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ON UNIQUENESS OF WEAK SOLUTION TO MIXED PROBLEM FOR INTEGRO-DIFFERENTIAL AGGREGATION EQUATION

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Abstract. In a well-known paper by A. Bertozzi, D. Slepcev (2010), there was established the existence and uniqueness of solution to a mixed problem for the aggregation equation

$$u_t - \Delta A(x,u) + \operatorname{div}(u\nabla K * u) = 0$$

describing the evolution of a colony of bacteria in a bounded convex domain Ω. In this paper we prove the existence and uniqueness of the solution to a mixed problem for a more general equation

$$\beta(x,u)_t = \operatorname{div}(\nabla A(x,u) - \beta(x,u)G(u)) + f(x,u).$$

The term $f(x,u)$ in the equation models the processes of "birth-destruction" of bacteria. The class of integral operators $G(v)$ is wide enough and contains, in particular, the convolution operators $\nabla K * u$. The vector kernel $g(x,y)$ of the operator $G(u)$ can have singularities.

Proof of the uniqueness of the solution in the work by A. Bertozzi, D. Slepcev was based on the conservation of the mass $\int_\Omega u(x,t)dx = const$ of bacteria and employed the convexity of Ω and the properties of the convolution operator. The presence of the "inhomogeneity" $f(x,u)$ violates the mass conservation. The proof of uniqueness proposed in the paper is suitable for a nonuniform equation and does not use the convexity of Ω.

Keywords: aggregation equation, integro-differential equation, global solution, uniqueness of solution.

Mathematics Subject Classification: 35K20, 35K55, 35K65

1. Introduction

In the last decade there appeared many works devoted to studying the aggregation equation

$$u_t = \operatorname{div}(\nabla A(x,u) - u\nabla K * u), \qquad K * u = \int\limits_{\mathbb{R}^n} K(x-y)u(y,t)dy, \tag{1}$$

where the kernel K can have singularity of Newtonian potential kind, see [1] and the references therein. A detailed survey of the results of such studies would have been too lengthy and this is why we mention only pioneering works devoted to equation (1).

In work [2], a model of bacteria chemotaxis was suggested as the system of equations

$$\begin{aligned} &u_t = \operatorname{div}(\nabla u - u\nabla v), \quad x \in \mathbb{R}^n, \quad t > 0, \\ &-\Delta v = u, \quad x \in \mathbb{R}^n, \quad t > 0. \end{aligned} \tag{2}$$

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It is obvious that for a bounded function v, system (2) is reduced to a single equation of form (1).

In work [3], system (2) was called Smolukhovsky-Poisson system and was used for studying gravitational collapse of a cloud of self-gravitating particles. In work [4], the same system was called Chavanis-Sommeria-Robert model with a reference to work [5].

Let Ω be a bounded domain in the space $\mathbb{R}^n$, $n \geqslant 2$, with the boundary in the class C^1. In the cylindrical domain $D^T = \Omega \times (0,T)$ we consider the equation

$$\beta(x,u)_t = \operatorname{div}(\nabla A(x,u) - \beta(x,u)G(u)) + f(x,u) \tag{3}$$

with initial and boundary conditions

$$u(x,0) = u_0(x), \qquad c\,m\,u_0(x) \geqslant 0, \quad x \in \Omega, \tag{4}$$

$$(\nabla A(x,u) - \beta(x,u)G(u)) \cdot \nu = 0 \qquad \text{on} \quad \partial\Omega \times (0,T), \tag{5}$$

where ν is the outward unit normal. The integral operator

$$G(v) = (G_1(v), G_2(v), \dots, G_n(v))$$

is defined by the formulae

$$G_i(v) = \int\limits_{\Omega} g_i(x,y) b(v(y)) dy.$$

The aim of the work is to prove the existence and uniqueness of the weak solution to problem (3)–(5) in the cylinder D^T with the height determined by data of the problem.

