

ON 2-GENERATENESS OF WEAKLY LOCALIZABLE SUBMODULES IN THE MODULE OF ENTIRE FUNCTIONS OF EXPONENTIAL TYPE AND POLYNOMIAL GROWTH ON THE REAL AXIS

N.F. ABUZYAROVA

Abstract. In the work we consider a topological module $\mathcal{P}(a; b)$ of entire functions, which is the isomorphic image under the Fourier-Laplace transform of the Schwarz space of distributions with compact supports in a finite or infinite interval $(a; b) \subset \mathbb{R}$. We prove that each weakly localizable module in $\mathcal{P}(a; b)$ is either generated by its two elements or is equal to the closure of two submodules of special form. We also provide dual results on subspaces in $C^\infty(a; b)$ invariant w.r.t. the differentiation operator.

Keywords: entire functions, subharmonic functions, Fourier-Laplace transform, finitely generated submodules, description of submodules, local description of submodules, invariant subspaces, spectral synthesis.

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1. INTRODUCTION

Let $[a_1; b_1] \Subset [a_2; b_2] \Subset \dots$ be a sequence of a finite or infinite interval $(a; b)$ of the real axis, P_k be a Banach space formed by all entire functions φ having a finite norm

$$\|\varphi\|_k = \sup_{z \in \mathbb{C}} \frac{|\varphi(z)|}{(1 + |z|)^k \exp(b_k y^+ - a_k y^-)}, \quad y^\pm = \max\{0, \pm y\}, \quad z = x + iy. \quad (1.1)$$

We denote by $\mathcal{P}(a; b)$ the inductive limit of the sequence $\{P_k\}$. Each of the embeddings $P_k \subset P_{k+1}$ is completely continuous and hence, $\mathcal{P}(a; b)$ is a locally convex space of type (LN^*) (see [1]). It is known (see, for instance, [2, Sect. 16.1]) that each element φ of space $\mathcal{P}(a; b)$ is an entire function of completely regular growth at order 1, and its indicator diagram is a segment of the imaginary axis $i[c_\varphi; d_\varphi] \subset i(a; b)$.

By $\mathcal{P}_0(a; b)$ we denote a linear subspace of space $\mathcal{P}(a; b)$ formed by all functions φ which decay fast on the real axis:

$$|\varphi(x)| = o(|x|^n), \quad n \in \mathbb{N}.$$

In the space $\mathcal{P}(a; b)$, the multiplication by the independent variable z is a continuous mapping, and this is why $\mathcal{P}(a; b)$ is a topological module over the ring of polynomials $\mathbb{C}[z]$. For the sake of brevity, if else is not said, we shall use the word “submodule” for a closed submodule of module $\mathcal{P}(a; b)$, i.e., a closed subspace invariant w.r.t. the multiplication by z .

We denote by $\mathcal{J}_{\varphi_1, \dots, \varphi_m}$ a submodule *generated by functions* $\varphi_1, \dots, \varphi_m \in \mathcal{P}(a; b)$ (or *m-generated*):

$$\mathcal{J}_{\varphi_1, \dots, \varphi_m} = \overline{\{p_1 \varphi_1 + \dots + p_m \varphi_m, \quad p_1, \dots, p_m \in \mathbb{C}[z]\}}, \quad (1.2)$$

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Functions $\varphi_1, \dots, \varphi_m$ are called *generators* of submodule $\mathcal{J}_{\varphi_1, \dots, \varphi_m}$. A submodule with a single generator is called *principle*.

In what follows we provide the definition of notions characterizing the properties of the submodules and used in the issues on the local description (see [3] – [6]).

Given a submodule $\mathcal{J} \subset \mathcal{P}(a; b)$, we let $c_{\mathcal{J}} = \inf_{\varphi \in \mathcal{J}} c_{\varphi}$, $d_{\mathcal{J}} = \sup_{\varphi \in \mathcal{J}} d_{\varphi}$. The set $[c_{\mathcal{J}}; d_{\mathcal{J}}]$ is called the *indicator segment* of submodule \mathcal{J} .

The *divisor* of a function $\varphi \in \mathcal{P}(a; b)$ is defined by the formula

$$n_{\varphi}(\lambda) = \begin{cases} 0, & \text{if } \varphi(\lambda) \neq 0, \\ m, & \text{if } \lambda \text{ is a zero of multiplicity } m, \end{cases}$$

for all $\lambda \in \mathbb{C}$.

The *divisor* of a submodule $\mathcal{J} \subset \mathcal{P}(a; b)$ is defined by the formula $n_{\mathcal{J}}(\lambda) = \min_{\varphi \in \mathcal{J}} n_{\varphi}(\lambda)$. Then we introduce the *zero set* Λ_{φ} of a function φ :

$$\Lambda_{\varphi} = \{(\lambda_k; m_k) : n_{\varphi}(\lambda_k) = m_k > 0\},$$

and the *zero set* $\Lambda_{\mathcal{J}}$ of submodule \mathcal{J} :

$$\Lambda_{\mathcal{J}} = \{(\lambda_k; m_k) : n_{\mathcal{J}}(\lambda_k) = m_k > 0\}.$$

Submodule \mathcal{J} is *weakly localizable* if it contains all functions $\varphi \in \mathcal{P}(a; b)$ satisfying the conditions:

- 1) $n_{\varphi}(z) \geq n_{\mathcal{J}}(z)$, $z \in \mathbb{C}$;
- 2) the indicator diagram of function φ is contained in the set $i[c_{\mathcal{J}}; d_{\mathcal{J}}]$. In the case $c_{\mathcal{J}} = a$ and $d_{\mathcal{J}} = b$ the weak localizable property of \mathcal{J} means that this submodule is *localizable (ample)*.

Let $\varphi \in \mathcal{P}(a; b)$, $c, d \in \overline{\mathbb{R}}$ and

$$a \leq c \leq c_{\varphi} \leq d_{\varphi} \leq d \leq b.$$

We denote by $\mathcal{J}(\varphi, \langle c; d \rangle)$ a submodule in $\mathcal{P}(a; b)$ formed by all functions $\psi \in \mathcal{P}(a; b)$ with the set of zeroes $\Lambda_{\psi} \supset \Lambda_{\varphi}$ and the indicator diagram $i[c_{\psi}; d_{\psi}] \subset i[c; d]$; hereinafter symbol “ $\langle \rangle$ ” stands for the bracket “[” or “(” subject to which of the relations $a = c$ or $a < c$ holds true. In the same way we treat the bracket “ \rangle ”. It is clear that the submodule $\mathcal{J}(\varphi, \langle c; d \rangle)$ is weakly localizable. For the submodule $\mathcal{J}(\varphi, [c_{\varphi}; d_{\varphi}])$ we shall employ a shorter notation $\mathcal{J}(\varphi)$.

A submodule \mathcal{J} is called *stable at a point* $\lambda \in \mathbb{C}$ if conditions $\varphi \in \mathcal{J}$ and $n_{\varphi}(\lambda) > n_{\mathcal{J}}(\lambda)$ imply the belonging $\frac{\varphi}{z-\lambda} \in \mathcal{J}$. A submodule \mathcal{J} is *stable* if it is stable at each point $\lambda \in \mathbb{C}$.

