# NEW CHARACTERIZATIONS OF BLOCH SPACES, BERS-TYPE AND ZYGMUND-TYPE SPACES AND RELATED QUESTIONS

## M. GARAYEV, H. GUEDIRI, H. SADRAOUI

**Abstract.** In terms of Berezin symbols, we give new characterizations of the Bloch spaces $\mathcal{B}$ and $\mathcal{B}_0$, Bers-type and the Zygmund-type spaces of analytic functions on the unit disc $\mathbb{D}$ in the complex plane $\mathbb{C}$. We discuss some properties of Toeplitz operators on the Bergman space $L_a^2(\mathbb{D})$. We provide a new characterization of certain function space with variable exponents. Namely, given a function $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathrm{Hol}(\mathbb{D})$ with a bounded sequence $\left\{ \widehat{f}(n) \right\}_{n \geqslant 0}$ of Taylor coefficients $\widehat{f}(n) = \frac{f^{(n)}(0)}{n!}$, $(n = 0, 1, 2, \ldots)$, we have $f \in H_{p(\cdot), q(\cdot), \gamma(\cdot)}$ if and only if

$$\int\limits_{0}^{1} \left( \frac{1}{2\pi} \int\limits_{0}^{2\pi} \left| \widetilde{D}_{(\widehat{f}(n) e^{int})}(\sqrt{r}) \right|^{p(t)} dt \right)^{\frac{q(t)}{p(t)}} (1-r)^{\frac{\gamma(t)p(t) - q(t)}{p(t)}} dr < +\infty.$$

Here $D_{(a_n)}$ denotes the associate diagonal operator on the Hardy-Hilbert space $H^2$ defined by the formula $D_{(a_n)} z^n = a_n z^n$ $(n = 0, 1, 2, \ldots)$.

**Keywords:** Bers-type space, Zygmund-type space, Bloch spaces, Berezin symbol.

**Mathematics Subject Classification:** 47B33, 30H30

## 1. INTRODUCTION

Let $\mathbb{D}$ be the unit disc in complex plane $\mathbb{C}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathrm{Hol}(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. We denote by $H^\infty = H^\infty(\mathbb{D})$ the space of bounded analytic functions on $\mathbb{D}$. Recall that a function $f \in \mathrm{Hol}(\mathbb{D})$ belongs to the Bloch space $\mathcal{B} = \mathcal{B}(\mathbb{D})$ if

$$\|f\|_b := \sup_{z \in \mathbb{D}} \left( 1 - |z|^2 \right) \left| f'(z) \right| < +\infty.$$

Being equipped with the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_b$, $\mathcal{B}$ is a Banach space. Let $\mathcal{B}_0 = \mathcal{B}_0(\mathbb{D})$ be the space consisting of all $f \in \mathcal{B}$ satisfying

$$\lim_{|z| \to 1} \left( 1 - |z|^2 \right) |f'(z)| = 0.$$

This space is called a little Bloch space.

Let $\alpha \geqslant 0$. The Bers-type space, denoted by $H_\alpha^\infty = H_\alpha^\infty(\mathbb{D})$, is a Banach space consisting of all $f \in \text{Hol}(\mathbb{D})$ such that

$$\|f\|_{\infty,\alpha} = \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^\alpha |f(z)| < +\infty.$$

Clearly, $H_0^\infty = H^\infty$.

Let $0 < p, q < +\infty$, $\gamma > -1$. If a function $f \in \text{Hol}(\mathbb{D})$ is such that

$$\|f\|_{Hp,q,\gamma}^q := \int_0^1 \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{\frac{q}{p}} (1-r)^\gamma dr < +\infty,$$

we say that $f$ belongs to a mixed norm space denoted by $H_{p,q,\gamma} = H_{p,q,\gamma}(\mathbb{D})$.

Let $\beta > 0$. The Zygmund-type space denoted by $\mathcal{Z}^\beta$ consist of all $f \in \text{Hol}(\mathbb{D})$ obeying

$$\|f\|_{\mathcal{Z}^\beta} := |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} \left(1 - |z|^2\right)^\beta \left|f''(z)\right| < +\infty.$$

The space $\mathcal{Z}^\beta$ becomes a Banach space with the above norm $\|\cdot\|_{\mathcal{Z}^\beta}$. Let $\beta = 1$. Then $\mathcal{Z}^1 = \mathcal{Z}$ is the classical Zygmund space. For more information on the Bers-type, Zygmund-type and Bloch spaces on the unit disc $\mathbb{D}$, see, e.g., K. Zhu [14, 15], X. Zhu [16], S. Stević [11], [12], Y. Ren [9], P. Duren [2], J. Shi [10].

Recall that the following old problem for most of functional spaces $X$ of analytic functions in $\mathbb{D}$, including the Hardy space $H^p(\mathbb{D})$, is open (see Privalov [8] and Duren [2]):

*How a function in $X$ can be recovered by the behavior of its Taylor coefficients?*

Ideally, one would like to find a condition on the $\widehat{f}(n) := \frac{f^n(0)}{n!}$ (Taylor coefficient), which is both necessary and sufficient for $f$ to be in $X$ Of course, for $X = H^p$, $p = 2$, the problem is completely solved: $f \in H^2$ if and only if $\sum_{n=0}^{\infty} \left|\widehat{f}(n)\right|^2 < +\infty$. For $p = \infty$, the problem of coefficients was solved by I. Schur in 1919 (see, Privalov [8, Ch. 2]). Some classical results on the Taylor coefficients of functions in Hardy and Bergman spaces are also known (see, for instance, [2, 8, 14]). Some recent results about Taylor coefficients of $H^1$ functions and entire functions in the Fock spaces $F_\alpha^p$ have been obtained, respectively, by Pavlović [6] and Tung [13]. But the general situation is much more complicated, and no complete answer is available (more informations about this are contained in [1] and references therein.)

