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# ON A HILBERT SPACE OF ENTIRE FUNCTIONS

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**Abstract.** We consider the Hilbert space $F^2_\varphi$ of entire functions of $n$ variables constructed by means of a convex function $\varphi$ in $\mathbb{C}^n$ depending on the absolute value of the variable and growing at infinity faster than $a|z|$ for each $a > 0$. We study the problem on describing the dual space in terms of the Laplace transform of the functionals. Under certain conditions for the weight function $\varphi$, we obtain the description of the Laplace transform of linear continuous functionals on $F^2_\varphi$. The proof of the main result is based on using new properties of Young-Fenchel transform and one result on the asymptotics of the multi-dimensional Laplace integral established by R.A. Bashmakov, K.P. Isaev, R.S. Yulmukhametov.

**Keywords:** Hilbert space, Laplace transform, entire functions, convex functions, Young-Fenchel transform.

**Mathematics Subject Classification:** 32A15, 42B10, 46E10, 46F05, 42A38

## 1. Introduction

**1.1. Problem.** Let $H(\mathbb{C}^n)$ be the space of entire functions in $\mathbb{C}^n$, $d\mu_n$ be the Lebesgue measure in $\mathbb{C}^n$ and for $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ $(\mathbb{C}^n)$ we define $\operatorname{abs} u := (|u_1|, \dots, |u_n|)$.

We denote by $\mathcal{V}(\mathbb{R}^n)$ the set of all convex functions $g$ in $\mathbb{R}^n$ such that

1) $g(x_1, \dots, x_n) = g(|x_1|, \dots, |x_n|)$, $(x_1, \dots, x_n) \in \mathbb{R}^n$;

2) the restriction $g$ on $[0, \infty)^n$ is non-decaying in each variable;

3) $\lim\limits_{x \to \infty} \dfrac{g(x)}{\|x\|} = +\infty$; $\|x\|$ is the Euclidean norm of a point $x \in \mathbb{R}^n$).

To each function $\varphi \in \mathcal{V}(\mathbb{R}^n)$ we associate the Hilbert space

$$F^2_\varphi = \left\{ f \in H(\mathbb{C}^n) : \|f\|_\varphi = \left( \int_{\mathbb{C}^n} |f(z)|^2 e^{-2\varphi(\operatorname{abs} z)} \, d\mu_n(z) \right)^{\frac{1}{2}} < \infty \right\}$$

with the scalar product

$$(f, g)_\varphi = \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-2\varphi(\operatorname{abs} z)} \, d\mu_n(z), \ f, g \in F^2_\varphi.$$

If $\varphi(x) = \frac{\|x\|^2}{2}$, then $F^2_\varphi$ is the Fock space.

It is obvious that for each function $\varphi \in \mathcal{V}(\mathbb{R}^n)$ and each $\lambda \in \mathbb{C}^n$, the function $f_\lambda(z) = e^{\langle \lambda, z \rangle}$ belongs to $F^2_\varphi$. This is why for each linear continuous functional $S$ on the space $F^2_\varphi$, the function

$$\hat{S}(\lambda) = S(e^{\langle \lambda, z \rangle}), \quad \lambda \in \mathbb{C}^n,$$

is well defined in $\mathbb{C}^n$; this function is the Laplace transform of the functional $S$. It is easy to see that $\hat{S}$ is an entire function.

By $(F^2_\varphi)^*$ we denote the dual space for $F^2_\varphi$.

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The aim of the work is to find the conditions for $\varphi \in \mathcal{V}(\mathbb{R}^n)$, under which the space $\widehat{(F^2_\varphi)^*}$ of the Laplace transforms of the linear continuous functionals on $F^2_\varphi$ can be described as $F^2_{\varphi^*}$.

If $\varphi(x) = \frac{\|x\|^2}{2}$, then $\widehat{(F^2_\varphi)^*} = F^2_\varphi$. Indeed, in this case the problem on describing the space $(F^2_\varphi)^*$ in terms of the Laplace transform of the functionals is easily solved thanks to the classical representation: for each $f \in F^2_\varphi$,

$$f(\lambda) = \pi^{-n} \int_{\mathbb{C}^n} f(z) e^{\langle \lambda, \overline{z} \rangle - \|z\|^2} \, d\mu_n(z), \quad \lambda \in \mathbb{C}^n.$$

If the function $\varphi \in \mathcal{V}(\mathbb{R}^n)$ is radial, the mentioned problem was solved by V.V. Napalkov and S.V. Popenov [5], [6].