It is easy to see that a *stability of submodule \mathcal{J} is a necessary condition for its weak localizable property*. However, not each stable submodule $\mathcal{P}(a; b)$ is weakly localizable. Indeed, it follows from the results of work [7, Sect. 4] that each principle submodule in $\mathcal{P}(a; b)$ is stable. It can be also checked straightforwardly by employing the definition of stability and the description of the topology in $\mathcal{P}(a; b)$. On the other hand, an example constructed in [8] as well as Theorem 3 of work [9] show that not all principle submodules in the submodule $\mathcal{P}(a; b)$ are weakly localizable. Thus, the statement that each stable finite generated submodule in $\mathcal{P}(a; b)$ is weakly localizable is wrong.

In the present work we prove the inverse statement: each weakly localizable submodule $\mathcal{J} \subset \mathcal{P}(a; b)$ either is generated by two (probably, coinciding) elements or is equal to the closure of the sum of two (probably, coinciding) submodules of the form $\mathcal{J}(\varphi, \langle c; d \rangle)$. In [3, Thms. 4, 5] we announced less general statements.

The issue on 2-degenerateness in a wide sense was studied earlier for localizable (ample) submodules in the module of entire functions of finite order determined by restrictions for the indicator [10], [11], for localized (ample) submodules in abstract weighted submodules of holomorphic functions [12], for submodules with a finite zero set in the module $\mathcal{P}(a; b)$ [4]. One of the results of work [12] is the theorem stating that localizable (ample) submodules of the

module $\mathcal{P}(a; b)$ are generated by two submodules of the form $\mathcal{J}(\varphi, (a; b))$. We note that by the abstract part of the paper [12] one can get Statement 1) of Theorem 1 in the present for the case $c_{\mathcal{J}} = a$ or (and) $d_{\mathcal{J}} = b$. Other statements on 2-generateness on weakly localizable submodules in $\mathcal{P}(a; b)$ proven here, namely, Statement 2) of Theorem 1, Theorem 3 and Statement 1) of Theorem 1 in the general formulation, can not be obtained by means of the results of the work [12].

The further presentation is as follows. The second section contains theorems on 2-generateness in a wide sense of an arbitrary localizable submodule \mathcal{J} in $\mathcal{P}(a; b)$ (Theorems 1 and 3). In the third section by these theorems we obtain dual statements on the structure of closed subspaces in space $C^\infty(a; b)$ invariant w.r.t. the differentiation operator.

2. STRUCTURE OF WEAKLY LOCALIZABLE SUBMODULES

Theorem 1. *Let $\mathcal{J} \subset \mathcal{P}(a; b)$ be a weakly localizable submodule.*

1) *If \mathcal{J} contains functions in $\mathcal{P}_0(a; b)$, then for each function $\varphi_1 \in \mathcal{J} \cap \mathcal{P}_0(a; b)$ there exists infinitely many functions $\varphi_2 \in \mathcal{J} \cap \mathcal{P}_0(a; b)$ possessing the property*

$$\mathcal{J} = \overline{\mathcal{J}(\varphi_1, \langle c_{\mathcal{J}}; d_{\mathcal{J}} \rangle) + \mathcal{J}(\varphi_2, \langle c_{\mathcal{J}}; d_{\mathcal{J}} \rangle)}. \quad (2.1)$$

2) *If $\mathcal{J} \cap \mathcal{P}_0(a; b) = \emptyset$, there exists a function $\varphi_0 \in \mathcal{J}$ such that*

$$\mathcal{J} = \mathcal{J}_{\varphi_0} = \{p\varphi_0, \quad p \in \mathbb{C}[z]\}. \quad (2.2)$$

Proof. 1) The first of the formulate statements can be proved in the same way as Theorem 2 in the work [4], where stable submodules with a finite set of zeroes were considered.

Without loss of generality we can assume that $0 \notin \Lambda_{\mathcal{J}}$ and $\varphi_1(0) = 1$. Let $\Lambda_{\varphi_1} = \{\lambda_j\}$, $|\lambda_1| \leq |\lambda_2| \leq \dots$, and each zeroes is taken counting the multiplicity.

We choose and fix two numbers $a', b' \in \mathbb{R}$ satisfying the conditions

$$a \leq a' < c_{\varphi_1} \leq d_{\varphi_1} < b' \leq b, \quad a' \leq c_{\mathcal{J}}, \quad d_{\mathcal{J}} \leq b',$$

where $c_{\varphi_1} = h_{\varphi_1}(-\pi/2)$, $d_{\varphi_1} = h_{\varphi_1}(\pi/2)$, h_{φ_1} is the indicator of the function φ_1 . We also choose and fix a sequence $\tilde{\Gamma} = \{\tilde{\gamma}_k\}$, $0 \notin \tilde{\Gamma}$ close to Λ_{φ_1} so that both sequences Λ_{φ_1} and $\tilde{\Gamma}$ satisfy the condition

$$\sum_{j=1}^{\infty} \frac{|\lambda_j - \tilde{\gamma}_j|}{1 + |\operatorname{Im} \lambda_j| + |\operatorname{Im} \tilde{\gamma}_j|} < +\infty. \quad (2.3)$$

We let

$$\tilde{C} = \sum_{j=1}^{\infty} \frac{|\lambda_j - \tilde{\gamma}_j|}{1 + |\operatorname{Im} \lambda_j|}, \quad \tilde{A}_m = e^{2\tilde{C}} \|s_1^{(m+1)}\|_{L^1(a'; b')},$$

where $s_1 \in C_0^\infty(a'; b')$ is the image of the function φ_1 under the Fourier-Laplace transform \mathcal{F} . The convergence of the series in the definition of the quantity \tilde{C} is implied by condition (2.3) (see the proof of Theorem 5.1.2 in [13]).

Let us consider an arbitrary sequence $\Gamma = \{\gamma_k\}$, $0 \notin \Gamma$, for which

$$|\gamma_k - \lambda_k| \leq |\tilde{\gamma}_k - \lambda_k|, \quad k = 1, 2, \dots \quad (2.4)$$

In accordance with Proposition 3 and Remark 1 in the work [4], a function φ_2 defined in terms of the function φ_1 and the sequence Γ by the identity

$$\varphi_2(z) = e^{-icz} \lim_{R \rightarrow \infty} \prod_{|\gamma_k| < R} \left(1 - \frac{z}{\gamma_k}\right), \quad \text{where } c = \frac{c_{\varphi_1} + d_{\varphi_1}}{2}, \quad (2.5)$$

is the Fourier-Laplace transform of some function $s_2 \in C_0^\infty(a'; b') \subset C_0^\infty(a; b)$, at that, $\text{ch supp } s_2 = [c_{\varphi_1}; d_{\varphi_1}]$ and

$$|s_2^{(m)}(t)| \leq \tilde{A}_m, \quad t \in (a; b), \quad m = 0, 1, \dots \quad (2.6)$$

Here $\text{ch supp } s_2$ is the closure of the convex hull of the support of the function s_2 .

