ISSN 2074-1871 Уфимский математический журнал. Том 9. № 4 (2017). С. 137-146.

УДК 517.576

MINIMUM MODULUS OF LACUNARY POWER SERIES AND h-MEASURE OF EXCEPTIONAL SETS

T.M. SALO, O.B. SKASKIV

Abstract. We consider some generalizations of Fenton theorem for the entire functions represented by lacunary power series. Let $f(z) = \sum_{k=0}^{+\infty} f_k z^{n_k}$, where (n_k) is a strictly increasing sequence of non-negative integers. We denote by

$$M_f(r) = \max\{|f(z)|\colon |z| = r\},$$
$$m_f(r) = \min\{|f(z)|\colon |z| = r\},$$
$$\mu_f(r) = \max\{|f_k|r^{n_k}\colon k \geqslant 0\}$$

the maximum modulus, the minimum modulus and the maximum term of f, respectively. Let $h(r)$ be a positive continuous function increasing to infinity on $[1, +\infty)$ with a non-decreasing derivative. For a measurable set $E \subset [1, +\infty)$ we introduce $h - \text{meas}\,(E) = \int_E \frac{dh(r)}{r}$. In this paper we establish conditions guaranteeing that the relations

$$M_f(r) = (1 + o(1))m_f(r), \quad M_f(r) = (1 + o(1))\mu_f(r)$$

are true as $r \to +\infty$ outside some exceptional set E such that $h-\text{meas}\,(E) < +\infty$. For some subclasses we obtain necessary and sufficient conditions. We also provide similar results for entire Dirichlet series.

Keywords: lacunary power series, minimum modulus, maximum modulus, maximal term, entire Dirichlet series, exceptional set, h-measure

Mathematics Subject Classification: 30B50

1. Introduction

Let L be the class of positive continuous functions increasing to infinity on $[0; +\infty)$. By L^+ we denote the subclass of L consisting of the differentiable functions with a non-decreasing derivative, and L^- stands for the subclass of functions with a non-increasing derivative.

Let f be an entire function of the form

$$f(z) = \sum_{k=0}^{+\infty} f_k z^{n_k}, \tag{1}$$

where (n_k) is a strictly increasing sequence of nonnegative integers. Given $r > 0$, we denote by $M_f(r) = \max\{|f(z)|\colon |z| = r\}$, $m_f(r) = \min\{|f(z)|\colon |z| = r\}$, $\mu_f(r) = \max\{|f_k|r^{n_k}\colon k \geqslant 0\}$ the maximum modulus, the minimum modulus and the maximum term of f, respectively.

P.C. Fenton [1] (see also [2]) proved the following statement.

Theorem 1 ([1]). *If*

$$\sum_{k=0}^{+\infty} \frac{1}{n_{k+1} - n_k} < +\infty, \tag{2}$$

T.M. Salo, O.B. Skaskiv, The minimum modulus of lacunary power series and h-measure of exceptional sets.

Поступила 22 июля 2016 г.

then for every entire function f of the form (1) *there exists a set $E \subset [1, +\infty)$ of finite logarithmic measure, i.e.* log-meas $E := \int_E d\log r < +\infty$*, such that the relations*

$$M_f(r) = (1 + o(1))m_f(r), \quad M_f(r) = (1 + o(1))\mu_f(r) \tag{3}$$

hold as $r \to +\infty$ ($r \notin E$).

P. Erdős and A.J. Macintyre [2] proved that condition (2) implies that (3) holds as $r = r_j \to +\infty$ for some sequence (r_j).

Denote by $D(\Lambda)$ the class of entire (absolutely convergent in the complex plane) Dirichlet series of the form

$$F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n}, \tag{4}$$

where $\Lambda = (\lambda_n)$ is a fixed sequence such that $0 = \lambda_0 < \lambda_n \uparrow +\infty$ $(1 \leqslant n \uparrow +\infty)$.

Let us introduce some notations. Given $F \in D(\Lambda)$ and $x \in \mathbb{R}$, we denote by

$$\mu(x, F) = \max\{|a_n|e^{x\lambda_n} : n \geqslant 0\}$$

the maximal term of series (4), by

$$M(x, F) = \sup\{|F(x + iy)| \colon y \in \mathbb{R}\}$$

we denote the maximum modulus of series (4), by

$$m(x, F) = \inf\{|F(x + iy)| \colon y \in \mathbb{R}\}$$

we denote the minimum modulus of series (4), and

$$\nu(x, F) = \max\{n \colon |a_n|e^{x\lambda_n} = \mu(x, F)\}$$

stands for the central index of series (4).