In work [6], the existence and uniqueness of the weak solution to problem (1),(4) with the boundary condition

$$(\nabla A(x,u) - u \nabla K * u) \cdot \nu = 0 \qquad \text{on} \quad \partial\Omega \times (0,T)$$

was proved in the case, when the kernel K belonged to $C^2(\mathbb{R}^n)$. In earlier work [7] there were established the existence and uniqueness of the solution to the same problem in the case $A = A(u) \in C^1[0,\infty]$ as the domain Ω was convex. But the proof of the existence in this work contained a gap. In work [7], there were also considered Cauchy problem and a problem with a periodic boundary condition.

In the papers the authors knows, the function $A(x,u)$ increases in u and among the power functions $A(x,u) = u^m$ only the case $m \geqslant 1$ was treated. We observe that the change $v = u^m$ in equation (1) with $A(x,u) = u^m$ leads us to equation (3) and this allows us to prove the existence of solution in the case $m \in (0,1)$, see Theorem 3 in Section 2 implied by the results in [8].

In work [8] there was prove the existence of a weak solution to the mixed problem in D^T for the equation

$$\beta(x,u)_t = \operatorname{div}(a(x,u,\nabla u) - \beta(x,u)G(u)) + f(x,u) \tag{6}$$

with the initial condition (4) and the boundary condition

$$(a(x,u,\nabla u) - \beta(x,u)G(u)) \cdot \nu = 0 \quad \text{on} \quad \partial\Omega \times (0,T). \tag{7}$$

The height T of the cylinder D^T is determined by the functions involved in the formulation of the problem. This result is discussed in details in Section 2.

The presence of a non-locality in equation (3) does not give a chance to employ the Kruzhkov's method of doubling variables applied in work [9] for proving the uniqueness of a renormalized solution to the mixed problem for the parabolic equation with a double nonlinearity.

A wider survey of works on aggregation equation can be found in work [6].

2. Main results

Suppose that $\beta(x,r)$, $f(x,r)$, $A(x,r)$ are Caratheodory functions. The function $A(x,u) \in C^1(\overline{\Omega}\times[0,\infty))$, $A(x,u)\geqslant 0$, has a positive derivative A_u,

$$A_u(x,u) > a_0 > 0, \quad x\in\Omega, \quad u>0, \tag{8}$$

$$(A(x,u)-A(x,v))(\beta(x,u)-\beta(x,v)) \geqslant a_0(u-v)^2 \quad \text{for all} \quad u,v\in\mathbb{R}_+. \tag{9}$$

We describe the conditions for the functions involved in the integral operator $G(v)$:

$$g_i(x,y)\in C^1(P), \qquad P=\{(x,y): x,y\in\overline{\Omega}, x\neq y\}.$$

We assume that for some $\lambda<n$ the inequalities hold:

$$\sum_{i=1}^n |(g_i(x,y))_{x_i}| + |g_i(x,y)| \leqslant C(1+|x-y|^{-\lambda}), \quad (x,y)\in P, \tag{10}$$

$$\sum_{i=1}^n \nu_i g_i(x,y)\leqslant 0, \quad x\in\partial\Omega, \quad y\in\Omega. \tag{11}$$

The functions $f(x,s)$, $b(s)$, $f(x,0)=0$, $b(0)=0$, satisfy the Lipschitz condition:

$$|f(x,s_1)-f(x,s_2)|\leqslant L_r|s_1-s_2|, \quad s_1,s_2\in[0,r] \quad \text{for all} \quad r>0, \tag{12}$$

$$|b(s_1)-b(s_2)|\leqslant L_k|s_1-s_2|, \quad s_1,s_2\in[0,k] \quad \text{for all} \quad k>0. \tag{13}$$

It follows from the condition $\lambda<n$ that there exists a number $\overline{q}>1$ such that $\lambda<\frac{n}{\overline{q}}$. We fix complex conjugate numbers q, $\overline{q}$ such that $\frac{1}{q}+\frac{1}{\overline{q}}=1$.

The weak solution is defined as follows.