Let $\{r_k\}_{k=0}^\infty$ be an increasing sequence of real numbers greater than 2 such that

$$|\varphi_1(x)| \leq |x|^{-k}, \quad x \in \mathbb{R}, \quad |x| \geq r_k. \quad (2.7)$$

We let

$$R_k = \max\{r_k, \tilde{A}_{k+1}(b' - a')\}, \quad k = 0, 1, 2, \dots \quad (2.8)$$

By the relation $\varphi_2 = \mathcal{F}(s_2)$, $s_2 \in C_0^\infty(a'; b')$ and estimates (2.6), for the function φ_2 we have

$$|\varphi_2(x)| \leq \frac{\tilde{A}_{k+1}(b' - a')}{|x|^{k+1}} \leq \frac{1}{|x|^k}, \quad |x| \geq R_k, \quad k = 0, 1, \dots \quad (2.9)$$

We observe that the latter estimates are valid for all functions φ_2 defined by formula (2.5) via the function φ_1 and the sequence Γ provided Γ satisfies (2.4).

The sequences Λ and Γ have the same density; we denote it by Δ_0 . For arbitrary fixed numbers $\Delta > \Delta_0$, $\delta > 0$ we let $R_j^* = \mu(\delta/\Delta) \max\{|\lambda_j|, |\gamma_j|\}$, where a function $\mu(\chi)$ is the inverse one for the function

$$\chi(\mu) = \frac{1}{\mu} \ln(1 + \mu) + \ln\left(1 + \frac{1}{\mu}\right). \quad (2.10)$$

Let us make use of the following statement valid for functions $\varphi_1, \varphi_2 \in \mathcal{F}(C_0^\infty(a; b))$ satisfying the conditions

$$\varphi_1(0) = \varphi_2(0) = 1, \quad h_{\varphi_1}(\theta) = h_{\varphi_2}(\theta), \quad \theta \in [0; 2\pi).$$

Theorem A [4, Thm. 1]. *Assume that for some numbers $\Delta > \Delta_0$, $\delta > 0$ and an increasing sequence $R_k \geq 2$, $k = 1, 2, \dots$, such that*

$$|\varphi(x)| \leq \frac{1}{|x|^k}, \quad |\psi(x)| \leq \frac{1}{|x|^k}, \quad x \in \mathbb{R}, \quad |x| \geq R_k, \quad k = 1, 2, \dots,$$

the relation

$$\limsup_{k \rightarrow \infty} \frac{\ln \frac{1}{S_{k+1}}}{\max\{R_k, R_k^*\}} > \delta \quad (2.11)$$

holds true, where

$$S_k = \sum_{j \geq k} \left| \frac{1}{\lambda_j} - \frac{1}{\gamma_j} \right|.$$

Then the submodule $\mathcal{J}_{\varphi_1, \varphi_2}$ generated by functions φ_1 and φ_2 in the module $\mathcal{P}(a; b)$ is stable.

We fix an arbitrary sequence Γ satisfying, apart of (2.4), additional requirements: the intersection $\Gamma \cap \Lambda$ is $\Lambda_{\mathcal{J}}$ and the sequences Λ and Γ satisfy relation (2.11). Since \mathcal{J} is a weakly localizable submodule, the function φ_2 defined by formula Relations (2.7), (2.9) and (2.11) mean that the assumptions of Theorem A hold true with the numbers R_k defined by formula (2.8). Hence, according to this theorem, 2-generated submodule $\mathcal{J}_{\varphi_1, \varphi_2}$ is stable or, that is equivalent in our case (see [7, Prop. 4.9]), *the identical zero can be approximated by the functions of the form $(p\varphi_1 - q\varphi_2)$ in the topology of $\mathcal{P}(a; b)$, where p, q are polynomials and $p(0) = q(0) = 1$. Due to [7, Prop. 4.8], this fact is a sufficient condition for the stability of the submodule*

$$\tilde{\mathcal{J}} := \overline{\mathcal{J}(\varphi_1, \langle c_{\mathcal{J}}; d_{\mathcal{J}} \rangle) + \mathcal{J}(\varphi_2, \langle c_{\mathcal{J}}; d_{\mathcal{J}} \rangle)}.$$

A stable submodule $\tilde{\mathcal{J}}$ contains a weakly localizable submodule $\mathcal{J}(\varphi_1)$. Theorem 1 in [3] states that then $\tilde{\mathcal{J}}$ is a weakly localizable submodule. Taking into consideration that submodules \mathcal{J} and $\tilde{\mathcal{J}}$ have the same indicator segments and zero sets, we conclude that $\mathcal{J} = \tilde{\mathcal{J}}$.

2) It is easy to check that if the submodule \mathcal{J} contains no functions in the subspace $\mathcal{P}_0(a; b)$, then $[c_{\mathcal{J}}; d_{\mathcal{J}}] \subset (a; b)$ and $2\rho_{\Lambda_{\mathcal{J}}} = d_{\mathcal{J}} - c_{\mathcal{J}}$. Moreover, in this case the set $\Lambda_{\psi} \setminus \Lambda_{\mathcal{J}}$ is finite for each function $\psi \in \mathcal{J}$. Indeed, if this is not the case, letting

$$\omega(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{\mu_j}\right),$$

where sequence $\{\mu_j\} \subset \Lambda_{\psi} \setminus \Lambda_{\mathcal{J}}$ is ‘‘sparse’’, i.e., $\lim_{j \rightarrow \infty} |\mu_{j+1}|/|\mu_j| = +\infty$, we obtain that $\frac{\psi}{\omega} \in \mathcal{J} \cap \mathcal{P}_0(a; b)$.

It follows from the above that for some $c \in \mathbb{R}$ the function

$$\varphi_0(z) = e^{icz} \lim_{R \rightarrow +\infty} \prod_{\lambda_j \in \Lambda_{\mathcal{J}}, |\lambda_j| < R} \left(1 - \frac{z}{\lambda_j}\right)$$

is contained in \mathcal{J} and generates this submodule, more precisely, relation (2.2) holds true. \square

In the rest of this section we prove the following fact: if the indicator segment of a weakly localizable submodule \mathcal{J} is the proper set of the interval $(a; b)$, then this submodule is either principal or 2-generated in the sense of (1.2).

Let a function $\Phi \in \mathcal{P}(a; b)$ be such that

$$\mathcal{J}(\Phi) = \mathcal{J}_{\Phi} = \{p\Phi, p \in \mathbb{C}[z]\}. \quad (2.12)$$

Then \mathcal{J}_{Φ} is weakly localizable submodule and in accordance with Theorem 2 [9], $\Phi \notin \mathcal{P}_0(a; b)$. As we shall see later in the proof of Theorem 3, in each weakly localizable submodule there exists a function with such properties.

Let us consider an arbitrary sequence $\{\mu_j\} \subset \Lambda_{\Phi} \setminus \{0\}$, for which

$$\liminf_{j \rightarrow \infty} \frac{|\mu_{j+1}|}{|\mu_j|} = \alpha_0 > 1. \quad (2.13)$$

We define the functions

$$\omega(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{\mu_j}\right), \quad \varphi = \frac{\Phi}{\omega}.$$

For $z \in \mathbb{C}$, $M \subset \mathbb{C}$, by the symbol $\text{dist}(z, M)$ we denote the distance from a point z to a set M .

Theorem 2. *Function φ belongs to $\mathcal{P}_0(a; b)$ and generates a weakly localizable principle submodule \mathcal{J}_{φ} .*

In order to prove this theorem, we shall make use of the three lemmata.