In [3] (see also [4]) we find the following theorem.

Theorem 2 ([3])**.** *For every entire function $F \in D(\Lambda)$ the relation*

$$F(x + iy) = (1 + o(1))a_{\nu(x,F)}e^{(x+iy)\lambda_{\nu(x,F)}} \tag{5}$$

holds as $x \to +\infty$ outside some set E of finite Lebesgue measure ($\int_E dx < +\infty$) uniformly in $y \in \mathbb{R}$, if and only if

$$\sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1} - \lambda_n} < +\infty. \tag{6}$$

Note, that in the paper [5] there were proved the analogues of other statements in the paper by P.C. Fenton [1] for subclasses of functions $F \in D(\Lambda)$ defined by various restrictions on the growth rate of the maximal term $\mu(x, F)$.

The finiteness of Lebesgue measure of an exceptional set E in theorem A is the best possible description. This is implied by the next statement.

Theorem 3 ([6])**.** *For every sequence $\lambda = (\lambda_k)$ (including those which satisfy* (6)*) and for every continuously differentiable function $h \colon [0, +\infty) \to (0, +\infty)$ such that $h'(x) \nearrow +\infty$ ($x \to +\infty$) there exist an entire Dirichlet series $F \in D(\lambda)$, a constant $\beta > 0$ and a measurable set $E_1 \subset [0, +\infty)$ of infinite h-measure ($h - \operatorname{meas}(E_1) \overset{def}{=} \int_{E_1} dh(x) = +\infty$) such that*

$$(\forall\ x \in E_1)\colon\ M(x, F) > (1 + \beta)\mu(x, F), \quad M(x, F) > (1 + \beta)m(x, F). \tag{7}$$

Recently, Ya.V. Mykytyuk remarked that in Theorem 3, it is sufficient to assume that a positive non-decreasing function h is such that

$$\frac{h(x)}{x} \to +\infty \quad \text{as} \quad x \to +\infty.$$

It follows from Theorem 3 that the finiteness of logarithmic measure of an exceptional set E in Fenton's Theorem 1 is also the best possible description.

It is easy to see that the relation

$$F(x+iy)=(1+o(1))a_{\nu(x,F)}e^{(x+iy)\lambda_{\nu(x,F)}}$$

holds as $x\to+\infty$ $(x\notin E)$ uniformly in $y\in\mathbb{R}$ if and only if

$$M(x,F)\sim\mu(x,F)\quad\text{and}\quad M(x,F)\sim m(x,F)\quad(x\to+\infty,\ x\notin E). \tag{8}$$

In view of Theorem 3, the natural question arises: *what conditions should an entire Dirichlet series satisfy in order to relation (5) be true as $x\to+\infty$ outside some set E_2 of finite h-measure, i.e.,*

$$h-\operatorname{meas}(E_2)<+\infty?$$

In this paper we provide the answer to this question as $h\in L^+$.

2. $h-$measure with non-decreasing density

According to Theorem 3, in the case $h\in L^+$, condition (6) must be fulfilled. Therefore, in the subclass

$$D(\Lambda,\Phi)=\{F\in D(\Lambda):\ \ln\mu(x,F)\geqslant x\Phi(x)\ (x>x_0)\},\quad \Phi\in L,$$

it should be strengthened. The following theorem indicates this.

Theorem 4. *Let $\Phi\in L$, $h\in L^+$ and φ be the inverse function for the function Φ. If*

$$(\forall b>0):\ \sum_{k=0}^{+\infty}\frac{1}{\lambda_{k+1}-\lambda_k}h'\Big(\varphi(\lambda_k)+\frac{b}{\lambda_{k+1}-\lambda_k}\Big)<+\infty, \tag{9}$$

then for all $F\in D(\Lambda,\Phi)$ identity (5) is true as $x\to+\infty$ outside some set E of a finite h-measure uniformly in $y\in\mathbb{R}$.

Before proving this theorem, we need additional notations and an auxiliary lemma.

Denote $\Delta_0=0$ and

$$\Delta_n=\sum_{j=0}^{n-1}(\lambda_{j+1}-\lambda_j)\sum_{m=j+1}^{\infty}\left(\frac{1}{\lambda_m-\lambda_{m-1}}+\frac{1}{\lambda_{m+1}-\lambda_m}\right).$$

for $n\geqslant 1$. The next lemma is similar to Lemma 1 in [8].