**1.2. Notations and definitions.** For $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n) \in \mathbb{R}^n(\mathbb{C}^n)$ we let $\langle u, v \rangle := u_1 v_1 + \dots + u_n v_n$, $\|u\|$ is the Euclidean norm of $u$.

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, by $|\alpha| := \alpha_1 + \ldots + \alpha_n$ we denote the length of the multi-index $\alpha$, $\tilde{\alpha} := (\alpha_1 + 1, \dots, \alpha_n + 1)$, and we denote $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $D^\alpha_z := \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}$.

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+$, $\varphi \in \mathcal{V}(\mathbb{R}^n)$, we define

$$c_\alpha(\varphi) := \int_{\mathbb{C}^n} |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n} e^{-2\varphi(\operatorname{abs} z)} \, d\mu_n(z).$$

For a function $u$ with a domain containing the set $(0, \infty)^n$, we define a function $u[e]$ in $\mathbb{R}^n$ by the rule:

$$u[e](x) = u(e^{x_1}, \dots, e^{x_n}), \ x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

By $\mathcal{B}(\mathbb{R}^n)$ we denote the set of all continuous functions $u : \mathbb{R}^n \to \mathbb{R}$ satisfying the condition

$$\lim_{x \to \infty} \frac{u(x)}{\|x\|} = +\infty.$$

The Young-Fenchel transform of the function $u : \mathbb{R}^n \to [-\infty, +\infty]$ is the function $u^* : \mathbb{R}^n \to [-\infty, +\infty]$ defined by the formula

$$u^*(x) = \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - u(y)), \quad x \in \mathbb{R}^n.$$

If $E$ is a convex domain in $\mathbb{R}^n$, $h$ is a convex set in $E$, $\tilde{E} = \{y \in \mathbb{R}^n : h^*(y) < \infty\}$, $p > 0$, then

$$D^h_y(p) := \{x \in E : h(x) + h^*(y) - \langle x, y \rangle \leqslant p\}, \quad y \in \tilde{E}.$$

By $V(D)$ we denote the $n$-dimensional volume of a set $D \subset \mathbb{R}^n$.

**1.3. Main result.**

**Theorem.** *Let $\varphi \in \mathcal{V}(\mathbb{R}^n)$ and for some $K > 0$ and each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ the inequalities*

$$\frac{1}{K} \leqslant V\left(D^{\varphi[e]}_\alpha\left(\frac{1}{2}\right)\right) V\left(D^{\varphi^*[e]}_\alpha\left(\frac{1}{2}\right)\right) \prod_{j=1}^n \alpha_j \leqslant K$$

*hold. Then the mapping $\mathcal{L} : S \in (F^2_\varphi)^* \to \hat{S}$ makes an isomorphism between the spaces $(F^2_\varphi)^*$ and $F^2_{\varphi^*}$.*

The proof of Theorem in Subsection 3.2 is based on new properties of Young-Fenchel transform, see Subsection 2.1, and one result on the asymptotics of the multi-dimensional Laplace integral in work [9], see Subsection 2.2.

## 2. Auxiliary data and results

**2.1. On some properties of Young-Fenchel transform.** It is easy to confirm that the following statement holds.

**Proposition 1.** *Let $u \in \mathcal{B}(\mathbb{R}^n)$. Then $(u[e])^*(x) > -\infty$ as $x \in \mathbb{R}^n$, $(u[e])^*(x) = +\infty$ as $x \notin [0,\infty)^n$ and $(u[e])^*(x) < +\infty$ as $x \in [0,\infty)^n$.*

We note that the last statement of Proposition 1 is implied, for instance, by the fact that for each $M > 0$ there exists a constant $A > 0$ such that

$$(u[e])^*(x) \leqslant \sum_{1 \leqslant j \leqslant n : x_j \neq 0} (x_j \ln \frac{x_j}{M} - x_j) + A, \quad x \in [0,\infty)^n.$$

**Proposition 2.** *Let $u \in \mathcal{B}(\mathbb{R}^n)$. Then*

$$\lim_{\substack{x \to \infty, \\ x \in [0,\infty)^n}} \frac{(u[e])^*(x)}{\|x\|} = +\infty.$$

*Proof.* For each $x \in [0,\infty)^n$ and $t \in \mathbb{R}^n$ we have

$$(u[e])^*(x) \geqslant \langle x, t \rangle - (u[e])(t).$$

Employing this inequality, we obtain that for each $M > 0$

$$(u[e])^*(x) \geqslant M\|x\| - u[e]\left(\frac{Mx}{\|x\|}\right), \quad x \in [0,\infty)^n \setminus \{0\}.$$

This completes the proof. □

The next three statements were proved in work [1], see there Lemma 6, Proposition 3, Proposition 4.