Lemma 1. *1) For each natural number n there exists a representation of function ω as a product of two entire functions $\omega_{1,n}$ and $\omega_{2,n}$ such that for all z , $\text{dist}(z, \Lambda_{\omega}) \geq \delta > 0$ the inequality*

$$|\ln |\omega_{1,n}(z)| - 2^{-n} \ln |\omega(z)|| \leq A \ln(e + |z|) \quad (2.14)$$

holds true, where A is a positive constant dependent only on the function ω and the quantity δ , $\Lambda_{\omega} = \{\mu_j\}$ is the zero set of function ω .

2) There exists a sequence $\{\omega_{2,n_k} \varphi\}_{k=1}^{\infty}$ converging to a function $\tilde{\Phi}$ in the sense of the topology in the space $\mathcal{P}(a; b)$ and $(\Phi/\tilde{\Phi})$ is a polynomial.

Proof. 1) Let

$$\begin{aligned}\widetilde{\mathcal{M}} &= \{\mu_j \in \Lambda_\omega : |\operatorname{Im} \mu_j| < 1\}, & \widehat{\mathcal{M}} &= \Lambda_\omega \setminus \widetilde{\mathcal{M}}, \\ \widetilde{\omega}(z) &= \prod_{\mu_j \in \widetilde{\mathcal{M}}} \left(1 - \frac{z}{\mu_j}\right), & \widehat{\omega}(z) &= \prod_{\mu_j \in \widehat{\mathcal{M}}} \left(1 - \frac{z}{\mu_j}\right).\end{aligned}$$

It is clear that $\omega = \widetilde{\omega}\widehat{\omega}$.

In order to obtain the representation $\widetilde{\omega} = \widetilde{\omega}_{1,n}\widetilde{\omega}_{2,n}$, we employ the following theorem.

Theorem B [15, Thm. 2]. *Let $\{z_k\}$, $k \in \mathbb{Z}$, be the zeroes of an entire function v taken so that $\operatorname{Re} z_k$ is an ascending sequence and*

$$\operatorname{Re} z_0 = \min_k \{\operatorname{Re} z_k, \operatorname{Re} z_k \geq 0\}.$$

If all points z_k are located in the strip $|\operatorname{Im} z| < 1$, and $|\operatorname{Re} z_k| > 1$, and each square

$$\Pi_j = \{z : |\operatorname{Im} z| < 1, 2j - 1 \leq \operatorname{Re} z < 2j + 1\}, \quad j \in \mathbb{Z},$$

contains at most one point z_k , then function v can be represented as the product of entire functions v_1, v_2 so that

$$|\ln |v_1(z)| - \ln |v_2(z)|| \leq C_1 \ln^+ |z| + C_2 \ln^+ \frac{1}{d(z)} + C_3,$$

where $d(z)$ is the distance from point z to the set of zeroes of the function v , and $C_i > 0$ are absolute constants independent of the function v .

Neglecting if needed finitely many zeroes of the function $\widetilde{\omega}$ and reordering the remaining zeroes so that their real parts ascend, we see that the sequence $\widetilde{\mathcal{M}} = \{\tilde{\mu}_k\}$, $k \in \mathbb{Z}$, satisfies the assumptions of Theorem B. According to this theorem, for all z , $\operatorname{dist}(z, \widetilde{\mathcal{M}}) \geq \delta > 0$, the function

$$\widetilde{\omega}_{1,n}(z) = \prod_{k \in \mathbb{Z}} \left(\left(1 - \frac{z}{\tilde{\mu}_{2^{n+1}k}}\right) \left(1 - \frac{z}{\tilde{\mu}_{2^{n+1}k+1}}\right) \right), \quad n \in \mathbb{N}, \quad (2.15)$$

satisfies the relation

$$|\ln |\widetilde{\omega}_{1,n}(z)| - 2^{-n} \ln |\widetilde{\omega}(z)|| \leq \tilde{A} \ln(e + |z|), \quad n \in \mathbb{N}, \quad (2.16)$$

where constant $\tilde{A} > 0$ depends only on δ , while the choice of indices $2^{n+1}k$, $2^{n+1}k+1$ in formula (2.15) is made in accordance with the arguments in the proof of Theorem B [15, Thm. 2]. We shall obtain the same statement for the function $\widehat{\omega}$ by employing one more result of work [15]. In order to do it, we recall needed notations. Let

$$P_k = \{z : 1 \leq \operatorname{Im} z \leq 2^k + 1, 0 \leq \operatorname{Re} z \leq 2^k\}, \quad k = 0, 1, 2, \dots$$

Then the difference $P_k \setminus P_{k-1}$, $k = 1, 2, \dots$, consists of three squares congruent to P_{k-1} . By the symbols P_k^m , $m = 1, 2, \dots, 12$, we denote these three squares as well as those symmetric w.r.t. both the axes and the origin. We locate the boundary segments and vertices so that the squares P_k^m are mutually disjoint and cover the set $\{z : |\operatorname{Im} z| \geq 1\}$.

Theorem C [15, Thm. 3]. *Let $\{z_k\}$, $k \in \mathbb{Z}$, be the zeroes of an entire function v taken so that $|z_k|$ ascend. Assume that $|\operatorname{Im} z| \geq 1$, and each square P_k^m contains at most one zero of function v . Then function v is represented as the product of two entire functions v_1, v_2 so that*

$$|\ln |v_1(z)| - \ln |v_2(z)|| \leq C_1 \ln^+ |z| + C_2 \ln^+ \frac{1}{d(z)} + C_3,$$

where $d(z)$ is the distance from a point z to the set of zeroes of the function v and C_i are absolute constants independent of the function v .

We fix a number $\alpha \in (1; \alpha_0)$. Neglecting if needed finitely many zeroes $\hat{\mu}_k$ of the function $\hat{\omega}$ and then reordering the remaining zeroes so that $|\hat{\mu}_k|$ ascend, in view of condition (2.13) we have

$$|\hat{\mu}_{k+1}| > \alpha |\hat{\mu}_k|, \quad k = 1, 2, \dots$$

We let

$$m = \left[\log_{\alpha} \sqrt{5} \right] + 1.$$

It is easy to confirm that all functions

$$\hat{\omega}_j(z) = \prod_{k=0}^{\infty} \left(1 - \frac{z}{\hat{\mu}_{mk+j}} \right), \quad j = 1, \dots, m,$$

satisfy the assumption of Theorem C. Applying this theorem n times to each function $\hat{\omega}_j$, $j = 1, \dots, m$, we obtain the representation

$$\hat{\omega}_j = \hat{\omega}_{j,1,n} \hat{\omega}_{j,2,n},$$

at that,

$$\left| \ln |\hat{\omega}_{j,1,n}(z)| - 2^{-n} \ln |\hat{\omega}_j(z)| \right| \leq \hat{A} \ln(e + |z|), \quad \text{dist}(z, \widehat{\mathcal{M}}) \geq \delta, \quad (2.17)$$

where constant $\hat{A} > 0$ depends only on δ and ω . Letting

$$\hat{\omega}_{1,n} = \hat{\omega}_{1,1,n} \dots \hat{\omega}_{m,1,n}, \quad \hat{\omega}_{2,n} = \frac{\hat{\omega}}{\hat{\omega}_{1,n}},$$

we obtain the required factorization

$$\hat{\omega} = \hat{\omega}_{1,n} \hat{\omega}_{2,n}.$$

By estimates (2.16), (2.17) we see that the functions

$$\omega_{1,n} = \tilde{\omega}_{1,n} \hat{\omega}_{1,n}, \quad \omega_{2,n} = \frac{\omega}{\omega_{1,n}}$$

satisfy the first statement of the lemma.