Lemma 1. *For all $n\geqslant 0$ and $k\geqslant 1$, the inequality*

$$\frac{\alpha_n}{\alpha_k}\ e^{\tau_k(\lambda_n-\lambda_k)}\leqslant e^{-q|n-k|}, \tag{10}$$

is true, where $\alpha_n=e^{q\Delta_n}$, $q>0$, and

$$\tau_k=\tau_k(q)=qx_k+\frac{q}{\lambda_k-\lambda_{k-1}},\quad x_k=\frac{\Delta_{k-1}-\Delta_k}{\lambda_k-\lambda_{k-1}}.$$

Proof. Since

$$\ln\alpha_n-\ln\alpha_{n-1}=q(\Delta_n-\Delta_{n-1})=-qx_n(\lambda_n-\lambda_{n-1}),$$

for $n \geqslant k+1$ we have

$$\begin{aligned}\ln\frac{\alpha_n}{\alpha_k}+\tau_k(\lambda_n-\lambda_k)&=-\,q\sum_{j=k+1}^{n}x_j(\lambda_j-\lambda_{j-1})+\tau_k\sum_{j=k+1}^{n}(\lambda_j-\lambda_{j-1})\\&=-\sum_{j=k+1}^{n}(qx_j-\tau_k)\,(\lambda_j-\lambda_{j-1})\\&\leqslant-\sum_{j=k+1}^{n}(qx_j-\tau_{j-1})\,(\lambda_j-\lambda_{j-1})\\&=-\,q\sum_{j=k+1}^{n}1=-q(n-k).\end{aligned}$$

Similarly, for $n \leqslant k-1$ we obtain

$$\begin{aligned}\ln\frac{\alpha_n}{\alpha_k}+\tau_k(\lambda_n-\lambda_k)&=-\ln\frac{\alpha_k}{\alpha_n}-\tau_k(\lambda_k-\lambda_n)\\&=q\sum_{j=n+1}^{k}x_j(\lambda_j-\lambda_{j-1})-\tau_k\sum_{j=n+1}^{k}(\lambda_j-\lambda_{j-1})\\&=-\sum_{j=n+1}^{k}(\tau_k-qx_j)\,(\lambda_j-\lambda_{j-1})\\&\leqslant-\sum_{j=n+1}^{k}(\tau_j-qx_j)\,(\lambda_j-\lambda_{j-1})=-q\sum_{j=n+1}^{k}1=-q(k-n),\end{aligned}$$

and this completes the proof. □

Proof of Theorem 4. We first note that condition (9) implies the convergence of series (6). We consider the function

$$f_q(z)=\sum_{n=0}^{+\infty}\frac{a_n}{\alpha_n}\;e^{z\lambda_n}.$$

Since $\Delta_n \geqslant 0$, we have $f_q \in D(\Lambda)$ and $\nu(x, f_q) \to +\infty$ $(x \to +\infty)$.

Let J be the range of the central index $\nu(x, f_q)$. Denote by (R_k) the sequence of the jump points of central index, numbered in such a way that $\nu(x, f_q) = k$ for all $x \in [R_k, R_{k+1})$ and $R_k < R_{k+1}$. Then for all $x \in [R_k, R_{k+1})$ and $n \geqslant 0$ we have

$$\frac{a_n}{\alpha_n}e^{x\lambda_n}\leqslant\frac{a_k}{\alpha_k}e^{x\lambda_k}.$$

According to Lemma 1, for $x \in [R_k + \tau_k, R_{k+1} + \tau_k)$ we obtain

$$\frac{a_ne^{x\lambda_n}}{a_ke^{x\lambda_k}}\leqslant\frac{\alpha_n}{\alpha_k}\;e^{\tau_k(\lambda_n-\lambda_k)}\leqslant e^{-q|n-k|}\qquad(n\geqslant 0).$$

Therefore,

$$\nu(x,F)=k,\quad \mu(x,F)=a_ke^{x\lambda_k}\quad(x\in[R_k+\tau_k,R_{k+1}+\tau_k))\tag{11}$$

and

$$\begin{aligned}|F(x+iy)-a_{\nu(x,F)}e^{(x+iy)\lambda_{\nu(x,F)}}|&\leqslant\sum_{n\neq\nu(x,F)}\mu(x,F)e^{-q|n-\nu(x,F)|}\\&\leqslant 2\;\frac{e^{-q}}{1-e^{-q}}\mu(x,F)\end{aligned}\tag{12}$$

for all $x \in [R_k + \tau_k, R_{k+1} + \tau_k)$ and $k \in J$. Thus, inequality (12) holds for all $x \notin E_1(q) \overset{def}{=} \bigcup_{k=0}^{+\infty} [R_{k+1} + \tau_k, R_{k+1} + \tau_{k+1})$.