Definition 1. *The function $u: D^T\to[0,\infty)$, $u\geqslant 0$, is called a weak solution to problem (3)–(5) if*

$$u\in L_\infty(D^T), \qquad \beta(x,u)\in L_\infty(0,T;L_q(\Omega)), \qquad A(x,u)\in L_2(0,T;H^1(\Omega))$$

and for all Lipschitz functions $\xi\in Lip(\overline{D^T})$ obeying $\xi(T)=0$ the identity holds:

$$\int\limits_{D^T}(-\beta(x,u)\xi_t+(\nabla A(x,u)-\beta(x,u)G(u))\cdot\nabla\xi-f(x,u)\xi)dxdt=\int\limits_{\Omega}u_0(x)\xi(x,0)dx. \tag{14}$$

Theorem 1. *Assume that conditions* (8)–(13) *hold and let there exists a non-negative solution to problem* (3)– (5). *Then this solution is unique.*

We shall use the following statements on the estimates of the potential type integrals (see, for instance, [10, Ch. I, Sect. 6]).

Lemma 1. *If*

$$\lambda<\frac{n}{\overline{q}}, \qquad \frac{1}{q}+\frac{1}{\overline{q}}=1, \qquad 1<q<\infty, \qquad f(x)\in L_q(\Omega),$$

then the function

$$v(x)=\int\limits_{\Omega}\frac{f(y)dy}{|x-y|^\lambda}$$

is continuous in $\mathbb{R}^n$ and satisfies the inequality

$$|v(x)|\leqslant C\|f\|_{q,\Omega}.$$

Lemma 2. *Assume that the domain Ω is bounded and $\lambda < n$, $f(x) \in L_2(\Omega)$. Then the function*

$$v(x) = \int\limits_{\Omega} \frac{f(y)dy}{|x-y|^{\lambda}}$$

is square summable and the inequality holds:

$$\|v(x)\|_{2,\Omega} \leqslant C\|f\|_{2,\Omega}.$$

Let $M_0 = \|u_0\|_{L_\infty(\Omega)}$ and $M_T > M_0$ be an arbitrary number. We are going to obtain some estimate for the integral operator $G(v)$. We consider a measurable function $v(x)$; this function obeys $|v(x)| \leqslant M_T$. It follows from Lemma 1 and conditions (10), (13) that

$$G(v) \in C^1(\mathbb{R}^n), \qquad |G(v)(x)| \leqslant C_G, \qquad |\nabla G(v)(x)| \leqslant d_G, \qquad x \in \Omega. \tag{15}$$

Let us provide a statement (see [8]) on existence of solution to problem (6), (4), (7).

In what follows, the nonlinearity exponents $p_i(x)$ satisfy the condition:

$$|p_i(x) - p_i(y)| \leqslant \frac{C}{-\ln|x-y|}, \quad i = \overline{1,n}, \tag{16}$$

as $|x-y| \leqslant \frac{1}{2}$, $x, y \in \overline{\Omega}$. In work [8], the conditions for the functions involved in equation (6) were as follows. The function $\beta(x,r)$ is odd in $r \in \mathbb{R}$ and for some M_0, M_T it satisfies the conditions

$$s\beta(x,r) \leqslant r\beta(x,s) \quad \text{as} \quad 0 < M_0 \leqslant r < s \leqslant M_T, \quad x \in \Omega; \tag{17}$$

$$\beta(x, M_T) \in L_q(\Omega), \quad \text{where} \quad q \geqslant \max_j(\overline{p}_j(x)), \quad x \in \Omega. \tag{18}$$

$$|\nabla\beta(x,r)| \leqslant N_g|\beta(x,r)|, \quad r \in [0, M_T] \quad x \in \Omega. \tag{19}$$

The function $q_1(x,r)$ is defined by the identity $f = \beta(x,r)q_1(x,r)$ and is bounded

$$|q_1(x,r)| \leqslant q_0 \quad \text{as} \quad |r| \leqslant M_T. \tag{20}$$

The functions $a_i(x,r,y)$ are continuous in $r \in \mathbb{R}, y \in \mathbb{R}^n$ and measurable in $x \in \Omega$. There exist a function $F(x) \in L_1(\Omega)$ and a continuous function $C(m)$, $m \geqslant 0$, such that

$$|a_j(x,r,y)|^{\overline{p}_j(x)} \leqslant C(m)(F(x) + \sum_{i=1}^{n} |y_i|^{p_i(x)}), \tag{21}$$

for all $r \in [-m, m]$, $y \in \mathbb{R}^n$, $x \in \Omega$.