2) It follows from the relations $\omega = \omega_{1,n} \omega_{2,n}$ and (2.14) that for all natural n and all $z \in \mathbb{C}$, $\text{dist}(z, \Lambda_{\omega}) \geq \delta$, the estimates

$$|\omega_{2,n}(z) \varphi(z)| \leq (e + |z|)^{[A]+1} |\Phi(z)|$$

hold true. By the topological properties of the space $\mathcal{P}(a; b)$, the sequence $\{\omega_{2,n} \varphi\}_{n=1}^{\infty}$ is relatively compact in this space. Hence, there exists a sequence $\{\omega_{2,n_k} \varphi\}_{k=1}^{\infty}$ converging to some function $\tilde{\Phi}$ in the topology of $\mathcal{P}(a; b)$ and the indicator of this function is equal to with to the coinciding indicator of the functions Φ and φ . The corresponding sequence of entire functions of minimal type at order 1

$$\omega_{1,n_k} = \frac{\Phi}{\omega_{2,n_k} \varphi}$$

converges to the entire function $(\Phi/\tilde{\Phi})$ having a minimal type at order 1. Passing to the limit, by means of estimates (2.14) we obtain the upper polynomial bound for $|\Phi/\tilde{\Phi}|$ on the real axis. Applying the corollary of Phragmén-Lindelöf theorem [2, Sect. 6.1], we conclude that $(\Phi/\tilde{\Phi})$ is a polynomial. \square

Let $n(r) = \sum_{|\mu_j| < r} 1$ be the counting function of the sequence Λ_{ω} , $N(r) = \int_0^r \frac{n(\tau)}{\tau} d\tau$, $M(r) = \max_{|z|=r} |\omega(z)|$, $m(r) = \min_{|z|=r} |\omega(z)|$.

Condition (2.13) for the sequence Λ_{ω} implies that

$$n(r) = C_0 \ln(1 + r), \quad r \geq 0, \quad (2.18)$$

where C_0 is a positive constant. By Lemma 3.5.8 in the monograph [22], in view of (2.18) and Jensen formula (see, for instance, [22, Sect. 1.2]) we obtain the double inequality

$$N(r) \leq M(r) \leq N(r) + C_0 \ln(1+r). \quad (2.19)$$

Lemma 2. 1) For all $z \in \mathbb{C}$ the estimate from above

$$\ln |\omega(z)| \leq N(|z|) + C_0 \ln(1+|z|) \quad (2.20)$$

holds true.

2) For all $\varepsilon > 0$ and $\delta > 0$ and all $z \in \mathbb{C}$, $\text{dist}(z, \Lambda_\omega) \geq \delta$ the estimate from below

$$\ln |\omega(z)| \geq (1-\varepsilon)N(|z|) - C_1 \ln(1+|z|) - C_{2,\varepsilon} \quad (2.21)$$

holds true, where constant $C_{2,\varepsilon} > 0$ depends on Λ_ω , δ and ε , while constant $C_1 > 0$ depends only on Λ_ω .

Proof. 1) Required estimate (2.20) is implied by the right inequality in (2.19).

2) It is known that for an entire function, whose zero set satisfies condition (2.18), the relation $\ln m(r) \sim \ln M(r)$ holds true as $r \rightarrow \infty$ over a set of unit relative measure [22, Thm. 3.6.1]. An exceptional set of values of r can be covered by a countable set of segments disjoint thanks to (2.13) and centred together with the set $\{|\mu_j|\}$ (i.e., each interval contains exactly one point $|\mu_j|$). This set of the intervals has a zero relative length. Without loss of generality we can assume that there exists a decreasing sequence of positive numbers δ_j , $j = 1, 2, \dots$, such that for each $\varepsilon > 0$ the inequality

$$\ln m(r) \geq (1-\varepsilon) \ln M(r), \quad r > r_\varepsilon, \quad r \in \bigcup_{j=1}^{\infty} ((1-\delta_j)|\mu_j|; (1+\delta_j)|\mu_j|)$$

holds true. By (2.13) and (2.18) one can get easily that

$$N(r) \leq N((1-\delta_j)r) + (C_0 \ln 2 + 1) \ln(1+r) + \tilde{C}_{2,\varepsilon}, \quad r > 0,$$

where constant $\tilde{C}_{2,\varepsilon} > 0$ depends only on Λ_ω , δ and ε .

Desired lower bound (2.21) is obtained by standard methods by two last estimates and the left inequality in (2.19). \square

Lemma 3. For each natural n function $\omega_{2,n}\varphi$ is contained in the submodule \mathcal{J}_φ .

Proof. For a fixed $n \in \mathbb{N}$ by (2.14) we have

$$\ln |\omega_{2,n}(z)| \leq (1-2^{-n}) \ln |\omega(z)| + A \ln(e+|z|), \quad \text{dist}(z, \Lambda_\omega) \geq \delta. \quad (2.22)$$

In view of (2.13) and (2.20) it implies the estimate

$$\ln |\omega_{2,n}(z)| \leq (1-2^{-n})N(|z|) + \tilde{A} \ln(e+|z|), \quad z \in \mathbb{C}. \quad (2.23)$$

We consider the weight function $\tilde{V}(x) = (e+|x|)^{\tilde{A}+1} \exp((1-2^{-n})N(|x|)) \geq 1$, $x \in \mathbb{R}$. This function is even, convex in $\ln|x|$, and for all $k = 0, 1, \dots$ the relation

$$|x|^k = o(\tilde{V}(x)), \quad |x| \rightarrow +\infty,$$

holds true. By estimate (2.23) it yields for the function $\omega_{2,n}$ that

$$\frac{|\omega_{2,n}(x)|}{\tilde{V}(x)} \rightarrow 0, \quad |x| \rightarrow +\infty.$$

Arguing as in the proof of Lemma 3 in the work [9], we obtain that there exists a sequence of polynomials $\{p_j\}$ converging to the function $\omega_{2,n}$ in the weighted norm $\|\cdot\| = \sup_{x \in \mathbb{R}} \frac{|\cdot|}{V(x)}$, where

$$V(x) = (1+|x|)^2 \tilde{V}(x).$$

We let $v(x) = \ln V(x)$,

$$P_v(z) = \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{v(\tau)}{(\tau - x)^2 + y^2} d\tau$$

is the Poisson integral of a function v , $z = x + iy$. By condition (2.13) it is easy to obtain that the function v belongs to the class of *slowly varying canonical weights* introduced in the monograph [19, §1.3]. This is why (see [19, Sect. 1.4]) the function P_v is harmonic in the upper and lower half-planes, is continuous and subharmonic in the whole complex plane and satisfies the estimate

$$P_v(z) \geq v(|z|), \quad z \in \mathbb{R},$$

and the relation

$$\limsup_{z \rightarrow \infty} \frac{P_v(z)}{v(|z|)} = 1. \quad (2.24)$$

Since $\mathcal{P}(a; b)$ is a locally convex space of type (LN^*) , the sequence $p_j\varphi$ in this space if and only if it is bounded in one of norms (1.1) (see [1]). Taking into account estimate (2.21), the definition of the weight V , relation (2.24) and the properties of the function $N(r)$ implied by condition (2.13), and employing the Phragmén-Lindelöf we establish

$$|p_j(z)\varphi(z)| \leq (e + |z|)^{\text{const}} \exp(d_\varphi y^+ - c_\varphi y^-),$$

where d_φ (c_φ) is the value of the indicator of the function φ at the point $\pi/2$ (respectively, at the point $-\pi/2$). The last estimate is equivalent to the boundedness of the sequence $\{p_j\varphi\}$ in one of norms (1.1).