Note that in his book [6], Pavlović proved the following characterization of functions belonging to the Hardy space $H^1$:

$$H^1 = \left\{ f \in \text{Hol}(\mathbb{D}) : \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|f(re^{it})\right| dt < +\infty \right\}.$$

**Theorem 1.1.** *For a function $f$ analytic in $\mathbb{D}$, the following statements are equivalent:*
*(a) $f \in H^1$;*

*(b) $\sup_{n \geqslant 0} \frac{1}{a_n} \sum_{j=0}^{n} \frac{1}{j+1} \|S_j(f)\|_{H^1} < +\infty$ ;*

*(c) $\sup_{n \geqslant 0} \|P_n f\|_{H^1} < +\infty$.*

*Here $P_n f := \frac{1}{a_n} \sum_{j=0}^{n} \frac{1}{j+1} S_j(f)$, where $a_n = \sum_{j=0}^{n} \frac{1}{j+1}$, $n = 0, 1, 2, \ldots$, and $S_j(f)$ are the partial sums of the Taylor series for $f$.*

Popa [7] gave some interesting generalization of this result of Pavlović by proving a similar characterization of upper triangular trace class matrices. Recently, Karaev [3] gave some characterizations of Hardy and Besov classes of analytic functions on $\mathbb{D}$ with the variable exponents.

In this paper, we use the so-called Berezin symbols technique to characterize the Bers-type, Zygmund-type spaces, and the Bloch spaces $\mathcal{B}$ and $\mathcal{B}_0$. We also consider some spaces defined by the variable exponent. We discuss Toeplitz operators on the Bergman space $L_a^2(\mathbb{D})$

## 2. BEREZIN SYMBOLS AND CHARACTERIZATIONS OF THE BERS-TYPE, ZYGMUND-TYPE AND BLOCH SPACES

In this section, we characterize the spaces $H_\alpha^\infty$, $\mathcal{Z}^\beta$, $\mathcal{B}$ and $\mathcal{B}_0$ in terms of Berezin symbols of diagonal operators (associated with the Taylor coefficients of the functions from the spaces $H_\alpha^\infty$, $\mathcal{Z}^\beta$, $\mathcal{B}$ and $\mathcal{B}_0$) acting on the Hardy space $H^2 = H^2(\mathbb{D})$.

Recall that Hardy-Hilbert space $H^2 = H^2(\mathbb{D})$ is the collection of analytic functions in $\mathbb{D}$ which satisfy the inequality

$$\|f\|_{H^2} := \left( \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{it}) \right|^2 d\theta \right)^{\frac{1}{2}} < +\infty.$$

It is well-known that $H^2$ is a reproducing kernel Hilbert space with the orthonormal basis $e_n(z) = z^n$ $(n = 0, 1, 2, \ldots)$, and consequently with the reproducing kernel

$$k_\lambda(z) = \sum_{n=0}^\infty \overline{e_n(\lambda)} e_n(z) = \sum_{n=0}^\infty \overline{\lambda}^n z^n = \frac{1}{1 - \overline{\lambda}z}, \quad \lambda \in \mathbb{D}.$$

The reproducing property means that $\langle f, k_\lambda \rangle_{H^2} = f(\lambda)$ for all $f \in H^2$ and $\lambda \in \mathbb{D}$. For any bounded sequence $(a_n)_{n \geqslant 0}$ of complex numbers $a_n$, we denote by $D_{(a_n)}$ the associate diagonal operator on the Hardy-Hilbert space $H^2$ which is defined by the formula

$$D_{(a_n)} z^n = a_n z^n, \qquad n = 0, 1, 2, \ldots \tag{1}$$

For any bounded linear operator $T$ on $H^2$, its Berezin symbol $\widetilde{T}$ is the following bounded complex-valued function in $\mathbb{D}$ :

$$\widetilde{T}(\lambda) := \langle T\mathcal{K}_\lambda, \mathcal{K}_\lambda \rangle \quad (\lambda \in \mathbb{D}),$$

where

$$\mathcal{K}_\lambda(z) := \frac{k_\lambda(z)}{\|k_\lambda(z)\|_{H^2}} = (1 - |\lambda|^2)^{\frac{1}{2}} (1 - \overline{\lambda}z)^{-1}$$

is the normalized reproducing kernel of $H^2$, and $ber(T) := \sup_{\lambda \in \mathbb{D}} \left| \widetilde{T}(\lambda) \right|$ is the Berezin number of $T$.

The following lemma is well-known.