The monotonicity and coercivity conditions are introduced as

$$(a(x,r,y) - a(x,r,z)) \cdot (y-z) \geqslant 0, \quad y \neq z; \tag{22}$$

$$a(x,r,y) \cdot y \geqslant \delta_0 \sum_{i=1}^{n} |y_i|^{p_i(x)} - F(x), \qquad \text{for all} \quad r \in \mathbb{R}, \quad y \in \mathbb{R}^n, \quad x \in \Omega. \tag{23}$$

The function $B(x,r)$ is defined by the identity

$$B(x,r) = \int\limits_0^r s d_s\beta(x,s),$$

grows in r and obeys the inequality

$$0 \leqslant B(x,r) \leqslant r\beta(x,r), \quad r \geqslant 0.$$

This is why $B(x, M_T) \in L_q(\Omega)$.

Theorem 2. *(see [8]). Assume that conditions (10), (11), (13), (16)–(23) hold and* $0 \leqslant u_0(x) \leqslant M_0$. *Then there exists* $T = T(M_0, M_T, q_0, d_G, N_g)$ *and a weak solution to problem (6), (4), (7) such that* $0 \leqslant u(x,t) \leqslant M_T$.

We do not provide here the definition of the weak solution to problem (6), (4), (7) since it is standard and for the considered in the work problem (3)–(5) it coincides with Definition 1.

The next statement follows Theorem 2.

Theorem 3. *Assume that conditions (8), (10), (11), (13), (17)–(20) hold and* $0 \leqslant u_0(x) \leqslant M_0$. *Then there exists a weak solution to problem (3)–(5) in the cylinder* D^T, *where* $T = T(M_0, M_T, q_0, d_G, N_g)$.

It is obvious that problem (3)–(5) is a particular case of problem (6), (4), (7). This is why it is sufficient to confirm that the assumptions of Theorem 3 ensures the assumptions of Theorem 3. For equation (3), the functions a_j are of the form $a_j = A_u u_{x_j} + A_{x_j}$, and this is why condition (21) holds as $p_j = 2$ thanks to the smoothness of the function $A(x,u)$. Conditions (22) and (23) are also satisfied by (8) and the boundedness of u.

As an important example, in D^T we consider the equation

$$u_t = \operatorname{div}(\nabla u^m - u\nabla K * u), \quad m \in (0,1),$$

where the kernel K satisfies the condition

$$\nabla_x^2 K(x,y) \leqslant C(1 + |x-y|^{-\lambda}), \quad (x,y) \in P, \quad \lambda < n.$$

The change $v = u^m$ leads us to the equation

$$(v^{\frac{1}{m}})_t = \operatorname{div}(\nabla v - u\nabla K * v^{\frac{1}{m}})$$

of form (3). By Theorem 3 we establish that the corresponding problem possesses a weak solution as $m \in (0,1)$, but we do not state the uniqueness.

3. Proof of Theorem 1 on uniqueness of solution

Let us establish an auxiliary statement.

Lemma 3. *Let functions* $u(x,t)$, $v(x,t)$ *be weak solution to problem (3)–(5),*

$$N(t) = \int\limits_{\Omega} |u(x,t) - v(x,t)|dx, \quad s(t) = \int\limits_{\Omega} (\beta(x, u(x,t)) - \beta(x, v(x,t)))dx.$$

Then for all $\tau \in [0,T]$ *we have:*

$$\int\limits_0^\tau s^2(t)dt \leqslant L_r^2 \tau^2 \int\limits_0^\tau N^2(t)dt$$

and

$$\int\limits_0^\tau s^2(t)dt \leqslant C(\Omega)\tau^2 \int\limits_0^\tau \int\limits_{\Omega} |u(x,t) - v(x,t)|^2 dxdt,$$

where $r = \|u+v\|_{L_\infty(D^T)}$, L_r *is a constant in (12).*

Proof. We write relation (14) for the function v, we deduct it from (14) and we obtain

$$\int_0^T \int_\Omega \big((\beta(x,v)-\beta(x,u))\xi_t + (\nabla A(x,u) - \nabla A(x,v) + \beta(x,v)G(v) \\ - \beta(x,u)G(u))\cdot\nabla\xi\big)dxdt = \int_0^T \int_\Omega (f(x,u)-f(x,v))\xi dxdt. \tag{24}$$