Employing once again the properties of locally convex space of type (LN^*) (see [1]), by the above fact we obtain that there exists a subsequence of this sequence converging to the function $\omega_{2,n}\varphi$ in $\mathcal{P}(a; b)$. \square

Proof of Theorem 2. The belonging $\varphi \in \mathcal{P}_0(a; b)$ is obvious. It follows from Statement 2) of Lemma 1 and Lemma 3 that

$$\Phi \in \mathcal{J}_\varphi. \quad (2.25)$$

By (2.12) we have $\mathcal{J}(\Phi) \subset \mathcal{J}_\varphi$. As it is stated in Theorem 1 of work [3], thanks to the stability of submodule \mathcal{J}_φ , this relation is equivalent to the weak localizable property of \mathcal{J}_φ . \square

Theorem 3. *Assume that the submodule \mathcal{J} is weakly localizable and $[c_{\mathcal{J}}; d_{\mathcal{J}}] \subset (a; b)$. Then either \mathcal{J} is the principal submodule or $\mathcal{J} = \mathcal{J}_{\varphi_1, \varphi_2}$, where $\varphi_1, \varphi_2 \in \mathcal{J} \cap \mathcal{P}_0(a; b)$.*

Proof. If $\mathcal{J} \cap \mathcal{P}_0(a; b) = \emptyset$, then, as it was shown in Statement 2) of Theorem 1, \mathcal{J} is the principal submodule. This is why we shall argue assuming that $\mathcal{J} \cap \mathcal{P}_0(a; b) \neq \emptyset$.

Let us show first that the submodule \mathcal{J} contains a function $\varphi_1 \in \mathcal{P}_0(a; b)$ with the properties: $c_{\varphi_1} = c_{\mathcal{J}}$, $d_{\varphi_1} = d_{\mathcal{J}}$, the principle submodule \mathcal{J}_{φ_1} is weakly localizable.

In order to do it, we consider an arbitrary function $\tilde{\varphi} \in \mathcal{J} \cap \mathcal{P}_0(a; b)$ and we let

$$\varphi = (e^{i(c_{\tilde{\varphi}} - c_{\mathcal{J}})z} + e^{i(d_{\mathcal{J}} - d_{\tilde{\varphi}})z}) \tilde{\varphi}.$$

It is clear that the function φ belongs to the set $\mathcal{J} \cap \mathcal{P}_0(a; b)$ and its indicator diagram is $i[c_{\mathcal{J}}; d_{\mathcal{J}}]$. If the principle submodule \mathcal{J}_φ is weakly localizable, we let $\varphi_1 = \varphi$. Otherwise we consider the maximal subharmonic minorant $v(z)$ of the function $(H(z) - \ln|\varphi(z)|)$, where $H(z) = d_{\mathcal{J}}(\text{Im } z)^+ - c_{\mathcal{J}}(\text{Im } z)^-$.

The function v satisfies the relation $vt \equiv -\infty$. Indeed, by the inclusion $\varphi \in \mathcal{P}_0(a; b)$, for each $k = 0, 1, 2, \dots$ we have $M_k = \max_{x \in \mathbb{R}} |\varphi(x)x^k| < +\infty$, as well as

$$\varphi(z) = \int_a^b s(t)e^{-itz} dt, \quad s \in C_0^\infty(a; b).$$

The class $\mathcal{C}_{(a;b)}(\{M_k\})$ (see, for instance, [21, Sect. IV.A]), contains a non-zero function s , and therefore, it is not quasi-analytic. In accordance with Carleman criterion, it is equivalent to the relation

$$\int_0^\infty \frac{\ln T(r)}{1+r^2} dr < +\infty,$$

where $T(r) = \sup_{k \geq 0} \frac{r^k}{M_k}$ is the trace function of the sequence $\{M_k\}$, (see, for instance, [21, Sect. IV.A]). Thus, $\ln T(e^t)$ is a finite function convex in $t \in \mathbb{R}$. Therefore, function $u(z) = \ln T(|z|)t \equiv -\infty$ is subharmonic in \mathbb{C} [23, Thm. 2.1.2]. The definition of u yields the estimate

$$u(x) + \ln |\varphi(x)| \leq 0, \quad x \in \mathbb{R}. \quad (2.26)$$

Function φ , as well as all the elements of the space $\mathcal{P}(a;b)$ has a completely regular growth in the entire plane, while the function u depends only on $|z|$. Applying the theorem on summation of indicators [24, Thm. 1], by (2.26) we obtain that u has the minimal type at order 1. This fact, estimate (2.26), Phragmén-Lindelöf theorem for subharmonic functions [2, Sect. 7.3] imply the estimate

$$u(z) + \ln |\varphi(z)| \leq H(z), \quad z \in \mathbb{C},$$

which yields the inequality $u(z) \leq v(z)$, $z \in \mathbb{C}$. Hence, $vt \equiv -\infty$.

Let ω be an entire function (of a minimal exponential type) satisfying the relation

$$|\ln |\omega(z)| - v(z)| \leq C \ln(1 + |z|), \quad zt \in E, \quad (2.27)$$

with some constant $C > 0$ whose exceptional set E can be covered by a countable union of circles with a finite sum of radii. The existence of such function is implied by Theorem 5 of the work [14]. We let $\Phi = \omega\varphi$. It is clear that $\Phi \in \mathcal{J}$. The fact that the function v is the *maximal* subharmonic minorant function ($H - \ln |\varphi|$) and estimate (2.27) imply relations (2.12) for the function Φ . We choose a sequence $\{\mu_j\} \subset \Lambda_\Phi \setminus \Lambda_{\mathcal{J}}$ satisfying conditions (2.13), $\mu_j \neq 0$. Let

$$\varphi_1 = \frac{\Phi}{\prod_{j=1}^{\infty} \left(1 - \frac{z}{\mu_j}\right)}.$$

For this function, Theorem 2 holds true and therefore, \mathcal{J}_{φ_1} is a weakly localizable submodule.