**Lemma 2.1.** *The Berezin symbol of the diagonal operator $D_{(a_n)}$ on the Hardy space $H^2$ is the following radial function:*

$$\widetilde{D}_{(a_n)}(|\lambda|) = (1 - |\lambda|^2) \sum_{k=0}^\infty a_k |\lambda|^{2k} \quad (\lambda \in \mathbb{D}) \tag{2}$$

*Proof.* Indeed, by using that

$$k_\lambda(z) = \frac{1}{1 - \overline{\lambda}z} = \sum_{k=0}^\infty \overline{\lambda}^k z^k$$

is the reproducing kernel of $H^2$, we have:

$$\widetilde{D}_{(a_n)}(\lambda) = \left\langle D_{(a_n)} \mathcal{K}_\lambda(z), \mathcal{K}_\lambda(z) \right\rangle = \left\langle D_{(a_n)} \frac{k_\lambda(z)}{\|k_\lambda(z)\|_2}, \frac{k_\lambda(z)}{\|k_\lambda(z)\|_2} \right\rangle$$

$$= \left(1 - |\lambda|^2\right) \left\langle D_{(a_n)} \sum_{k=0}^{\infty} \overline{\lambda}^k z^k, \sum_{k=0}^{\infty} \overline{\lambda}^k z^k \right\rangle = \left(1 - |\lambda|^2\right) \left\langle \sum_{k=0}^{\infty} \overline{\lambda}^k a_k z^k, \sum_{k=0}^{\infty} \overline{\lambda}^k z^k \right\rangle$$

$$= \left(1 - |\lambda|^2\right) \sum_{k=0}^{\infty} a_k |\lambda|^{2k}.$$

Hence,

$$\widetilde{D}_{(a_n)}(\lambda) = \left(1 - |\lambda|^2\right) \sum_{k=0}^{\infty} a_k |\lambda|^{2k}, \qquad \lambda \in \mathbb{D},$$

which proves formula (2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Our next result characterizes the spaces $H_\alpha^\infty$, $\mathcal{Z}^\beta$, $\mathcal{B}$ and $\mathcal{B}_0$ in terms of behavior of the Berezin symbols of the corresponding diagonal operators mentioned above.

**Theorem 2.1.** *Let a function* $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in \text{Hol}(\mathbb{D})$ *have a bounded sequence* $\left\{\widehat{f}(n)\right\}_{n \geqslant 0}$ *of Taylor coefficients* $\widehat{f}(n) = \frac{f^{(n)}(0)}{n!}$, $(n = 0, 1, 2, \ldots)$. *Then* $f \in H_\alpha^\infty$ $(\alpha \geqslant 0)$ *if and only if*

$$\sup_{\substack{0 \leqslant r < 1 \\ 0 \leqslant \theta < 2\pi}} (1 - r)^{\alpha - 1} \left| \widetilde{D}_{(\widehat{f}(n) e^{in\theta})}(\sqrt{r}) \right| < +\infty.$$

*Proof.* Indeed, let a function $f$ be as in the statement of the theorem. Then, rewriting $f$ and using Lemma 2.1, we obtain:

$$f(z) = f(|z| e^{i \arg(z)}) = f(re^{i\theta}) = \sum_{n=0}^{\infty} \widehat{f}(n)(re^{i\theta})^n$$

$$= \sum_{n=0}^{\infty} \widehat{f}(n) e^{in\theta} r^n = \frac{(1 - r) \sum_{n=0}^{\infty} \widehat{f}(n) e^{in\theta} r^n}{1 - r} = \frac{\widetilde{D}_{(\widehat{f}(n) e^{in\theta})}(\sqrt{r})}{1 - r}.$$

Hence, for any $z = re^{i\theta}$, with $r = |z|$ and $\theta = \arg(z)$, $0 \leqslant r < 1$, $0 \leqslant \theta < 2\pi$, we have

$$f(z) = \frac{\widetilde{D}_{(\widehat{f}(n) e^{in\theta})}(\sqrt{r})}{1 - r}. \tag{3}$$

Using (3), we get

$$\left(1 - |z|^2\right)^\alpha |f(z)| = (1 - r^2)^\alpha \frac{\widetilde{D}_{(\widehat{f}(n) e^{in\theta})}(\sqrt{r})}{1 - r} = (1 - r)^{\alpha - 1}(1 + r)^\alpha \left| \widetilde{D}_{(\widehat{f}(n) e^{in\theta})}(\sqrt{r}) \right| \tag{4}$$

for all $r$, $0 \leqslant r < 1$ and all $\theta \in [0, 2\pi)$. Hence, we obtain

$$(1 - r)^{\alpha - 1} \left| \widetilde{D}_{(\widehat{f}(n) e^{in\theta})}(\sqrt{r}) \right| \leqslant \left(1 - |z|^2\right)^\alpha |f(z)| \leqslant 2^\alpha (1 - r)^{\alpha - 1} \left| \widetilde{D}_{(\widehat{f}(n) e^{in\theta})}(\sqrt{r}) \right|$$

for all $z = re^{i\theta} \in \mathbb{D}$. In particular, by (5) we conclude that that $f \in H_\alpha^\infty$ if and only if

$$\sup_{\substack{0 \leqslant r < 1 \\ \theta \in [0, 2\pi)}} (1 - r)^{\alpha - 1} \left| \widetilde{D}_{(\widehat{f}(n) e^{in\theta})}(\sqrt{r}) \right| < +\infty.$$

This completes the proof. $\square$

An immediate corollary of inequalities (5) is as follows.