Substituting here $\xi=\xi(t)\in C_0^\infty(0,T)$, we obtain:

$$\int_0^T \xi'(t)\int_\Omega(\beta(x,u)-\beta(x,v))dxdt = \int_0^T \xi(t)\int_\Omega (f(x,v)-f(x,u))dxdt.$$

This means that the function $s(t)$ is absolutely continuous in t. This is why, to complete the proof of the lemma, it is sufficient to substitute the function

$$\xi(t)=\int_t^\tau s(r)dr,\quad t\in[0,\tau],\quad \xi(t)=0,\quad t>\tau,$$

into (24). By (12) and Steklov-Friedrichs inequality we have:

$$\int_0^\tau s^2(t)dt \leqslant \int_0^\tau L_r N(t)|\xi(t)|dt \leqslant L_r\tau\left(\int_0^\tau N^2(t)dt\int_0^\tau s^2(t)dt\right)^{\frac{1}{2}}.$$

This implies easily the stated inequalities. □

Proof of Theorem 1. Let u and v be solutions to problem (3)–(5). Given $t\in(0,T)$, we define the function $\phi(x,t)$ as the solution to the Neumann problem

$$\Delta\phi(x,t)=\beta(x,u(x,t))-\beta(x,v(x,t))-\overline{s}(t),\quad x\in\Omega,\qquad \frac{\partial\phi}{\partial\nu}=0,\quad x\in\partial\Omega, \tag{25}$$

where $\overline{s}(t)=\frac{s(t)}{\operatorname{meas}\Omega}$. The solvability condition of this problem is the orthogonality in $L_2(\Omega)$ of the right hand in this equation to the solutions of the homogeneous equation (see, for instance, [11, Ch. II, Thm. 5.2]). The solutions of the problem for the homogeneous equation are constant functions only. According Lemma 3, we have the identity

$$\int_\Omega(\beta(x,u(x,t))-\beta(x,v(x,t))-\overline{s}(t))dx=0,$$

that is, the solvability condition of the Neumann problem holds. We can also assume that

$$\int_\Omega \phi(x,t)dx=0.$$

We observe that $\phi(x,0)=0$. Since

$$(\beta(x,u(x,t))-\beta(x,v(x,t)))\in L_\infty(0,T;L_q(\Omega))\cap H^1(0,T;H^{-1}(\Omega)),$$

then

$$\phi\in L_\infty(0,T;H^2(\Omega))\cap H^1(0,T;H^1(\Omega)),$$

see, for instance, [11, Ch. II, Eq. (5.4)]. Then $\nabla\phi\in C(0,T;L_2(\Omega))$ and the identity holds:

$$-\int_\Omega \nabla\phi\cdot\nabla w dx = \int_\Omega(\beta(x,u(x,t))-\beta(x,v(x,t))-\overline{s}(t))w(x)dx, \tag{26}$$

for each $w \in W_2^1(\Omega)$.

Let us transform equation (24). We denote

$$P(x,t) = \nabla A(x,u) - \nabla A(x,v) + \beta(x,v)G(v) - \beta(x,u)G(u).$$

It follows from Definition 1 that $P \in L_2(D^T)$. We let $F(x,t) = f(x,u) - f(x,v) \in L_2(D^T)$. We substitute

$$\xi = \varphi_{-h} = \frac{1}{h}\int\limits_{t-h}^{t} \varphi(x,r)dr,$$

into (24), where $\varphi \in C^\infty(\overline{D^T})$, $\varphi = 0$ as $t \geqslant T - \delta$. Taking into consideration (26), we obtain:

$$\int\limits_{D^T} (\nabla\phi \cdot \nabla(\varphi_{-h})_t - \overline{s}(t)(\varphi_{-h})_t + P \cdot \nabla\varphi_{-h} - F\varphi_{-h})dxdt = 0.$$