**Proposition 3.** *Let $u \in \mathcal{B}(\mathbb{R}^n)$. Then*

$$(u[e])^*(x) + (u^*[e])^*(x) \leqslant \sum_{\substack{1 \leqslant j \leqslant n: \\ x_j \neq 0}} (x_j \ln x_j - x_j), \quad x = (x_1, \dots, x_n) \in [0,\infty)^n \setminus \{0\};$$

$$(u[e])^*(0) + (u^*[e])^*(0) \leqslant 0.$$

**Proposition 4.** *Let $u \in \mathcal{B}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ be a convex function. Then*

$$(u[e])^*(x) + (u^*[e])^*(x) = \sum_{j=1}^{n} (x_j \ln x_j - x_j), \quad x = (x_1, \dots, x_n) \in (0,\infty)^n.$$

**Proposition 5.** *Let $u \in \mathcal{V}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ be a convex function. Then*

$$(u[e])^*(x) + (u^*[e])^*(x) = \sum_{\substack{1 \leqslant j \leqslant n: \\ x_j \neq 0}} (x_j \ln x_j - x_j), \quad x = (x_1, \dots, x_n) \in [0,\infty)^n \setminus \{0\};$$

$$(u[e])^*(0) + (u^*[e])^*(0) = 0.$$

Propositions 4 and 5 can be strengthen by employing the results by D. Azagra [2], [3]. He proved the following theorem.

**Theorem A.** *Let $U \subseteq \mathbb{R}^n$ be an open convex set. For each convex function $f : U \to \mathbb{R}$ and each $\varepsilon > 0$ there exists a real analytic convex function $g : U \to \mathbb{R}$ such that*

$$f(x) - \varepsilon \leqslant g(x) \leqslant f(x), \quad x \in U.$$

Thus, the following corollary hold [3].

**Corollary A.** *Let $U \subseteq \mathbb{R}^n$ be an open convex set. For each convex function $f : U \to \mathbb{R}$ and each $\varepsilon > 0$ there exists an infinitely differentiable convex function $g : U \to \mathbb{R}$ such that*

$$f(x) - \varepsilon \leqslant g(x) \leqslant f(x), \quad x \in U.$$

Employing Proposition 4 and Corollary A, we easily confirm the following statement.

**Proposition 6.** *Let $u \in \mathcal{B}(\mathbb{R}^n)$ be a convex function. Then*

$$(u[e])^*(x) + (u^*[e])^*(x) = \sum_{j=1}^{n} (x_j \ln x_j - x_j), \quad x = (x_1, \ldots, x_n) \in (0, \infty)^n.$$

Moreover, the following proposition is true.

**Proposition 7.** *Let $u \in \mathcal{V}(\mathbb{R}^n)$ be a convex function. Then*

$$(u[e])^*(x) + (u^*[e])^*(x) = \sum_{\substack{1 \leqslant j \leqslant n: \\ x_j \neq 0}} (x_j \ln x_j - x_j), \quad x = (x_1, \ldots, x_n) \in [0, \infty)^n \setminus \{0\};$$

$$(u[e])^*(0) + (u^*[e])^*(0) = 0.$$

*Proof.* According Proposition 6, our statement is true for the points $x \in (0, \infty)^n$. Assume that $x = (x_1, \ldots, x_n)$ belongs to the boundary of $[0, \infty)^n$ and $x \neq 0$. For the sake of simplicity we consider the case when the first $k$ $(1 \leqslant k \leqslant n - 1)$ coordinates of $x$ are positive and all other are equal to zero. For each $\xi = (\xi_1, \ldots, \xi_n)$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ we have

$$(u[e])^*(x) + (u^*[e])^*(x) \geqslant \sum_{j=1}^{k} x_j(\xi_j + \mu_j) - (u(e^{\xi_1}, \ldots, e^{\xi_n}) + u^*(e^{\mu_1}, \ldots, e^{\mu_n})).$$

By this inequality we obtain that

$$\begin{aligned}(u[e])^*(x) + (u^*[e])^*(x) \geqslant &\sum_{j=1}^{k} x_j(\xi_j + \mu_j) \\ &- (u(e^{\xi_1}, \ldots, e^{\xi_k}, 0, \ldots, 0) + u^*(e^{\mu_1}, \ldots, e^{\mu_k}, 0, \ldots, 0)).\end{aligned}$$