**Corollary 1.** *If* $f \in H_\alpha^p$ *and* $\alpha > 0$, *then*

$$\sup_{\substack{0 \leqslant r < 1 \\ 0 \leqslant \theta < 2\pi}} (1-r)^{\alpha-1} \left| \widetilde{D}_{(\widehat{f}(n)e^{in\theta})}(\sqrt{r}) \right| \leqslant \|f\|_{p,\alpha} \leqslant 2^\alpha \sup_{\substack{0 \leqslant r < 1 \\ 0 \leqslant \theta < 2\pi}} (1-r)^{\alpha-1} \left| \widetilde{D}_{(\widehat{f}(n)e^{in\theta})}(\sqrt{r}) \right|.$$

**Theorem 2.2.** *Let a function* $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n \in \mathrm{Hol}(\mathbb{D})$ *have the sequence* $\left\{ \widehat{f}(n) \right\}_{n \geqslant 0}$ *of Taylor coefficients such that* $\widehat{f}(n) = O\left(\frac{1}{n}\right)$ *as* $n \to \infty$. *Then*
(a) *f belongs to Bloch space* $\mathcal{B}$ *if and only if*

$$\sup_{\substack{0 \leqslant r < 1 \\ 0 \leqslant \theta < 2\pi}} \left| \widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r}) \right| < +\infty;$$

(b) $f \in \mathcal{B}_0$ *if and only if*

$$\lim_{r \to 1} \left| \widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r}) \right| = 0$$

*for all* $\theta \in [0, 2\pi)$.

*Proof.* It follows from the condition $\widehat{f}(n) = O\left(\frac{1}{n}\right)$ $(n \to \infty)$ that $(n\widehat{f}(n))_n$ is bounded, and hence, the diagonal operator $D_{(n\widehat{f}(n))}$ is bounded on the Hardy-Hilbert space $H^2$. This implies immediately that $D_{((n+1)\widehat{f}(n+1)e^{in\theta})}$ is also bounded in $H^2$ for every fixed $\theta \in [0, 2\pi)$. Therefore, by formula (2) in Lemma 2.1, we have

$$\left(1 - |z|^2\right) f'(z) = (1-r^2)\sum_{n=0}^{\infty} \widehat{f}(n)z^n = (1-r^2)\sum_{n=1}^{\infty} n\widehat{f}(n)z^{n-1}$$

$$= (1-r^2)\sum_{n=0}^{\infty} (n+1)\widehat{f}(n+1)z^n$$

$$= (1-r)(1+r)\sum_{n=0}^{\infty} (n+1)\widehat{f}(n+1)e^{in\theta}r^n$$

$$= (1+r)\left[(1-r)\sum_{n=0}^{\infty} \left((n+1)\widehat{f}(n+1)e^{in\theta}\right) r^n\right]$$

$$= (1+r)\widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r}).$$

Hence,

$$\left(1 - |z|^2\right) f'(z) = (1+r)\widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r}) \tag{6}$$

for all $z = re^{i\theta} \in \mathbb{D}$, where $r \in [0, 1)$ and $\theta \in [0, 2\pi)$. Now formula (6) shows that $f \in \mathcal{B}$ if and only if

$$\sup_{\substack{0 \leqslant r < 1 \\ 0 \leqslant \theta < 2\pi}} \left| \widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r}) \right| < +\infty$$

and this proves (a).

On the other hand, it follows also immediately from (6) that $f \in \mathcal{B}_0$ if and only if

$$\lim_{r \to 1} \left| \widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r}) \right| = 0$$

for all $\theta$, $0 \leqslant \theta < 2\pi$. This proves (b). $\square$

Formula (6) implies also the following results.

**Corollary 2.** *Assume that* $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n$ *satisfies the condition of Theorem 2.2. If* $f \in \mathcal{B}$, *then*

$$\inf_{\substack{0 \leqslant r < 1 \\ 0 \leqslant \theta < 2\pi}} \left| \widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r}) \right| + |f(0)| \leqslant \|f\|_{\mathcal{B}} \leqslant |f(0)| + 2 \sup_{\substack{0 \leqslant r < 1 \\ 0 \leqslant \theta < 2\pi}} \left| \widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r}) \right|.$$

**Corollary 3.** *If* $f$ *is the same as in Corollary 2, then we have:*
(a) $\|f\|_{\mathcal{B}} \leqslant |f(0)| + 2 \sup_{0 \leqslant \theta < 2\pi} ber(D_{((n+1)\widehat{f}(n+1)e^{in\theta})})$.
(b) $\|f\|_{\mathcal{B}} \leqslant |f(0)| + 2 \sup_{n \geqslant 0}(n+1)\left| \widehat{f}(n+1) \right|$.

Clearly, Statement (a) implies Statement (b) since for each arbitrary fixed $\theta \in [0, 2\pi)$

$$ber(D_{((n+1)\widehat{f}(n+1)e^{in\theta})}) \leqslant \left\| D_{((n+1)\widehat{f}(n+1)e^{in\theta})} \right\| = \sup_{n \geqslant 0}(n+1)\left| \widehat{f}(n+1) \right|.$$

**Corollary 4.** *Each function* $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n \in \mathrm{Hol}(\mathbb{D})$ *obeying* $\widehat{f}(n) = o(\frac{1}{n})$ *as* $n \to \infty$ *belongs to the little Bloch space* $\mathcal{B}_0$.