Employing the properties of Steklov averaging, we rewrite this as

$$\int\limits_{D^T} (-(\nabla\phi_h)_t \cdot \nabla\varphi - (\overline{s}_h)_t\varphi + P_h \cdot \nabla\varphi - F_h\varphi)dxdt = 0.$$

Let $\chi_{(0,\tau)}$ be the characteristic function of the interval $(0,\tau)$. We substitute $\varphi = \chi(0 < t < \tau)\phi_h$ into the latter equation and we pass to the limit as $h \to 0$. We obtain:

$$-\frac{1}{2}\int\limits_{\Omega} |\nabla\phi(\tau)|^2 + \int\limits_0^\tau\int\limits_{\Omega} (-\overline{s}_t\phi + P \cdot \nabla\phi - F\phi)dxdt = 0.$$

Taking into consideration that

$$\int\limits_{\Omega} \phi(x,t)dx = 0,$$

we get:

$$\begin{aligned} \frac{1}{2}\int\limits_{\Omega} |\nabla\phi(\tau)|^2 dx = &\int\limits_0^\tau\int\limits_{\Omega} \nabla(A(x,u) - A(x,v)) \cdot \nabla\phi dxdt \\ &- \int\limits_0^\tau\int\limits_{\Omega} (G(u)\beta(x,u) - G(v)\beta(x,v)) \cdot \nabla\phi dxdt \\ &- \int\limits_0^\tau\int\limits_{\Omega} (f(x,u) - f(x,v))\phi dxdt = I_1 + I_2 + I_3. \end{aligned} \tag{27}$$

Since $A(x,u) - A(x,v) \in L_2(0,T,H^1(\Omega))$ and the function A increases in the second variable, employing (26) and Lemma 3 and taking into consideration (9), we can write the relations:

$$\begin{aligned} I_1 &= -\int\limits_0^\tau\int\limits_{\Omega} (A(x,u) - A(x,v))(\beta(x,u) - \beta(x,v) - \overline{s}(t))dxdt \\ &\leqslant -\int\limits_0^\tau\int\limits_{\Omega} (a_0(u-v)^2 - A_M|(u-v)\overline{s}(t)|dxdt \leqslant \int\limits_0^\tau\int\limits_{\Omega} (C\tau - a_0)(u-v)^2 dxdt. \end{aligned} \tag{28}$$

We rewrite integral I_2 as

$$I_2 = -\int_0^\tau \int_\Omega (G(u)\beta(x,u) - G(v)\beta(x,v)) \cdot \nabla\phi dxdt = -\int_0^\tau \int_\Omega (\beta(x,u) - \beta(x,v))G(u) \cdot \nabla\phi dxdt$$

$$+\int_0^\tau \int_\Omega \beta(x,v)(G(u) - G(v)) \cdot \nabla\phi dxdt = I_4 + I_5.$$

Employing (26) for the first term, we obtain

$$I_4 = \int_0^\tau \int_\Omega \nabla\phi \cdot \nabla(G(u) \cdot \nabla\phi)dxdt - \int_0^\tau \int_\Omega \overline{s}(t)G(u) \cdot \nabla\phi dxdt = I_{41} + I_{42}.$$

By Lemma 3 we have the estimate

$$I_{42} \leqslant C\tau\|u - v\|_{L_2(D_0^\tau)}\|\nabla\phi\|_{L_2(D_0^\tau)},$$

where $D_0^\tau = \Omega \times (0,\tau)$. Then,

$$I_{41} = \int_0^\tau \int_\Omega \partial_i\phi(x)\partial_j\phi(x) \int_\Omega \partial_i g_j(x,y)b(u(y,t))dydxdt$$

$$+\int_0^\tau \int_\Omega \partial_i\phi(x)\partial_i\partial_j\phi(x) \int_\Omega g_j(x,y)b(u(y,t))dydxdt = I_6 + I_7.$$