We define a function $u_k$ on $\mathbb{R}^k$ by the rule: $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k \to u(\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$. We observe that for each $t = (t_1, \ldots, t_k) \in \mathbb{R}^k$, $\breve{t} = (t_1, \ldots, t_k, 0, \ldots, 0) \in \mathbb{R}^n$ we have

$$\begin{aligned}u^*(\breve{t}) &= \sup_{v \in \mathbb{R}^n} (\langle \breve{t}, v \rangle - u(v)) \\ &\leqslant \sup_{v_1, \ldots, v_k \in \mathbb{R}} (\sum_{j=1}^{k} t_j v_j - u(v_1, \ldots, v_k, 0, \ldots, 0)) = \sup_{v \in \mathbb{R}^k} (\langle t, v \rangle - u_k(v)) = u_k^*(t).\end{aligned}$$

Employing this and the above inequality, for $\tilde{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ and each $\tilde{\xi} = (\xi_1, \ldots, \xi_k), \tilde{\mu} = (\mu_1, \ldots, \mu_k) \in \mathbb{R}^k$ we have

$$(u[e])^*(x) + (u^*[e])^*(x) \geqslant \langle \tilde{x}, \tilde{\xi} \rangle - u_k[e](\tilde{\xi}) + \langle \tilde{x}, \tilde{\mu} \rangle - u_k^*[e](\tilde{\mu}).$$

Therefore,

$$(u[e])^*(x) + (u^*[e])^*(x) \geqslant (u_k[e])^*(\tilde{x}) + (u_k^*[e])^*(\tilde{x}).$$

Since by the Proposition 6,

$$(u_k[e])^*(\tilde{x}) + (u_k^*[e])^*(\tilde{x}) = \sum_{j=1}^{k} (x_j \ln x_j - x_j),$$

then $(u[e])^*(x)+(u^*[e])^*(x) \geqslant \sum_{j=1}^{k}(x_j \ln x_j - x_j)$. By Proposition 3 this implies the first statement of the proposition.

If $x = 0$, then

$$(u[e])^*(0) = -\inf_{\xi\in\mathbb{R}^n} u[e](\xi) = -u(0),$$
$$(u^*[e])^*(0) = -\inf_{\xi\in\mathbb{R}^n} u^*[e](\xi) = -u^*(0) = \inf_{\xi\in\mathbb{R}^n} u(\xi) = u(0).$$

Therefore, $(u[e])^*(0) + (u^*[e])^*(0) = 0$. □

**2.2. Asymptotics of multi-dimensional Laplace integral.** In work [9] there was established the following theorem.

**Theorem B.** *Let $E$ be a convex domain in $\mathbb{R}^n$, $h$ be a convex function in $E$, $\tilde{E} = \{y \in \mathbb{R}^n : h^*(y) < \infty\}$ and the interior of $\tilde{E}$ is non-empty. Let*

$$D^h = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : h(x) + h^*(y) - \langle x, y\rangle \leqslant 1\},$$
$$D^h_y = \{x \in \mathbb{R}^n : (x, y) \in D\}, \ y \in \mathbb{R}^n.$$

*Then*

$$e^{-1}V(D^h_y)e^{h^*(y)} \leqslant \int_{\mathbb{R}^n} e^{\langle x,y\rangle - h(x)}\, dx \leqslant (1 + n!)V(D^h_y)e^{h^*(y)}, \quad y \in \tilde{E}.$$

Here we assume that $h(x) = +\infty$ as $x \notin E$.

## 3. Description of dual space

**3.1. Auxiliary lemmata.** In the proof of Theorem the following four lemmata will be useful.

**Lemma 1.** *Let $\varphi \in \mathcal{V}(\mathbb{R}^n)$. Then the system $\{\exp\langle\lambda, z\rangle\}_{\lambda\in\mathbb{C}^n}$ is complete in $F^2_\varphi$.*

*Proof.* Let $S$ be a linear continuous functional on the space $F^2_\varphi$ such that $S(e^{\langle\lambda,z\rangle}) = 0$ for each $\lambda \in \mathbb{C}^n$. Since for each multi-index $\alpha \in \mathbb{Z}^n_+$ we have $(D^\alpha_\lambda \hat{S})(\lambda) = S(z^\alpha e^{\langle z,\lambda\rangle})$, this identity implies that $S(z^\alpha) = 0$. Since the function $\varphi(|z_1|, \dots, |z_n|)$ is convex in $\mathbb{C}^n$, it follows from the result by B.A. Taylor on the weight approximation of entire functions by polynomials [4, Thm. 2] that the polynomials are dense in $F^2_\varphi$. Hence, $S$ is the zero functional. By the known corollary of Khan-Banach theorem we obtain that the system $\{\exp\langle\lambda, z\rangle\}_{\lambda\in\mathbb{C}^n}$ is complete in $F^2_\varphi$. □

We note that the system $\{z^\alpha\}_{|\alpha|\geqslant 0}$ is orthogonal in $F^2_\varphi$. Moreover, it is dense in $F^2_\varphi$. Therefore, the system $\{z^\alpha\}_{|\alpha|\geqslant 0}$ is a basis in $F^2_\varphi$.