Since

$$\tau_{k+1} - \tau_k = \frac{2q}{\lambda_{k+1} - \lambda_k},$$

and by the Lagrange theorem

$$h(R_{k+1} + \tau_{k+1}) - h(R_{k+1} + \tau_k) = (\tau_{k+1} - \tau_k)h'(R_{k+1} + \tau_k + \theta_k(\tau_{k+1} - \tau_k)),$$

where $\theta_k \in (0;1)$, for each $q > 0$ we have

$$\begin{aligned} h - \operatorname{meas}(E_1(q)) &= \sum_{k=0}^{+\infty} \int_{R_{k+1}+\tau_k}^{R_{k+1}+\tau_{k+1}} dh(x) \\ &= \sum_{k=0}^{+\infty} (h(R_{k+1} + \tau_{k+1}) - h(R_{k+1} + \tau_k)) \qquad (13) \\ &\leqslant 2q \sum_{k=0}^{+\infty} \frac{1}{\lambda_{k+1} - \lambda_k} h'\Big(R_{k+1} + \tau_k + 2q \frac{1}{\lambda_{k+1} - \lambda_k}\Big). \end{aligned}$$

Here we have employed the condition $h \in L^+$.

For $F \in D(\Lambda, \Phi)$ and $x > \max\{x_0, 1\}$ we have

$$x\Phi(x) \leqslant \ln \mu(x, F) = \ln \mu(1, F) + \int_1^x \lambda_{\nu(x,f)} dx \leqslant \ln \mu(1, F) + (x - 1)\lambda_{\nu(x-0,F)}.$$

This implies

$$x\Phi(x) \leqslant x\lambda_{\nu(x-0,F)} \qquad (14)$$

for all $x \geqslant x_1 \geqslant x_0$, i.e.

$$x \leqslant \varphi\left(\lambda_{\nu(x-0,F)}\right) \quad (x \geqslant x_1).$$

Thus, according to (11), for $k \geqslant k_0$ we obtain

$$R_{k+1} + \tau_k \leqslant \varphi\left(\lambda_{\nu(R_{k+1}+\tau_k-0,F)}\right) = \varphi(\lambda_k).$$

Applying this inequality to inequality (13), by the condition $h \in L^+$ we have

$$h - \operatorname{meas}(E_1(q)) \leqslant 2q \sum_{k=0}^{+\infty} \frac{1}{\lambda_{k+1} - \lambda_k} h'\Big(\varphi(\lambda_k) + 2q \frac{1}{\lambda_{k+1} - \lambda_k}\Big). \qquad (15)$$

Therefore, using (9) we conclude that $h - \operatorname{meas}(E_1(q)) < +\infty$.

Let $q_k = k$. Since $h - \operatorname{meas}(E_1(q_k)) < +\infty$, we have

$$h - \operatorname{meas}(E_1(q_k) \cap [x, +\infty)) = o(1) \quad (x \to +\infty),$$

hence, it is possible to choose an increasing to $+\infty$ sequence (x_k) such that

$$h - \operatorname{meas}\big(E_1(q_k) \cap [x_k; +\infty)\big) \leqslant \frac{1}{k^2}$$

for all $k \geqslant 1$. Denote $E_1 = \bigcup_{k=1}^{+\infty} \big(E_1(q_k) \cap [x_k; x_{k+1})\big)$. Then

$$h - \operatorname{meas}(E_1) = \sum_{k=1}^{+\infty} h - \operatorname{meas}(E_1(q_k) \cap [x_k; x_{k+1})) \leqslant \sum_{k=1}^{+\infty} \frac{1}{k^2} < +\infty,$$

On the other hand, by inequality (12), for $x \in [x_k; x_{k+1}) \setminus E_1$ we get

$$|F(x+iy) - a_{\nu(x,F)}e^{(x+iy)\lambda_{\nu(x,F)}}| \leqslant 2\ \frac{e^{-q_k}}{1-e^{-q_k}}\mu(x,F),$$

and therefore, as $x \to +\infty$ $(x \notin E_1)$, we obtain (5). The proof is complete. □

We observe that if $h(x) \equiv x$, then condition (9) becomes condition (6), and h-measure of the set E is its Lebesgue measure.