Now we argue as in the proof of Statement 1 of Theorem 1. We introduce the function φ_2 by formula (2.5), where the sequence Γ satisfies the condition $\Gamma \cap \Lambda_{\varphi_1} = \Lambda_{\mathcal{J}}$ and is so close to the sequence Λ_{φ_1} that the submodule $\mathcal{J}_{\varphi_1, \varphi_2}$ is stable. Moreover, this stable submodule contains a weakly localizable submodule $\mathcal{J}(\Phi)$. Theorem 1 in the work [3] states that then the submodule $\mathcal{J}_{\varphi_1, \varphi_2}$ is weakly localizable. The indicator segment and the zero set of the submodule $\mathcal{J}_{\varphi_1, \varphi_2}$ are the same as for the original submodule \mathcal{J} . Therefore, $\mathcal{J} = \mathcal{J}_{\varphi_1, \varphi_2}$. \square

3. REPRESENTATION FOR INVARIANT SUBSPACES ADMITTING A WEAK SPECTRAL SYNTHESIS

We consider the Schwarz space $\mathcal{E}(a;b) = C^\infty(a;b)$ equipped with the metrizable topology of the projective limit of the Banach spaces $C^k[a_k; b_k]$, where $[a_1; b_1] \Subset [a_2; b_2] \Subset \dots$ is some sequence of the segments exhausting the interval $(a;b)$. It is known that $\mathcal{E}(a;b)$ is the reflexive Fréchet space. By W is a closed subspace of this space invariant w.r.t. the differentiation operator $D = \frac{d}{dt}$ (shortly, *D-invariant*). If else is not said, in what follows we consider only closed subspaces in $\mathcal{E}(a;b)$.

Let $\text{Exp } W$ be all root elements of the operator D (exponential monomials $t^j e^{-i\lambda t}$) contained in W . For a non-trivial not coinciding with the entire space $\mathcal{E}(a;b)$ subspace W , the set $\text{Exp } W$ is at most countable.

We let

$$W_I = \{f \in \mathcal{E} : f^{(k)}(t) = 0, t \in I, k = 0, 1, 2, \dots\}, \quad (3.1)$$

where $I \subset (a; b)$ is a relatively closed non-empty segment and denote by I_W the minimal relatively closed in $(a; b)$ non-empty segment satisfying the condition $W_I \subset W$ (the existence of such segment is implied by Theorem 4.1 in [16]).

The Fourier-Laplace transform \mathcal{F} acting in the strongly dual space $\mathcal{E}'(a; b)$ by the rule

$$\mathcal{F}(S)(z) = (S, e^{-itz}), \quad S \in (C^\infty(a; b))',$$

is a linear topological isomorphism of space $(C^\infty(a; b))'$ and $\mathcal{P}(a; b)$ [17, Thm. 7.3.1]. We have the following

Duality principle *There exists a one-to-one correspondence between the set $\{\mathcal{J}\}$ of weakly localizable submodules of the module $\mathcal{P}(a; b)$ and the set $\{W\}$ of D -invariant subspaces of the space $\mathcal{E}(a; b)$ determined by rule $\mathcal{J} \longleftrightarrow W$ if and only if $\mathcal{J} = \mathcal{F}(W^0)$, where a closed subspace $W^0 \subset \mathcal{E}'(a; b)$ is formed by all the distributions $S \in \mathcal{E}'(a; b)$ annullating W ; at that,*

$$\bar{I}_W = [c_{\mathcal{J}}; d_{\mathcal{J}}], \quad \text{Exp } W = \{t^j e^{-i\lambda_k t}, j = 0, \dots, m_k - 1, (\lambda_k, m_k) \in \Lambda_{\mathcal{J}}\},$$

where $\Lambda_{\mathcal{J}}$ is the set of the zeroes of the submodule \mathcal{J} ([3, Duality principle], [4, Prop. 1]).

It is known (see [16, Thm. 2.1]) that given a nontrivial D -invariant subspace W , the spectrum σ_W of the operator $D : W \rightarrow W$ either coincides with the entire complex plane or is discrete; in the second case $\sigma_W = \Lambda_{\mathcal{J}}$ according the **duality principle**.

A nontrivial D -invariant subspace admits a *weak spectral synthesis* if

$$W = \overline{W_{I_W} + \mathcal{L}(\text{Exp } W)}, \quad \mathcal{L}(\cdot) \text{ is the linear span of a set.} \quad (3.2)$$

It is clear that a D -invariant subspace W admitting a weak spectral synthesis is minimal among all D -invariant subspaces \widetilde{W} satisfying

$$I_{\widetilde{W}} = I_W, \quad \text{Exp } \widetilde{W} = \text{Exp } W.$$

By the **duality principle**, the *annullying submodule* $\mathcal{J} = \mathcal{F}(W^0)$ of such subspace is the maximal one among all maximal submodules $\widetilde{\mathcal{J}} \subset \mathcal{P}(a; b)$, with the zero set and the indicator diagram satisfying the conditions:

$$\Lambda_{\widetilde{\mathcal{J}}} = \Lambda_{\mathcal{J}}, \quad [c_{\widetilde{\mathcal{J}}}; d_{\widetilde{\mathcal{J}}}] = [c_{\mathcal{J}}; d_{\mathcal{J}}].$$

Therefore, \mathcal{J} is a weakly localizable submodule. It is clear that the opposite is true as well: if an annulling submodule of a D -invariant subspace is weakly localizable, then this subspace admits a weak spectral synthesis.

We recall that the *completeness radius* $\rho(\Lambda)$ of a sequence of multiple points $\Lambda = \{(\lambda_j, m_j)\}$ is defined as the infimum of the radii of open intervals $I \subset \mathbb{R}$, for which the system of the exponential monomials $\{t^k e^{-i\lambda_j t}, k = 0, \dots, m_j - 1, j \in \mathbb{N}\}$, is incomplete in the spaces $\mathcal{E}(I)$, $C(I)$, $L^p(I)$, $1 \leq p < \infty$ (see [18]).

Given arbitrary subsets $A, B \subset \mathbb{R}$, we denote by $A \div B$ their *geometric difference*, i.e., the set of all $x \in \mathbb{R}$ obeying $x + B \subset A$. Let $S \in \mathcal{E}'(a; b)$ and $h \in (a; b) \div \text{ch supp } S$, where $\text{ch supp } S$ is the convex hull of $\text{supp } S$. We define a functional $S_h \in \mathcal{E}'(a; b)$ by the formula

$$(S_h, f) = (S, f(t + h)), \quad f \in \mathcal{E}(a; b).$$

Given a distribution $S \in \mathcal{E}'(a; b)$ and a non-empty relatively closed in $(a; b)$ segment $\langle c; d \rangle$ satisfying the condition

$$\text{ch supp } S \Subset \langle c; d \rangle, \quad (3.3)$$

we let

$$W(S, \langle c; d \rangle) = \{f \in \mathcal{E}(a; b) : (S * f)(h) = (S_h, f) = 0, \forall h \in \langle c; d \rangle \div \text{ch supp } S\}.$$

It is clear that $W(S, \langle c; d \rangle)$ is a D -invariant subspace.