*Proof.* Indeed, since by assumption $\lim_{n \to \infty} n\left| \widehat{f}(n) \right| = 0$, we see that $D_{((n+1)\widehat{f}(n+1)e^{in\theta})}$ is a compact diagonal operator, and since $H^2$ is the standard reproducing kernel Hilbert space (which means that the normalized reproducing kernel $\mathcal{K}_\lambda(z) = \frac{(1-|\lambda|^2)^{\frac{1}{2}}}{1-\overline{\lambda}z}$ weakly tends to zero as $\lambda$ approaches any boundary point of the unit disc $\mathbb{D}$), we obtain that $\lim_{r \to 1} \left| \widetilde{D}_{((n+1)\widehat{f}(n+1)e^{in\theta})}(\sqrt{r}) \right| = 0$ for all $\theta \in [0, 2\pi)$, which implies that $f \in \mathcal{B}_0$. The proof is complete. $\square$

**Theorem 2.3.** *Let* $\beta > 0$ *and* $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n \in \mathrm{Hol}(\mathbb{D})$ *be a function such that* $\widehat{f}(n) = O\left(\frac{1}{(n-1)n}\right)$ *as* $n \to \infty$. *Then* $f \in \mathcal{Z}^\beta$ *if and only if*

$$\sup_{\substack{0 \leqslant r < 1 \\ 0 \leqslant \theta < 2\pi}} (1-r)^{\beta-1} \left| \widetilde{D}_{((n+1)(n+2)\widehat{f}(n+2)e^{in\theta})}(\sqrt{r}) \right| < +\infty.$$

*Proof.* Since by assumption $((n-1)n\widehat{f}(n))_{n \geqslant 2}$ is a bounded sequence, the diagonal operator $D_{((n-1)n\widehat{f}(n)e^{in\theta})}$ is a bounded operator on the Hardy space $H^2$ for all $\theta \in [0, 2\pi)$. Then, by Lemma 2.1 we have:

$$(1 - |z|^2)^\beta |f''(z)| = (1 - r^2)^\beta \left| \sum_{n=2}^{\infty}(n-1)n\widehat{f}(n)z^{n-2} \right|$$

$$= (1 - r^2)^\beta \left| \sum_{n=0}^{\infty}(n+1)(n+2)\widehat{f}(n+2)z^n \right|$$

$$= (1 - r^2)^\beta \left| \sum_{n=0}^{\infty}(n+1)(n+2)\widehat{f}(n+2)e^{in\theta}r^n \right|$$

$$= (1 + r)^\beta(1 - r)^{\beta-1} \left| (1-r)\sum_{n=0}^{\infty}\left[(n+1)(n+2)\widehat{f}(n+2)e^{in\theta}\right]r^n \right|$$

$$= (1+r)^\beta (1-r)^{\beta-1} \left| \widetilde{D}_{\left((n+1)(n+2)\widehat{f}(n+2)e^{in\theta}\right)} \sqrt{r} \right|$$

and hence,

$$(1 - |z|^2)^\beta |f''(z)| = (1+r)^\beta (1-r)^{\beta-1} \left| \widetilde{D}_{\left((n+1)(n+2)\widehat{f}(n+2)e^{in\theta}\right)} \left(\sqrt{r}\right) \right| \tag{7}$$

for all $z = re^{i\theta} \in \mathbb{D}$. This equality shows that $f \in \mathscr{Z}^\beta$ if and only if

$$\sup_{\substack{0 \leqslant r < 1 \\ 0 \leqslant \theta < 2\pi}} (1 - r)^{\beta-1} \left| \widetilde{D}_{\left((n+1)\widehat{f}(n+1)e^{in\theta}\right)} \left(\sqrt{r}\right) \right| < +\infty,$$

which proves the theorem. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

## 3. Generalized subharmonicity and Toeplitz operators on Bergman space

In this section we apply representations (4), (6) and (7) for studying the boundedness, compactness and belonging to Schatten-Neumann class for Toeplitz operators acting in the Bergman space $L_a^2 = L_a^2(\mathbb{D})$.

Let $dA(z)$ be the area measure on $\mathbb{D}$ normalized so that the area of $\mathbb{D}$ is 1. In terms of Cartesian and polar coordinates is reads as

$$dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} rdrd\theta$$

For $1 \leqslant p < +\infty$, the usual Lebesgue space $L^p(\mathbb{D}, dA)$ denote the Banach space of Lebesgue measurable functions $f$ on $\mathbb{D}$ with the norm

$$\|f\|_p = \left[ \int_{\mathbb{D}} |f(z)|^p \, dA(z) \right]^{\frac{1}{p}} < +\infty.$$

The Bergman space $L_a^p = L_a^p(\mathbb{D})$ is defined to be the subspace of $L^p(\mathbb{D}, dA)$ consisting of analytic functions. For $p = 2$, $L_a^2$ is a reproducing kernel Hilbert space with the reproducing kernel

$$k(z, w) = \frac{1}{(1 - z\overline{w})^2}$$

Recall that $P : L^2(\mathbb{D}, dA) \to L_a^2$ is the Bergman projection and this is an integral operator given by the formula

$$Pf(z) = \int_{\mathbb{D}} k(z, w) f(w) dA(w) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^2} dA(w).$$

Given a function $\varphi \in L^\infty(\mathbb{D})$, we define the Toeplitz operator $T_\varphi$ on $L_a^2$ by $T_\varphi f = P(\varphi f)$, $f \in L_a^2$. Since the Bergman projection has the unit norm, we clearly get $\|T_\varphi\| \leqslant \|\varphi\|_\infty$. More information about Bergman space Toeplitz operators can be found, for example, in the book by K.Zhu [14].