Let $\Phi \in L$. Consider the classes

$$D_0(\Lambda,\Phi) = \big\{F \in D(\Lambda):\ (\exists K>0)[\ln\mu(x,\Phi) \geqslant Kx\Phi(x)\ (x>x_0)]\big\},$$
$$D_1(\Lambda,\Phi) = \{F \in D(\Lambda):\ (\exists K_1,K_2>0)[\ln\mu(x,\Phi) \geqslant K_1x\Phi(K_2x)\ (x>x_0)]\}.$$

Theorem 5. *Let $\Phi_0 \in L$, $h \in L^+$ and φ_0 be the inverse function for the function Φ_0. If*

$$(\forall b>0):\ \sum_{n=0}^{+\infty}\frac{1}{\lambda_{n+1}-\lambda_n}h'\left(\varphi_0(b\lambda_n)+\frac{b}{\lambda_{n+1}-\lambda_n}\right) < +\infty, \tag{16}$$

then for each function $F \in D_0(\Lambda,\Phi_0)$ relation (5) holds as $x \to +\infty$ outside some set E of finite h - measure uniformly in $y \in \mathbb{R}$.

Theorem 6. *Let $\Phi_1 \in L$, $h \in L^+$, and φ_1 be the inverse function to the function Φ_1. If*

$$(\forall b>0):\ \sum_{n=0}^{+\infty}\frac{h'(b\varphi_1(b\lambda_n))}{\lambda_{n+1}-\lambda_n} < +\infty, \tag{17}$$

then for every function $F \in D_1(\Lambda,\Phi_1)$ relation (5) holds as $x \to +\infty$ outside some set E of finite h-measure uniformly in $y \in \mathbb{R}$.

Proof of Theorems 5 and 6. Theorems 5 and 6 are implied immediately by Theorem 4.

Indeed, if $F \in D_0(\Lambda,\Phi_0)$, then $F \in D(\Lambda,\Phi)$ as $\Phi(x) = K\Phi_0(x)$. But in this case $\varphi(x) = \varphi_0(x/K)$ and condition (9) follows condition (16). Then it remains to apply Theorem 4.

In the same way, if $F \in D_1(\Lambda,\Phi_1)$, then $F \in D(\Lambda,\Phi)$ as $\Phi(x) = K_1\Phi_1(K_2x)$. But in this case $\varphi(x) = \varphi_1(x/K_1)/K_2$ and hence, condition (9) follows condition (17). It remains to employ Theorem 4 once again. □

Remark 1. *It is easy to see that for each fixed functions $h \in L^+$ and $\Phi \in L$ there exists a sequence Λ such that conditions* (9), (16) *and* (17) *hold.*

The next theorem shows that condition (17) is necessary for relations (5), (8) to hold for each $F \in D_1(\Lambda,\Phi_1)$ as $x \to +\infty$ outside a set of a finite h-measure. Here we assume that condition (6) is satisfied.

Theorem 7. *Let $\Phi_1 \in L$, $h \in L^+$, and φ_1 be the inverse function for the function Φ_1. For each sequence Λ such that*

$$(\exists b>0):\ \sum_{n=0}^{+\infty}\frac{h'(b\varphi_1(b\lambda_n))}{\lambda_{n+1}-\lambda_n} = +\infty, \tag{18}$$

there exist a function $F \in D_1(\Lambda,\Phi_1)$, a set $E \subset [0,+\infty)$ and a constant $\beta > 0$ such that inequalities (7) hold for all $x \in E$ and $h-\operatorname{meas}(E) = +\infty$.

Proof. We denote $\varkappa_1 = \varkappa_2 = 1$, $\varkappa_n = \sum\limits_{k=1}^{n-2} r_k$, $(n \geqslant 3)$, where

$$r_1 = \max\Big\{b\varphi_1(b\lambda_2), \frac{1}{\lambda_2-\lambda_1}\Big\},$$
$$r_k = \max\Big\{b\varphi_1(b\lambda_{k+1}) - b\varphi_1(b\lambda_k), \frac{1}{\lambda_{k+1}-\lambda_k}\Big\} \quad (k \geqslant 2),$$

and we also choose

$$a_0 = 1, \quad a_n = \exp\Big\{-\sum_{k=1}^{n} \varkappa_k(\lambda_k - \lambda_{k-1})\Big\} \ (n \geqslant 1).$$

We prove that the function F defined by series (4) with the above defined coefficients (a_n) and the exponents (λ_n) belongs to the class $D_1(\Lambda, \Phi_1)$.