Lemma 4. *A D -invariant subspace $W(S, \langle c; d \rangle)$ admits a weak spectral synthesis, its annihilating submodule is $\mathcal{J}(\varphi, \langle c; d \rangle)$, where $\varphi = \mathcal{F}(S)$.*

Proof. By the arguments given after the **duality principle** we see that the first statement of the lemma is implied by the first one. We denote by \mathcal{J}_1 the annihilating submodule of the subspace $W(S, \langle c; d \rangle)$. In accordance with the **duality principle** we have the inclusion

$$\mathcal{J}_1 \subset \mathcal{J}(\varphi, \langle c; d \rangle).$$

Since the zero set $\Lambda_{\mathcal{J}_1}$ of the submodule \mathcal{J}_1 coincides with the zero set Λ_φ of the function φ and the indicator segment of this submodule is equal to $[c; d]$, it follows from (3.3) that the quantity $\rho(\Lambda_{\mathcal{J}_1})$ is less than the half of the length of the segment $[c; d]$. Statement 3) of Theorem 2 in work [3] states that in this case the submodule \mathcal{J}_1 is weakly localizable only if it is stable.

As we mentioned earlier [4, Introduction], the module $\mathcal{P}(a; b)$ is a bornological and b -stable space (the latter notion was introduced in the work [5]). This is why it belongs to the class of the topological modules, for which it was proved in the work [7] (Proposition 4.2 and Remark 1 in the end of Subsection 1 of Section 4) that the stability of the submodule $\mathcal{J} \subset \mathcal{P}(a; b)$ at each point $\lambda \in \mathbb{C}$ is implied by its stability at some single point. Thus, in order to prove the identity

$$\mathcal{J}_1 = \mathcal{J}(\varphi, \langle c; d \rangle)$$

(which is equivalent to the weak localizable property of the submodule \mathcal{J}_1), it is sufficient to check the stability of the submodule \mathcal{J}_1 at some point $\mu \notin \Lambda_\varphi$. Without loss of generality we can assume that $\mu = 0$, $\varphi(0) = 1$.

Let $\psi \in \mathcal{J}_1$, $\psi(0) = 0$. The function ψ is the limit of a generalized sequence of the form $(a_1 e^{ih_1 z} + \dots + a_m e^{ih_m z})\varphi$ in the topology of the space $\mathcal{P}(a; b)$, where $h_j \in \langle c; d \rangle \div [c_\varphi; d_\varphi]$, $j = 1, \dots, m$; $i[c_\varphi; d_\varphi]$ is the indicator diagram of the function φ (coinciding with $\text{ch supp } S$ by Paley-Wiener theorem). Since it is obvious that $e^{ih'z}\varphi \rightarrow e^{ihz}\varphi$ as $h' \rightarrow h$ in the topology of $\mathcal{P}(a; b)$, we can assume that

$$h_j \in (c; d) \div [c_\varphi; d_\varphi], \quad j = 1, \dots, m. \quad (3.4)$$

By the definition of the topology in $\mathcal{P}(a; b)$ it is easy to obtain that the generalized sequence

$$\left(a_1 \frac{e^{ih_1 z} - 1}{z} + \dots + a_m \frac{e^{ih_m z} - 1}{z} \right) \varphi \quad (3.5)$$

converges to the function $\frac{\psi}{z}$.

By belongings (3.4), each element of generalized sequence (3.5) belongs to a *localizable* submodule $\mathcal{J}(\varphi, (c; d))$ of the module $\mathcal{P}(c; d)$. By the duality principle and a well-known result on a spectral synthesis in the kernel of the convolution operator (see, for instance, [20, Thm. 16.4.1]), this submodule coincides with the annihilating submodule $\mathcal{J}_2 \subset \mathcal{P}(c; d)$ of a D -invariant subspace $W(S, (c; d)) \subset \mathcal{E}(c; d)$, where

$$W(S, (c; d)) = \{f \in \mathcal{E}(c; d) : (S * f)(h) = (S_h, f) = 0, \forall h \in (c; d) \div \text{ch supp } S\}.$$

In view of the above, we conclude that each function in generalized sequence (3.5) belongs to the submodule $\mathcal{J}_2 = \mathcal{J}(\varphi, (c; d))$, which, in its turn, is contained in \mathcal{J}_1 . And therefore, the limiting function $\frac{\psi}{z}$ of generalized sequence (3.5) satisfies also the belonging $\frac{\psi}{z} \in \mathcal{J}_1$, that is, the submodule \mathcal{J}_1 is stable at point 0. This fact implies both statements of the lemma. \square

Remark 1. *We note that the proven lemma is true if we replace condition (3.3) by a weaker condition: the length of the segment $\text{ch supp } S$ is less than $(d - c)$ (the latter can be equal to $+\infty$).*

Theorem 4. *Each D -invariant subspace with a discrete spectrum σ_W satisfying the condition $2\rho(\sigma_W) < |I_W|$, where $|I_W| \leq +\infty$ is the length of the segment I_W , can be represented as the set of the solutions to two (probably, coinciding) homogeneous convolution equations:*

$$f \in W \iff \begin{cases} (S_1 * f)(h) = 0, & h \in I_W \div \text{ch supp } S_1, \\ (S_2 * f)(h) = 0, & h \in I_W \div \text{ch supp } S_2. \end{cases} \quad (3.6)$$

Proof. Corollary 2 in the work [3] states that a D -invariant subspace W satisfying the assumptions of the proven theorem admits a weak spectral synthesis and its annihilating submodule \mathcal{J} is weakly localizable. It is easy to see that this submodule contains a function φ_1 in $\mathcal{P}_0(a; b)$ with an indicator diagram compactly embedded in the segment iI_W . And thus, in accordance with Statement 1 of Theorem 1 and the duality principle

$$\mathcal{J} = \overline{\mathcal{J}(\varphi_1, I_W) + \mathcal{J}(\varphi_2, I_W)},$$

where function $\varphi_2 \in \mathcal{J} \cap \mathcal{P}_0(a; b)$ has the same indicator diagram as the function φ_1 . Applying Lemma 4, in view of the reflexivity of the space $\mathcal{E}(a; b)$ we obtain relation (3.6) with $S_1 = \mathcal{F}^{-1}(\varphi_1)$, $S_2 = \mathcal{F}^{-1}(\varphi_2)$. \square

Theorem 5. *If a D -invariant subspace W admits a weak spectral synthesis and $\bar{I}_W \subset (a; b)$, then there exist distributions $S_1, S_2 \in W^0$ (probably, $S_1 = S_2$) such that*

$$f \in W \iff \begin{cases} (S_1, D^j f) = 0, & j = 0, 1, 2, \dots, \\ (S_2, D^j f) = 0, & j = 0, 1, 2, \dots \end{cases} \quad (3.7)$$

Proof. The annihilating submodule $\mathcal{J} = \mathcal{F}(W^0)$ is weakly localizable and satisfies the assumptions of Theorem 3. Therefore, either $\mathcal{J} = \mathcal{J}_\varphi$ or $\mathcal{J} = \mathcal{J}_{\varphi_1, \varphi_2}$. In view of the duality principle and the reflexivity of the space $\mathcal{E}(a; b)$, it follows that (3.7) holds true with $S_1 = \mathcal{F}^{-1}(\varphi_1)$, $S_2 = \mathcal{F}^{-1}(\varphi_2)$ (at that $S_1 = S_2$, if \mathcal{J} is the principle submodule). \square

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Natalia Fairbakhovna Abuzyarova,
Bashkir State University,
Zaki Validi str. 32,
450074, Ufa, Russia
E-mail: abnatf@gmail.com