**Definition 3.1** ([14])**.** *Suppose $f$ is a nonnegative function on $\mathbb{D}$. We say that $f$ has a generalized subharmonic property if there exists a constant $C > 0$ such that*

$$f(z) \leqslant \frac{C}{|D(z, r)|} \int_{D(z,r)} f(w) dA(w)$$

*for all $z \in \mathbb{D}$. Here for each $r > 0$ and $a \in \mathbb{D}$*

$$D(a, r) := \{ z \in \mathbb{D} : \beta(z, a) < r \}$$

*is the Bergman disc with the Bergman metric*

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)},$$

*where $\rho(z, w) = \left| \frac{z-w}{1-\bar{z}w} \right|$ $(z, w \in \mathbb{D})$ is the pseudo-hyperbolic distance on $\mathbb{D}$) and $|D(a, r)|$ is the normalized area of $D(a, r)$.*

Before proving a next theorem, we mention the following known result [14].

**Lemma 3.1.** *If $\varphi \geqslant 0$ possesses the generalized subharmonic property, then*
*(1) $T_\varphi$ is bounded on $L_a^2$ if and only if $\varphi$ is bounded as $|z| \to 1^-$;*
*(2) $T_\varphi$ is compact on $L_a^2$ if and only if $\varphi(z) \to 0$ as $|z| \to 1^-$;*
*(3) $T_\varphi$ is in $S_p(L_a^2)$ if and only if $\varphi \in L^p(\mathbb{D}, d\lambda)$, where $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}$ is the Möbius invariant measure on $\mathbb{D}$.*

**Theorem 3.1.** *Let a function $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathrm{Hol}(\mathbb{D})$ have a bounded sequence $\left\{ \widehat{f}(n) \right\}_{n \geqslant 0}$ of Taylor coefficients. Let $\alpha \geqslant 0$. Then*
*(a) $T_{(1+|z|)^\alpha (1-|z|)^{\alpha-1} \left| \widetilde{D}_{(\widehat{f}(n) e^{in \arg(z)})}(\sqrt{|z|}) \right|}$ is bounded on $L_a^2$ if and only if*

$$(1 - |z|)^{\alpha-1} \left| \widetilde{D}_{(\widehat{f}(n) e^{in \arg(z)})}(\sqrt{|z|}) \right|$$

*is bounded as $|z| \to 1^-$;*
*(b) $T_{(1+|z|)^\alpha (1-|z|)^{\alpha-1} \left| \widetilde{D}_{(\widehat{f}(n) e^{in \arg(z)})}(\sqrt{|z|}) \right|}$ is compact on $L_a^2$ if and only if*

$$(1 - |z|)^{\alpha-1} \left| \widetilde{D}_{(\widehat{f}(n) e^{in \arg(z)})}(\sqrt{|z|}) \right| \to 0$$

*as $|z| \to 1^-$;*
*(c) $T$ belongs to the Schatten-Neumann class $S_p := S_p(L_a^2)$ if and only if*

$$\int_{\mathbb{D}} (1 + |z|)^{\alpha p} (1 - |z|)^{p(\alpha-1)} \left| \widetilde{D}_{(\widehat{f}(n) e^{in \arg(z)})}(\sqrt{|z|}) \right|^p d_\lambda(z) < +\infty,$$

*where $d_\lambda(z) = \frac{dA(z)}{(1-|z|)^2}$ is the Möbius invariant measure on $\mathbb{D}$.*

*Proof.* Indeed, one can prove that if $f$ is analytic, $p > 0$, and $\alpha$ is real, then the function $(1-|z|)^\alpha |f(z)|^p$ possesses the generalized subharmonic property (see K.Zhu [14]). In particular, the function $(1 - |z|^2)^\alpha |f(z)|$ possesses the same property. On the other hand, by formula (4),

$$(1 - |z|^2)^\alpha |f(z)| = (1 + |z|)^\alpha (1 - |z|)^{\alpha-1} \left| \widetilde{D}_{(\widehat{f}(n) e^{in \arg(z)})}(\sqrt{|z|}) \right|$$

for all $z \in \mathbb{D}$. Now the statement of theorem is immediately implied by Lemma 3.1. $\square$

By using formulae (6) and (7), the nextg two results can be proved by the same method as in the above proof, and therefore the proof is omitted.