Since the condition

$$\sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1}-\lambda_n} < +\infty$$

implies $n^2 = o(\lambda_n)$ $(n \to +\infty)$, we have $\dfrac{\ln n}{\lambda n} \to 0$ $(n \to +\infty)$. By the construction,

$$\varkappa_n = \frac{\ln a_{n-1} - \ln a_n}{\lambda_n - \lambda_{n-1}} \ (n \geqslant 1)$$

and $\varkappa_n \uparrow +\infty$ $(n \to +\infty)$. Therefore Stolz theorem yields that $-\dfrac{\ln a_n}{\lambda_n} \to +\infty$ $(n \to +\infty)$ and by Valiron formula [9] the abscissa of the absolute convergence of series (4) is equal to $+\infty$, i.e., $F \in D(\Lambda)$.

Moreover, it is known that in the case $\varkappa_n \uparrow +\infty$ $(n \to +\infty)$ we have

$$\forall x \in [\varkappa_n, \varkappa_{n+1}) : \quad \mu(x, F) = a_n e^{x\lambda_n}, \quad \nu(x, F) = n. \tag{19}$$

Since by the construction

$$\varkappa_n \leqslant b\varphi_1(b\lambda_{n-1}) + \sum_{k=1}^{n-2} \frac{1}{\lambda_{k+1}-\lambda_k} \leqslant 2b\varphi_1(b\lambda_{n-1}) \quad (n > n_0),$$

for sufficiently large n for all $x \in [\varkappa_n, \varkappa_{n+1})$ we have

$$\ln \mu(2x, F) = \ln \mu(x, F) + \int_x^{2x} \lambda_{\nu(t)} dt \geqslant x\lambda_{\nu(x)}$$
$$= x\lambda_n \geqslant \frac{x}{b}\Phi_1\left(\frac{\varkappa_{n+1}}{2b}\right) \geqslant \frac{x}{b}\Phi_1\left(\frac{x}{2b}\right).$$

Hence, for $x \geqslant x_0$ we have

$$\ln \mu(x, F) \geqslant \frac{1}{2b} x\Phi_1\Big(\frac{x}{4b}\Big),$$

and thus $F \in D_1(\Lambda, \Phi_1)$.

We observe that

$$\varkappa_{n+1} - \varkappa_n = r_{n-1} \geqslant \frac{1}{\lambda_n - \lambda_{n-1}} \quad (n \geqslant 1).$$

For $x \in \Big[\varkappa_n, \varkappa_n + \frac{1}{\lambda_n-\lambda_{n-1}}\Big]$ we have

$$\frac{a_{n-1}e^{x\lambda_{n-1}}}{\mu(x, F)} = \frac{a_{n-1}e^{x\lambda_{n-1}}}{a_n e^{x\lambda_n}} = \exp\{(\lambda_n - \lambda_{n-1})(\varkappa_n - x)\} \geqslant e^{-1} := \beta, \tag{20}$$

and, therefore, for $x \in E = \bigcup\limits_{n=1}^{\infty} \left[\varkappa_n, \varkappa_n + \frac{1}{\lambda_n - \lambda_{n-1}}\right]$, by choosing $n = \nu(x, F)$ we get

$$F(x) \geqslant a_{n-1}e^{x\lambda_{n-1}} + a_n e^{x\lambda_n} = \mu(x, F)\left(1 + \frac{a_{n-1}e^{x\lambda_{n-1}}}{a_n e^{x\lambda_n}}\right) \geqslant (1+\beta)\mu(x, F).$$

Hence, inequalities (7) are true.

Now we prove that $h - \text{meas}\,(E) = +\infty$. By the construction of $(\varkappa_n)$ for all $n \geqslant 1$ we have

$$\varkappa_n \geqslant b\varphi_1(b\lambda_{n-1}). \tag{21}$$

Taking into consideration the Lagrange theorem, the condition $h \in L^+$ and inequality (21), we obtain

$$h - \text{meas}\,(E) = \sum_{n=1}^{+\infty} \int\limits_{\varkappa_n}^{\varkappa_n + \frac{1}{\lambda_n - \lambda_{n-1}}} dh(x) = \sum_{n=1}^{+\infty}\left(h(\varkappa_n + \frac{1}{\lambda_n - \lambda_{n-1}}) - h(\varkappa_n)\right)$$
$$\geqslant \sum_{n=1}^{+\infty} \frac{h'(\varkappa_n)}{\lambda_n - \lambda_{n-1}} \geqslant \sum_{n=1}^{+\infty} \frac{h'(b\varphi_1(b\lambda_{n-1}))}{\lambda_n - \lambda_{n-1}} = +\infty.$$

The proof is complete. □

The next criterion is implied immediately by Theorems 6 and 7.

