n}{p}}\right)^n - Q^n\left(qQ^{\frac{n}{p}}\right)^{n-p+1} + Q^n \\ &= q^{n+1}\left(Q^{\frac{n}{p}}\right)^{n+1} - q^n\left(Q^{\frac{n}{p}}\right)^n - q^{n-p+1}\left(Q^{\frac{n}{p}}\right)^{n+1} + Q^n \\ &= q^n\left(Q^{\frac{n}{p}}\right)^{n+1}\left(q - \frac{1}{Q^{\frac{n}{p}}} - \frac{1}{q^{p-1}}\right) + Q^n. \end{aligned} \tag{2.1}$$
Denote
$$g(q) = f\left(qQ^{\frac{n}{p}}\right) = q^{n+1}\left(Q^{\frac{n}{p}}\right)^{n+1} - q^n\left(Q^{\frac{n}{p}}\right)^n - q^{n-p+1}\left(Q^{\frac{n}{p}}\right)^{n+1} + Q^n.$$
We have
$$g(0) = \left(Q^{\frac{n}{p}}\right)^p$$
and
$$g(1) = \left(Q^{\frac{n}{p}}\right)^p - \left(Q^{\frac{n}{p}}\right)^n,$$
which is negative as $Q > 1$ and is positive as $0 < Q < 1$. In particular, as $Q = 1$, the value $g(q)$ can be written as
$$g(q) = (q-1)\,r(q),$$
where
$$r(q) = q^n - \left(1 + q + \cdots + q^{n-p}\right)$$
with
$$r(1) = -(n-p) < 0, \qquad r(2) = 2^n - 2^{n-p+1} + 1 > 0,$$
which shows that $g(q) = 0$ has two positive roots, one of which is 1 and other, say δ_0 in $(1,2)$. Therefore,
$$g(q) > 0$$
if $q > \delta_0$. Hence,
$$f(t) > 0$$
as $t > \delta_0 Q^{\frac{n}{p}}$ and $Q = 1$. This implies the desired result.

As $Q > 1$, the equation $g(q) = 0$ has two positive roots, one of which is $Q^{-\frac{n}{p}} \in (0,1)$ and the other, say δ_0, belongs to $(1,2)$ by (2.1). In this case,
$$g(q) > 0$$
if $q > \delta_0$, which implies that
$$f(t) > 0$$

if $t > \delta_0 Q^{\frac{n}{p}}$. This leads us to the desired result.

As $0 < Q < 1$, the equation $g(q) = 0$ has two positive roots, one of which is $Q^{-\frac{n}{p}} > 1$ and the other, say t_0, lies in $(1, \infty)$. Let δ_0 be the greatest positive root of $g(q) = 0$. Then

$$g(q) > 0$$

if $q > \delta_0$. Thus,

$$f(t) > 0$$

if $t > \delta_0 Q^{\frac{n}{p}}$ and this leads us to the desired result. □

Proof of Theorem 1.2. Consider

$$R(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-p} z^p + 1.$$

As $|z| > 1$, we have

$$\begin{aligned}|R(z)| \geqslant& |a_0| |z|^n - Q^n |z|^p \frac{|z|^{n-p} - 1}{|z| - 1} - 1 > |a_0| |z|^n - \frac{Q^n |z|^n}{|z| - 1} - 1 \\ =& \frac{1}{|z| - 1} \left\{ |a_0| |z|^{n+1} - (|a_0| + Q^n) |z|^n - |z| + 1 \right\}.\end{aligned}$$

The equation

$$|a_0| t^{n+1} - (|a_0| + Q^n) t^n - t + 1 = 0$$

obviously has exactly two positive roots, one lies in $(0, 1)$ and the other is in $(1, \infty)$. Let $t_0 > 1$ be the greatest positive root of the above equation. Then

$$|a_0| t^{n+1} - (|a_0| + Q^n) t^n - t + 1 \geqslant 0 \qquad \text{for all} \quad t \geqslant t_0.$$

So, $|R(z)| > 0$ if $|z| \geqslant t_0$ and this proves the desired result. □

Proof of Theorem 1.3. Consider

$$R(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-p} z^p + 1$$

As $|z| > 1$, we get

$$\begin{aligned}|R(z)| \geqslant& |a_0| |z|^n - Q^n |z|^p \frac{|z|^{n-p} - 1}{|z| - 1} - 1 \\ =& \frac{1}{|z| - 1} \left\{ |a_0| |z|^{n+1} - (|a_0| + Q^n) |z|^n - Q^n |z|^p - |z| + 1 \right\}.\end{aligned}$$

We consider the function

$$g(t) = \frac{f(t)}{t - 1},$$

where

$$f(t) = |a_0| t^{n+1} - (|a_0| + Q^n) t^n - Q^n t^p - t + 1.$$

We obtain

$$\begin{aligned}g\left(1 + \frac{Q}{|a_0|}\right) =& |a_0| \left(1 + \frac{Q}{|a_0|}\right)^n - Q^{n-1} |a_0| \left\{ \left(1 + \frac{Q}{|a_0|}\right)^n - \left(1 + \frac{Q}{|a_0|}\right)^p \right\} - 1 \\ =& |a_0| \left(1 + \frac{Q}{|a_0|}\right)^n - Q^{n-1} |a_0| \left(1 + \frac{Q}{|a_0|}\right)^n + Q^{n-1} |a_0| \left(1 + \frac{Q}{|a_0|}\right)^p - 1 \\ >& |a_0| \left(1 + \frac{Q}{|a_0|}\right)^n - Q^{n-1} |a_0| \left(1 + \frac{Q}{|a_0|}\right)^n + \frac{|a_0|^2}{Q} \left(1 + \frac{|a_0|}{|a_0|}\right)^p - 1 \\ =& |a_0| \left(1 + \frac{Q}{|a_0|}\right)^n \left(1 - Q^{n-1}\right) + \left(\frac{2^p |a_0|^2}{Q} - 1\right) \geqslant 0\end{aligned}$$

provided

$$Q \leqslant \min\left\{1, 2^p\left|a_0\right|^2\right\}.$$

Clearly, $f(t) = 0$ has exactly two positive roots with

$$f(0) > 0, \qquad f(1) < 0, \qquad f(\infty) > 0.$$

In this case,

$$f\left(1 + \frac{Q}{|a_0|}\right) > 0 \quad \text{if} \quad Q \leqslant \min\left\{1, 2^p\left|a_0\right|^2\right\},$$

which implies that

$$g(t) > 0 \qquad \text{for all} \qquad t \geqslant 1 + \frac{Q}{|a_0|}$$

provided $Q \leqslant \min\left\{1, 2^p\left|a_0\right|^2\right\}$. Therefore,

$$|R(z)| > 0 \quad \text{for all} \quad t \geqslant 1 + \frac{Q}{|a_0|} \quad \text{if} \quad Q \leqslant \min\left\{1, 2^p\left|a_0\right|^2\right\}$$

and this completes the proof. □

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