**Lemma 2.** *Let $\varphi \in \mathcal{V}(\mathbb{R}^n)$. Then*

$$c_\alpha(\varphi) \geqslant \frac{\pi^n}{\tilde{\alpha}_1 \cdots \tilde{\alpha}_n} e^{2(\varphi[e])^*(\tilde{\alpha})}, \quad \alpha \in \mathbb{Z}^n_+.$$

*In particular, for each $M > 0$ there exists a constant $C_M > 0$ such that $c_\alpha(\varphi) \geqslant C_M M^{|\alpha|}$ for each $\alpha \in \mathbb{Z}^n_+$*

*Proof.* For each $\alpha \in \mathbb{Z}_+^n$ and each positive numbers $R_1, \dots, R_n$ we have

$$\begin{aligned} c_\alpha(\varphi) =&(2\pi)^n \int\limits_0^\infty \cdots \int\limits_0^\infty r_1^{2\alpha_1+1} \cdots r_n^{2\alpha_n+1} e^{-2\varphi(r_1,\cdots,r_n)}\, dr_1 \cdots dr_n \\ \geqslant&(2\pi)^n \int\limits_0^{R_1} \cdots \int\limits_0^{R_n} r_1^{2\alpha_1+1} \cdots r_n^{2\alpha_n+1} e^{-2\varphi(R_1,\cdots,R_n)}\, dr_1 \cdots dr_n \\ =&(2\pi)^n \frac{R_1^{2\alpha_1+2}}{2\alpha_1+2} \cdots \frac{R_n^{2\alpha_n+2}}{2\alpha_n+2} e^{-2\varphi(R_1,\cdots,R_n)}. \end{aligned}$$

This implies that for each $t \in \mathbb{R}^n$

$$c_\alpha(\varphi) \geqslant \frac{\pi^n}{\tilde\alpha_1 \cdots \tilde\alpha_n} e^{\langle 2\tilde\alpha, t\rangle - 2\varphi[e](t)}.$$

Therefore,

$$c_\alpha(\varphi) \geqslant \frac{\pi^n}{\tilde\alpha_1 \cdots \tilde\alpha_n} e^{2(\varphi[e])^*(\tilde\alpha)}.$$

Employing now Proposition 2, we obtain easily the second statement of the lemma. □

**Lemma 3.** *Assume that an entire in $\mathbb{C}^n$ function satisfies $f(z) = \sum\limits_{|\alpha|\geqslant 0} a_\alpha z^\alpha \in F_\varphi^2$. Then*

$$\sum_{|\alpha|\geqslant 0} |a_\alpha|^2 c_\alpha(\varphi) < \infty \quad \textit{and} \quad \|f\|_\varphi^2 = \sum_{|\alpha|\geqslant 0} |a_\alpha|^2 c_\alpha(\varphi).$$

*And vice versa, let the sequence $(a_\alpha)_{|\alpha|\geqslant 0}$ of complex number $a_\alpha$ is such that the series $\sum\limits_{|\alpha|\geqslant 0} |a_\alpha|^2 c_\alpha(\varphi)$ converges. Then $f(z) = \sum\limits_{|\alpha|\geqslant 0} a_\alpha z^\alpha \in H(\mathbb{C}^n)$. At that, $f \in F_\varphi^2$.*

*Proof.* Let

$$f(z) = \sum_{|\alpha|\geqslant 0} a_\alpha z^\alpha$$

be an entire function in $\mathbb{C}^n$ in the class $F_\varphi^2$. Then

$$\begin{aligned} \|f\|_\varphi^2 &= \int_{\mathbb{C}^n} |f(z)|^2 e^{-2\varphi(\text{abs } z)}\, d\lambda(z) = \int_{\mathbb{C}^n} \sum_{|\alpha|\geqslant 0} a_\alpha z^\alpha \sum_{|\beta|\geqslant 0} \overline{a}_\beta \overline{z}^\beta e^{-2\varphi(\text{abs } z)}\, d\mu_n(z) \\ &= \sum_{|\alpha|\geqslant 0} |a_\alpha|^2 \int_{\mathbb{C}^n} |z_1|^{2\alpha_1} \cdots |z_n|^{2\alpha_n} e^{-2\varphi(\text{abs } z)}\, d\mu_n(z) = \sum_{|\alpha|\geqslant 0} |a_\alpha|^2 c_\alpha(\varphi). \end{aligned}$$