Theorem 8. *Let $\Phi_1 \in L$, $h \in L^+$ and φ_1 be the inverse function for the function Φ_1. For each entire function $F \in D_1(\Lambda, \Phi_1)$ relation (5) holds as $x \to +\infty$ outside some set E of a finite h-measure uniformly in $y \in \mathbb{R}$ if and only if (17) is true.*

It is worth noting that if condition (16) of Theorem 5 is not fulfilled, that is

$$(\exists b_1 > 0):\ \sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1} - \lambda_n} h'\left(\varphi_0(b_1\lambda_n) + \frac{b_1}{\lambda_{n+1} - \lambda_n}\right) = +\infty,$$

then for $b = \max\{b_1; 2\}$ we have

$$\sum_{n=0}^{+\infty} \frac{h'(b\varphi_0(b\lambda_n))}{\lambda_{n+1} - \lambda_n} = +\infty.$$

Therefore, condition (18) holds and according Theorem 7, there exist a function $F \in D_1(\Lambda, \Phi_0)$, a set $E \subset [0, +\infty)$ and a constant $\beta > 0$ such that inequalities (7) hold for all $x \in E$ and $h - \text{meas}\,(E) = +\infty$.

Since for $\Phi_0(x) = x^\alpha$, $\alpha > 0$, we have $D_0(\Lambda, \Phi_0) = D_1(\Lambda, \Phi_0)$, from Theorem 5 and 7 we obtain the following theorem.

Theorem 9. *Let $\Phi_0(x) = x^\alpha$ $(\alpha > 0)$, $h \in L^+$. For each entire function $F \in D_0(\Lambda, \Phi_0)$ relation (5) holds as $x \to +\infty$ outside some set E of a finite h-measure uniformly in $y \in \mathbb{R}$ if and only if*

$$(\forall b > 0):\ \sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1} - \lambda_n} h'\left(b(\lambda_n)^{1/\alpha} + \frac{b}{\lambda_{n+1} - \lambda_n}\right) < +\infty,$$

is true.

3. h-measure with a non-increasing density

We note that for each differentiable function $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ with a bounded derivative $h'(x) \leqslant c < +\infty$ $(x > 0)$ we have

$$\int_E dh(x) = \int_E h'(x)dx \leqslant c\int_E dx.$$

Hence, the finiteness of Lebesgue measure of a set $E \subset \mathbb{R}_+$ implies $h - \text{meas}\,(E) < +\infty$. Therefore, according Theorem A, condition (6) provides that the exceptional set E is of a finite h-measure. However, we conjecture that for $h \in L^-$ in the subclass

$$D_\varphi(\Lambda) = \big\{F \in D(\Lambda)\colon\ (\exists n_0)(\forall n \geqslant n_0)[|a_n| \leqslant \exp\{-\lambda_n\varphi(\lambda_n)\}]\big\}, \quad \varphi \in L,$$

condition (6) can be weakened significantly. The following conjecture seems to be true.

Conjecture 1. *Let $\varphi \in L$, $h \in L^-$. If*

$$\sum_{n=0}^{+\infty}\frac{h'(\varphi(\lambda_n))}{\lambda_{n+1}-\lambda_n} < +\infty,$$

then for all $F \in D_\varphi(\Lambda)$ relation (5) is true as $x \to +\infty$ outside some set E of finite h-measure uniformly in $y \in \mathbb{R}$.

4. $h-$measure and lacunary power series

The important corollaries for entire functions represented by a lacunary power series of the form (1) are implied by the proven theorems.

For an entire function f of the form (1) we let $F(z) = f(e^z)$, $z \in \mathbb{C}$.

We observe that as $x = \ln r$, $y = \varphi$,

$$F(x+iy) = F(\ln r + i\varphi) = f(re^{i\varphi})$$

and $M(x,F) = M_f(r)$, $m(x,F) = m_f(r)$, $\mu(x,F) = \mu_f(r)$, $\nu(x,F) = \nu_f(r)$. In addition, for $E_2 \stackrel{def}{=} \{r \in \mathbb{R} : \ln r \in E_1\}$ and h_1 such that $h_1'(x) = h'(e^x)$ we have

$$h - \log - \text{meas}(E_2) \stackrel{def}{=} \int_{E_2}\frac{dh(r)}{r} = \int_{E_1}\frac{dh(e^x)}{e^x} = \int_{E_1} dh_1(x) = h_1 - \text{meas}(E_1).$$

The next corollary is implied by Theorem B.