**Theorem 3.2.** *Let a function $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathrm{Hol}(\mathbb{D})$ have the sequence $\left\{ \widehat{f}(n) \right\}_{n \geqslant 0}$ of Taylor coefficients such that $\widehat{f}(n) = O\left(\frac{1}{n}\right)$ as $n \to \infty$. Then*
*(a) $T_{(1-|z|^2)|f'(z)|}$ is bounded on $L_a^2$ if and only if $\left| \widetilde{D}_{((n+1)\widehat{f}(n+1) e^{in \arg(z)})}(\sqrt{|z|}) \right|$ is bounded as $|z| \to 1^-$;*

(b) $T_{\left(1-|z|^2\right)|f'(z)|}$ *is compact on* $L_a^2$ *if and only if*

$$\left|\widetilde{D}_{\left((n+1)\widehat{f}(n+1)e^{in\,\arg(z)}\right)}\left(\sqrt{|z|}\right)\right| \to 0$$

*as* $|z| \to 1^-$;
(c) $T_{\left(1-|z|^2\right)|f'(z)|}$ *is in* $S_p(L_a^2)$ *if and only if*

$$\int_{\mathbb{D}} (1+|z|)^p \left|\widetilde{D}_{\left((n+1)\widehat{f}(n+1)e^{in\,\arg(z)}\right)}\left(\sqrt{|z|}\right)\right|^p d\lambda(z) < +\infty.$$

**Theorem 3.3.** *Let* $\beta > 0$ *and* $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)z^n \in \mathrm{Hol}(\mathbb{D})$ *be a function such that* $\widehat{f}(n) = O(((n-1)n)^{-1})$ *as* $n \to \infty$. *Then*

(a) $T_{\left(1-|z|^2\right)^{\beta}|f''(z)|}$ *is bounded on* $L_a^2$ *if and only if* $(1-|z|)^{\beta-1}\left|\widetilde{D}_{\left((n+1)(n+2)\widehat{f}(n+2)e^{in\,\arg(z)}\right)}\left(\sqrt{|z|}\right)\right|$ *is bounded as* $|z| \to 1^-$;
(b) $T_{\left(1-|z|^2\right)^{\beta}|f''(z)|}$ *is compact on* $L_a^2$ *if and only if*

$$(1-|z|)^{\beta-1}\left|\widetilde{D}_{\left((n+1)(n+2)\widehat{f}(n+2)e^{in\,\arg(z)}\right)}\left(\sqrt{|z|}\right)\right| \to 0$$

*as* $|z| \to 1^-$;
(c) $T_{\left(1-|z|^2\right)^{\beta}|f''(z)|}$ *is in* $S_p(L_a^2)$ *if and only if*

$$(1+|z|)^{\beta}(1-|z|)^{\beta-1}\left|\widetilde{D}_{\left((n+1)(n+2)\widehat{f}(n+2)e^{in\,\arg(z)}\right)}\left(\sqrt{|z|}\right)\right| \in L^p(\mathbb{D}, d\lambda).$$

4. CHARACTERIZATION OF MIXED NORM SPACE $H_{p(t),q(t),\gamma(t)}$ WITH VARIABLE EXPONENTS

Let $\mathbb{T} = \partial\mathbb{D}$ and let $p = p(t)$ and $q(t)$, $t \in \mathbb{T}$, be bounded positive measurable functions defined on $\mathbb{T}$, and let $\gamma(t) > -1$ on $\mathbb{T}$. Following by Kokilashvili and Paatashvili [4, 5] (see also Karaev [2]), we say that the analytic function $f$ in $\mathbb{D}$ belongs to the Hardy class $H^{p(t)}$ if

$$\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{it})\right|^{p(t)} dt < +\infty,$$

where $p(t) = p(e^{it})$, $t \in [0, 2\pi)$.

Similarly, we say that a function $f \in \mathrm{Hol}(\mathbb{D})$ belongs to the mixed norm space $H_{p(t),q(t),\gamma(t)}$ with the variable exponents if

$$\|f\|_{H_{p(t),q(t),\gamma(t)}}^{q(t)} := \int_0^1 \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{it})\right|^{p(t)} dt\right)^{\frac{q(t)}{p(t)}} (1-r)^{\gamma(t)} dr < +\infty,$$

where $p(t) = p(e^{it})$, $q(t) = q(e^{it})$ and $\gamma(t) = \gamma(e^{it})$, $t \in [0, 2\pi)$.

For $p(t) = p = const > 0$, $q(t) = q = const > 0$ and $\gamma(t) = \gamma = const > -1$, the class $H_{p(\cdot),q(\cdot),\gamma(\cdot)}$ coincides with the class $H_{p,q,\gamma}$.

The following theorem characterize the spaces $H_{p(\cdot),q(\cdot),\gamma(\cdot)}$ in terms of Berezin symbols and Taylor coefficients.