Vice versa, the convergence of the series $\sum\limits_{|\alpha|\geqslant 0} |a_\alpha|^2 c_\alpha(\varphi)$ and Lemma 2 implies that for each $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that $|a_\alpha| \leqslant c_\varepsilon \varepsilon^{|\alpha|}$ for each $\alpha \in \mathbb{Z}_+^n$. This means that $f(z) = \sum\limits_{|\alpha|\geqslant 0} a_\alpha z^\alpha$ is an entire function in $\mathbb{C}^n$. It is easy to see that $f \in F_\varphi^2$. □

**Lemma 4.** *Let $\varphi \in \mathcal{V}(\mathbb{R}^n)$. Then*

$$(2\pi)^n e^{-1} V\left(D_{\tilde\alpha}^{\varphi[e]}\left(\frac{1}{2}\right)\right) e^{2(\varphi[e])^*(\tilde\alpha)} \leqslant c_\alpha(\varphi) \leqslant (2\pi)^n (1+n!) V\left(D_{\tilde\alpha}^{\varphi[e]}\left(\frac{1}{2}\right)\right) e^{2(\varphi[e])^*(\tilde\alpha)}$$

*for each $\alpha \in \mathbb{Z}_+^n$.*

*Proof.* Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$. Then

$$\begin{aligned} c_\alpha(\varphi) =&(2\pi)^n \int\limits_0^\infty \cdots \int\limits_0^\infty r_1^{2\alpha_1+1} \cdots r_n^{2\alpha_n+1} e^{-2\varphi(r_1,\cdots,r_n)}\, dr_1 \cdots dr_n \\ =&(2\pi)^n \int\limits_{-\infty}^\infty \cdots \int\limits_{-\infty}^\infty e^{(2\alpha_1+2)t_1+\cdots+(2\alpha_n+2)t_n-2\varphi[e](t_1,\dots,t_n)}\, dt_1 \cdots dt_n. \end{aligned}$$

That is,

$$c_\alpha(\varphi) = (2\pi)^n \int_{\mathbb{R}^n} e^{\langle 2\tilde{\alpha}, t\rangle - 2\varphi[e](t)}\, dt.$$

By Theorem B we have

$$(2\pi)^n e^{-1} V\big(D_{2\tilde{\alpha}}^{2\varphi[e]}\big) e^{2(\varphi[e])^*(\tilde{\alpha})} \leqslant c_\alpha(\varphi) \leqslant (2\pi)^n (1+n!) V\big(D_{2\tilde{\alpha}}^{2\varphi[e]}\big) e^{2(\varphi[e])^*(\tilde{\alpha})}$$

Since $D_{2\tilde{\alpha}}^{2\varphi[e]} = D_{\tilde{\alpha}}^{\varphi[e]}\left(\frac{1}{2}\right)$, by the previous inequality this completes the proof. □

**3.2. Proof of Theorem.** Let us prove that the mapping $\mathcal{L}$ acts from $(F_\varphi^2)^*$ into $F_{\varphi^*}^2$. Let $S \in (F_\varphi^2)^*$. Then there exists a function $g_S \in F_\varphi^2$ such that $S(f) = (f, g_S)_\varphi$, that is,

$$S(f) = \int_{\mathbb{C}^n} f(z) \overline{g_S(z)} e^{-2\varphi(\text{abs } z)}\, d\mu_n(z), \quad f \in F_\varphi^2.$$

At that, $\|S\| = \|g_S\|_\varphi$. If $g_S(z) = \sum\limits_{|\alpha|\geqslant 0} b_\alpha z^\alpha$, then $\hat{S}(\lambda) = \sum_{|\alpha|\geqslant 0} \frac{c_\alpha(\varphi)\overline{b_\alpha}}{\alpha!} \lambda^\alpha, \ \lambda \in \mathbb{C}^n$. Therefore,

$$\|\hat{S}\|_{\varphi^*}^2 = \sum_{|\alpha|\geqslant 0} \left(\frac{c_\alpha(\varphi)|b_\alpha|}{\alpha!}\right)^2 c_\alpha(\varphi^*). \tag{1}$$

By Lemma 3,

$$\begin{aligned} c_\alpha(\varphi) &\leqslant (2\pi)^n (1+n!) V\left(D_{\tilde{\alpha}}^{\varphi[e]}\left(\frac{1}{2}\right)\right) e^{2(\varphi[e])^*(\tilde{\alpha})}, \\ c_\alpha(\varphi^*) &\leqslant (2\pi)^n (1+n!) V\left(D_{\tilde{\alpha}}^{\varphi^*[e]}\left(\frac{1}{2}\right)\right) e^{2(\varphi^*[e])^*(\tilde{\alpha})} \end{aligned}$$

for each $\alpha \in \mathbb{Z}_+^n$.