We apply Gauss-Ostrogradsky formula to the integral I_7:

$$I_7 = -\int_0^\tau \int_\Omega \partial_j\partial_i\phi(x)\partial_i\phi(x) \int_\Omega g_j(x,y)b(u(y,t))dydxdt$$

$$-\int_0^\tau \int_\Omega \partial_i\phi(x)\partial_i\phi(x) \int_\Omega \partial_j g_j(x,y)b(u(y,t))dydxdt$$

$$+\int_0^\tau \int_{\partial\Omega} \partial_i\phi(x)\partial_i\phi(x)\nu_j \int_\Omega g_j(x,y)b(u(y,t))dydSdt.$$

Then in view of (11) we obtain

$$I_7 \leqslant -\frac{1}{2}\int_0^\tau \int_\Omega |\nabla\phi|^2 \int_\Omega \partial_j g_j(x,y)b(u(y,t))dydxdt.$$

Employing conditions (10), (13), we estimate the integral:

$$\int_\Omega |\partial_i g_j(x,y)b(u(y,t))|dy \leqslant \int_\Omega C(1 + |x-y|^{-\lambda})|b(u(y,t))|dy$$

$$\leqslant C\|u_0\|_{L_1(\Omega)} + C\int_\Omega \frac{|b(u(y,t))|}{|x-y|^\lambda}dy \leqslant C(M_T).$$

We have

$$I_6 \leqslant C(M_T)\int_0^\tau\int_\Omega |\nabla\phi|^2 dxdt.$$

We substitute the obtained estimates for I_6 and I_7 into I_4:

$$I_4 \leqslant \int_0^\tau\int_\Omega (C|\nabla\phi|^2 + \tau^2(u-v)^2)dxdt. \tag{29}$$

Employing conditions (10), (13) and Lemma 2, we establish the inequalities:

$$\|G_j(u(t)) - G_j(v(t))\|_{L_2(\Omega)} < C(\Omega)\|u(t)-v(t)\|_{L_2(\Omega)}, \quad j=\overline{1,n}.$$

Hence,

$$I_5 \leqslant C\int_0^\tau \|u(t)-v(t)\|_{L_2(\Omega)}\|\nabla\phi(t)\|_{L_2(\Omega)}dt.$$

Then

$$I_5 \leqslant C(\tau)\int_0^\tau \|\nabla\phi(t)\|^2_{L_2(\Omega)}dt + \tau\int_0^\tau\int_\Omega (u-v)^2 dxdt.$$

Letting $\eta(t) = (\int_\Omega |\nabla\phi(t)|^2 dx)^{\frac{1}{2}}$, by (27) and previous estimates we obtain

$$\begin{aligned}\frac{1}{2}\eta^2(\tau) + (a_0 - C(\tau+\tau^2))\int_0^\tau\int_\Omega (u-v)^2 dxdt \\ \leqslant C\int_0^\tau \eta^2(t)dt + \int_0^\tau\int_\Omega |(f(x,u)-f(x,v))\phi|dxdt.\end{aligned} \tag{30}$$

Employing condition (12) for the function $f(x,u)$ and Poincaré inequality, we establish that

$$\begin{aligned}\int_0^\tau\int_\Omega |f(x,u)-f(x,v)||\phi|dxdt &\leqslant \frac{a_0}{2}\int_0^\tau\int_\Omega (u-v)^2 dxdt + \frac{L_r^2}{a_0}\int_0^\tau\int_\Omega |\phi|^2 dxdt \\ &\leqslant \frac{a_0}{2}\int_0^\tau\int_\Omega (u-v)^2 dxdt + \frac{C_1 L_r^2}{a_0}\int_0^\tau\int_\Omega |\nabla\phi|^2 dxdt.\end{aligned} \tag{31}$$

Hence, for sufficiently small τ it follows from (30), (31) that

$$\eta^2(\tau) \leqslant C_2\int_0^\tau \eta^2(t)dt.$$

By means of Grönwall's lemma we hence conlcude that $\eta(t) = 0$ for all $0 \leqslant t \leqslant \tau$. Therefore, $u \equiv v$ in the cylinder $\Omega\times(0,\tau)$. In the same way we establish the identity $u \equiv v$ in the cylinder $\Omega\times(\tau,2\tau)$ and so forth. The proof is complete. □

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