**Theorem 4.1.** *Let* $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathrm{Hol}(\mathbb{D})$ *be a function with the bounded sequence* $\left\{\widehat{f}(n)\right\}_{n \geqslant 0}$ *of Taylor coefficients* $\widehat{f}(n) = \frac{f^{(n)}(0)}{n!}$, $(n = 0, 1, 2, \ldots)$. *Then* $f \in H_{p(\cdot), q(\cdot), \gamma(\cdot)}$ *if and only if*

$$\int\limits_0^1 \left( \frac{1}{2\pi} \int\limits_0^{2\pi} \left| \widetilde{D}_{(\widehat{f}(n)e^{int})}(\sqrt{r}) \right|^{p(t)} dt \right)^{\frac{q(t)}{p(t)}} (1-r)^{\frac{\gamma(t)p(t)-q(t)}{p(t)}} dr < +\infty.$$

*Proof.* Using formula (2) in Lemma 2.1, we rewrite the function $f(z) = \sum\limits_{n=0}^{\infty} \widehat{f}(n) z^n$ as follows:

$$f(z) = f(re^{it}) = \frac{\widetilde{D}_{(\widehat{f}(n)e^{int})}(\sqrt{r})}{1-r} \tag{8}$$

for each $z = re^{it} \in \mathbb{D}$. Now the statement of the theorem follows the definition of the space $H_{p(\cdot), q(\cdot), \gamma(\cdot)}$. Namely, in view of (8), we obtain

$$\int\limits_0^1 \left( \frac{1}{2\pi} \int\limits_0^{2\pi} \left| f(re^{it}) \right|^{p(t)} dt \right)^{\frac{q(t)}{p(t)}} (1-r)^{\gamma(t)} dr = \int\limits_0^1 \left( \frac{1}{2\pi} \int\limits_0^{2\pi} \left| \frac{\widetilde{D}_{(\widehat{f}(n)e^{int})}(\sqrt{r})}{1-r} \right|^{p(t)} dt \right)^{\frac{q(t)}{p(t)}} (1-r)^{\gamma(t)} dr$$

$$= \int\limits_0^1 \left( \frac{1}{2\pi} \int\limits_0^{2\pi} \left| \widetilde{D}_{(\widehat{f}(n)e^{int})}(\sqrt{r}) \right|^{p(t)} dt \right)^{\frac{q(t)}{p(t)}} (1-r)^{\gamma(t) - \frac{q(t)}{p(t)}} dr.$$

This yields that $f \in H_{p(\cdot), q(\cdot), \gamma(\cdot)}$ if and only if

$$\int\limits_0^1 \left( \frac{1}{2\pi} \int\limits_0^{2\pi} \left| \widetilde{D}_{(\widehat{f}(n)e^{int})}(\sqrt{r}) \right|^{p(t)} dt \right)^{\frac{q(t)}{p(t)}} (1-r)^{\gamma(t) - \frac{q(t)}{p(t)}} dr < +\infty.$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

## REFERENCES

1. J.M. Ash, M.T. Karaev. *On the boundary behavior of special classes of $C^\infty$-functions and analytic functions* // Intern. Math. Forum. **7**:1-4, 153–166 (2012).
2. P. Duren. *Theory of $H^p$ spaces.* Dover Publications, Mineola (2000).
3. M.T. Karaev. *A characterization of the some function classes* // J. Funct. Spaces. Appl. **2012**, id 796798 (2012).
4. V. Kokilashvili and V. Paatashvili. *On Hardy classes of analytic functions with a variable exponent* // Proc. A. Razmadze Math. Inst. **142**:1, 134–137 (2006).
5. V. Kokilashvili and V. Paatashvili. *On the convergence of sequences of functions in Hardy classes with a variable exponent* // Proc. A. Razmadze Math. Inst. **146**:1, 124–126 (2008).
6. M. Pavlović. *Introduction to Function Spaces on the Disk.* Matematicki Institut SANU, Belgrade (2004).
7. N. Popa. *A characterization of upper triangular trace class matrices* // Compt. Rend. Math. **347**:1-2, 59–62 (2009).
8. I.I. Privalov. *Granicnye svoistva analiticeskih funkciĭ.* GosTekhIzdat, Moscow (1950) (in Russian).
9. Y. Ren. *New criteria for generalized weighted composition operators from mixed norm spaces into Zygmund-type spaces* // Filomat. **26**:6, 1171–1178 (2012).

10. J. Shi. *Inequalities for the integral means of holomorphic functions and their derivatives in the unit ball of* $\mathbb{C}^n$ // Trans, Amer. Math. Soc. **328**:2, 619–637 (1991).

11. S. Stević. *Weighted composition operators between mixed norm spaces and* $H_\alpha^\infty$ *spaces in the unit ball* // J. Ineq. Appl. **2007**:1, 1–9 (2008).

12. S. Stević. *Norm and essential norm of composition followed by differentiation from* $\alpha-Bloch$ *spaces to* $H_\mu^\infty$ // Appl. Math. Comput. **207**:1, 225–229 (2009).

13. J. Tung. *Taylor coefficients of functions in Fock spaces* // J. Math. Anal. Appl. **318**:2, 397–409 (2006).

14. K. Zhu, *Operator theory in function spaces.* Marcel, New York (1990).

15. K. Zhu. *Bloch type spaces of analytic functions* // Rocky Mount. J. Math. **23**:3, 1143–1177 (1993).

16. X. Zhu, *Generalized weighted composition operators from Bloch spaces into Bers- type spaces* // Filomat. **26**:6, 1163–1169 (2012).

Mubariz Garayev,
Department of Mathematics, College of Science,
King Saud University,
P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-mail: mgarayev@ksu.edu.sa

Hocine Guediri,
Department of Mathematics, College of Science,
King Saud University,
P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-mail: hguediri@ksu.edu.sa

Houcine Sadraoui,
Department of Mathematics, College of Science,
King Saud University,
P.O. Box 2455, Riyadh 11451, Saudi Arabia
E-mail: sadrawi@ksu.edu.sa