Therefore,

$$c_\alpha(\varphi) c_\alpha(\varphi^*) \leqslant (2\pi)^{2n} (1+n!)^2 V\left(D_{\tilde{\alpha}}^{\varphi[e]}\left(\frac{1}{2}\right)\right) V\left(D_{\tilde{\alpha}}^{\varphi^*[e]}\left(\frac{1}{2}\right)\right) e^{2(\varphi[e])^*(\tilde{\alpha}) + 2(\varphi^*[e])^*(\tilde{\alpha})}$$

for each $\alpha \in \mathbb{Z}_+^n$.

According Proposition 6, for each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ we have

$$(\varphi[e])^*(\tilde{\alpha}) + (\varphi^*[e])^*(\tilde{\alpha}) = \sum_{j=1}^n ((\alpha_j+1)\ln(\alpha_j+1) - (\alpha_j+1)).$$

Since by the Stirling's formula [10], for each $m \in \mathbb{Z}_+$ we have

$$(m+1)\ln(m+1) - (m+1) = \ln \Gamma(m+1) - \ln\sqrt{2\pi} + \frac{1}{2}\ln(m+1) - \frac{\theta}{12(m+1)},$$

where $\theta \in (0,1)$ depends on $m$, then

$$(\varphi[e])^*(\tilde{\alpha}) + (\varphi^*[e])^*(\tilde{\alpha}) = -n\ln\sqrt{2\pi} + \sum_{j=1}^n \left(\ln\Gamma(\alpha_j+1) + \frac{1}{2}\ln(\alpha_j+1) - \frac{\theta_j}{12(\alpha_j+1)}\right),$$

where $\theta_j \in (0,1)$ depends on $\alpha_j$. Then

$$\frac{e^{2((\varphi[e])^*(\tilde{\alpha})+(\varphi^*[e])^*(\tilde{\alpha}))}}{\alpha!^2} = \frac{1}{(2\pi)^n}\prod_{j=1}^{n}(\alpha_j+1)e^{-\frac{\theta_j}{6(\alpha_j+1)}}. \qquad (2)$$

Thus,

$$\frac{c_\alpha(\varphi)c_\alpha(\varphi^*)}{\alpha!^2} \leqslant (2\pi)^n(1+n!)^2 V\left(D_{\tilde{\alpha}}^{\varphi[e]}\left(\frac{1}{2}\right)\right) V\left(D_{\tilde{\alpha}}^{\varphi^*[e]}\left(\frac{1}{2}\right)\right)\prod_{j=1}^{n}\tilde{\alpha}_j.$$

Employing the condition for $\varphi$, we obtain that

$$\frac{c_\alpha(\varphi)c_\alpha(\varphi^*)}{\alpha!^2} \leqslant (2\pi)^n(1+n!)^2 K$$

for each $\alpha \in \mathbb{Z}_+^n$. Letting $M_1 = (2\pi)^n(1+n!)^2K$, by (1) we obtain

$$\|\hat{S}\|_{\varphi^*}^2 \leqslant M_1 \sum_{|\alpha|\geqslant 0} c_\alpha(\varphi)|b_\alpha|^2 = M_1\|g_S\|_\varphi^2 = M_1\|S\|^2.$$

Hence, $\hat{S} \in F_{\varphi^*}^2$. Moreover, the latter estimate implies that the linear mapping $\mathcal{L}$ acts continuously from $(F_\varphi^2)^*$ into $F_{\varphi^*}^2$.

We observe that the mapping $\mathcal{L}$ is injective from $(F_\varphi^2)^*$ into $F_{\varphi^*}^2$ since by Lemma 1 the system $\{\exp\langle\lambda, z\rangle\}_{\lambda\in\mathbb{C}^n}$ is complete in $F_\varphi^2$.