Corollary 1. *For each sequence (n_k) such that condition (6) holds and for each function $h \in L^+$ there exist an entire function f of the form (1), a constant $\beta > 0$ and a set E_2 of an infinite h-log-measure, i.e.$\big(\int_{E_2}\frac{dh(r)}{r} = +\infty\big)$ such that*

$$(\forall r \in E_2):\ M_f(r) \geqslant (1+\beta)\mu_f(r), \qquad M_f(r) \geqslant (1+\beta)m_f(r). \tag{22}$$

By Theorem 4 we obtain the following corollary.

Corollary 2. *Let $\Phi \in L$, $h \in L^+$ and φ be the inverse function for the function Φ. If for an entire function f of the form (1)*

$$\ln\mu_f(r) \geqslant \ln r\Phi(\ln r) \quad (r \geqslant r_0) \tag{23}$$

and

$$(\forall b > 0):\quad \sum_{k=0}^{+\infty}\frac{1}{n_{k+1}-n_k}h'\Big(\exp\Big\{\varphi(n_k) + \frac{b}{n_{k+1}-n_k}\Big\}\Big) < +\infty, \tag{24}$$

then the relation

$$f(re^{i\varphi}) = (1+o(1))a_{\nu_f(r)}r^{n_{\nu_f(r)}}e^{i\varphi n_{\nu_f(r)}} \tag{25}$$

holds as $r \to +\infty$ outside some set E_2 of finite h-log-measure uniformly in $\varphi \in [0, 2\pi]$.

In fact, it follows from condition (23) that $F \in D(\Lambda, \Phi)$ with $\Lambda = (n_k)$ and it remains to apply Theorem 4 with the function h_1.

Denote by $\mathcal{E}$ the class of entire functions of positive lower order, i.e.

$$\lambda_f := \varliminf_{r\to+\infty} \ln\ln M_f(r)/\ln r > 0.$$

By Theorem 8 we obtain the following corollary.

Corollary 3. *Let $h \in L^+$. In order the relations* (3) *hold for each function $f \in \mathcal{E}$ of the form* (1) *as $r \to +\infty$ outside a set of a finite h-log-measure, it is necessary and sufficient to have*

$$(\forall b > 0): \quad \sum_{k=0}^{+\infty} \frac{1}{n_{k+1} - n_k} h'\big((n_k)^b\big) < +\infty.$$

Acknowledgements

We are grateful to Prof. I.E. Chyzhykov and Dr. A.O. Kuryliak for helpful comments and corrections in the previous versions of this paper.

REFERENCES

1. P.C. Fenton. *The minimum modulus of gap power series* // Proc. Edinburgh Math. Soc. II. **21**:1, 49–54 (1978).
2. P. Erdös, A.J. Macintyre. *Integral functions with gap power series* // Proc. Edinburgh Math. Soc. II. **10**:2, 62–70 (1954).
3. O.B. Skaskiv. *Maximum of the modulus and maximal term of an integral Dirichlet series* // Dop. Akad. Nauk Ukr. SSR, Ser. A. 11, 22–24 (1984). (in Ukrainian)
4. R.P. Srivastava. *On the entire functions and their derivatives represented by Dirichlet series* // Ganita. **9**:2, 82–92 (1958).
5. O.B. Skaskiv, M.N. Sheremeta. *On the asymptotic behavior of entire Dirichlet series* // Matem. Sborn. **131**(173):3(11), 385–402 (1986). [Math. USSR. Sb. **59**:2, 379–396 (1988).
6. T.M. Salo, O.B. Skaskiv, O.M. Trakalo. *On the best possible description of exceptional set in Wiman–Valiron theory for entire functions* // Matem. Stud. **16**:2, 131–140 (2001).
7. T. Salo, O. Skaskiv. *On the maximum modulus and maximal term absolute convergent Dirichlet series* // Mat. Visn. Nauk. Tov. Im. Shevchenka. **4**, 564–574 (2007). (in Ukrainian).
8. M.R. Lutsyshyn, O.B. Skaskiv. *Asymptotic properties of a multiple Dirichlet series* // Matem. Stud. **3**, 41–48 (1994). (in Ukrainian).
9. S.Mandelbrojt. *Dirichlet series. Principles and methods.* Reidel Publishing Company, Dordrecht (1972).

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