Let us show that the mapping $\mathcal{L}$ acts from $(F_\varphi^2)^*$ onto $F_{\varphi^*}^2$. Assume that $G \in F_{\varphi^*}^2$. Employing the representation of an entire function $G$ by the Taylor series

$$G(\lambda) = \sum_{|\alpha|\geqslant 0} d_\alpha\lambda^\alpha, \quad \lambda \in \mathbb{C}^n,$$

we get

$$\|G\|_{\varphi^*}^2 = \sum_{|\alpha|\geqslant 0} |d_\alpha|^2 c_\alpha(\varphi^*).$$

For each $\alpha \in \mathbb{Z}_+^n$ we define the numbers $g_\alpha = \frac{\overline{d_\alpha}\alpha!}{c_\alpha(\varphi)}$ and consider the convergence of the series $\sum\limits_{|\alpha|\geqslant 0} |g_\alpha|^2 c_\alpha(\varphi)$. We have

$$\sum_{|\alpha|\geqslant 0}|g_\alpha|^2c_\alpha(\varphi) = \sum_{|\alpha|\geqslant 0}\left|\frac{\overline{d_\alpha}\alpha!}{c_\alpha(\varphi)}\right|^2 c_\alpha(\varphi) = \sum_{|\alpha|\geqslant 0}\frac{\alpha!^2}{c_\alpha(\varphi)c_\alpha(\varphi^*)}|d_\alpha|^2c_\alpha(\varphi^*).$$

By Lemma 4,

$$c_\alpha(\varphi) \geqslant e^{-1}V\left(D_{\tilde{\alpha}}^{\varphi[e]}\left(\frac{1}{2}\right)\right)e^{2(\varphi[e])^*(\tilde{\alpha})}, \quad c_\alpha(\varphi^*) \geqslant e^{-1}V\left(D_{\tilde{\alpha}}^{\varphi^*[e]}\left(\frac{1}{2}\right)\right)e^{2(\varphi^*[e])^*(\tilde{\alpha})}$$

for each $\alpha \in \mathbb{Z}_+^n$. Therefore,

$$c_\alpha(\varphi)c_\alpha(\varphi^*) \geqslant e^{-2}V\left(D_{\tilde{\alpha}}^{\varphi[e]}\left(\frac{1}{2}\right)\right)V\left(D_{\tilde{\alpha}}^{\varphi^*[e]}\left(\frac{1}{2}\right)\right)e^{2((\varphi[e])^*(\tilde{\alpha})+(\varphi^*[e])^*(\tilde{\alpha}))}$$

for each $\alpha \in \mathbb{Z}_+^n$. By identity (2) this implies

$$\frac{\alpha!^2}{c_\alpha(\varphi)c_\alpha(\varphi^*)} \leqslant \frac{e^2(2e\pi)^n}{V\left(D_{\tilde{\alpha}}^{\varphi[e]}\left(\frac{1}{2}\right)\right)V\left(D_{\tilde{\alpha}}^{\varphi^*[e]}\left(\frac{1}{2}\right)\right)\prod\limits_{j=1}^{n}(\alpha_j+1)}$$

for each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$.

Employing the condition for $\varphi$, we obtain that $\frac{\alpha!^2}{c_\alpha(\varphi)c_\alpha(\varphi^*)} \leqslant Ke^2(2e\pi)^n$, $\forall \alpha \in \mathbb{Z}_+^n$. Therefore, for the considered series we have

$$\sum_{|\alpha|\geqslant 0} |g_\alpha|^2 c_\alpha(\varphi) \leqslant Ke^2(2e\pi)^n \sum_{|\alpha|\geqslant 0} |d_\alpha|^2 c_\alpha(\varphi^*) = Ke^2(2e\pi)^n \|G\|^2_{\varphi^*}. \tag{3}$$

Thus, the series $\sum\limits_{|\alpha|\geqslant 0} |g_\alpha|^2 c_\alpha(\varphi)$ converges. But by Lemma 3 the function

$$g(\lambda) = \sum_{|\alpha|\geqslant 0} g_\alpha \lambda^\alpha, \quad \lambda \in \mathbb{C}^n,$$

is entire and by (3), $g$ belongs to $F^2_\varphi$ and

$$\|g\|^2_\varphi \leqslant Ke^2(2e\pi)^n \|G\|^2_{\varphi^*}. \tag{4}$$

We define a functional $S$ on $F^2_\varphi$ by the formula

$$S(f) = \int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-2\varphi(\mathrm{abs} z)} \, d\mu_n(z), \quad f \in F^2_\varphi.$$

It is clear that $S$ is a linear continuous functional on $F^2_\varphi$. At that, $\hat{S} = G$. Since $\|S\| = \|g\|_\varphi$, estimate (4) shows that the inverse mapping $\mathcal{L}^{-1}$ is continuous. Thus, $\mathcal{L}$ makes an isomorphism between the spaces $(F^2_\varphi)^*$ and $F^2_{\varphi^*}$. The proof is complete.

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