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INVERTING OF GENERALIZED RIEMANN-LIOUVILLE OPERATOR BY MEANS OF INTEGRAL LAPLACE TRANSFORM

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Abstract. We employ the integral Laplace transform to invert the generalized Riemann-Liouville operator in a closed form. We establish that the inverse generalized Riemann-Liouville operator is a differential or integral-differential operator. We establish a relation between Riemann-Liouville operator and Temlyakov-Bavrin operator. We provide new examples of generalized Riemann-Liouville operator.

Keywords: Riemann-Liouville operator, fractional integral, Laplace transform.

Mathematics Subject Classification: 26A33, 44A10

1. Introduction

Let $f(x) \in L_1(0,1)$, then the function

$$J^{\alpha}[f](x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt \in L_1(0,1)$$

is called a fractional integral of order α, see [8, 11, 15]. The fractional integral can be written as

$$J^{\alpha}[f](x) = \frac{x^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-\varepsilon)^{\alpha-1} f(\varepsilon x)\, d\varepsilon \in L_1(0,1).$$

The latter formula allows one to extend the fractional integration for the case of the functions $w = f(z)$ analytic in the unit circle $B = \{z : |z| < 1\}$. As a result, we arrive at the generalized Riemann-Liouville operator

$$L_{\omega}[f](z) = \alpha \int_0^1 (1-\varepsilon)^{\alpha-1} f(\varepsilon z)\, d\varepsilon. \tag{1.1}$$

The extension of the notion of fractional integral (1.1) was made by M.M. Džrbašjan. In the work [5], the generalized Riemann-Liouville operator was introduced.

We shall say a function $\omega(x)$ belongs to Ω if it is nonnegative and continuous on $[0,1)$ and

$$\omega(0) = 1, \int_0^1 \omega(x)\, dx < \infty,$$

for all $r \in [0,1)$ the inequality

$$\int_r^1 \omega(x)\, dx > 0$$

holds true. Let $w = f(z)$ be a function analytic in the unit circle $B = \{z : |z| < 1\}$.

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Definition 1.1. *The generalized Riemann-Liouville operator is the following operator [5]*

$$L_\omega [f] = f(0) + z \int_0^1 \omega(\varepsilon) f'(\varepsilon z) \, d\varepsilon.$$

We define the sequence of the numbers

$$\Delta_1 = 1, \Delta(k) = k \int_0^1 r^{k-1} \omega(x) \, dx < \infty, \quad k = 1, 2, \ldots$$

M.M. Džrbašjan established the following statements [5]:
i)

$$L_\omega \left[z^k\right] = \Delta_k z^k;$$

ii) Let $w = f(z)$ be a function analytic in the unit circle and assume that its Taylor series is of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Then

$$L_\omega [f](z) = \sum_{k=0}^{\infty} a_k \Delta_k z^k;$$

iii) The operator L_ω is invertible and

$$L_\omega^{-1} [f](z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{\Delta_k},$$

where

$$f(z) = \sum_{k=0}^{\infty} a_k z^k.$$

iiii) If a function $\omega(x) \in \Omega$ is a continuously differentiable in $[0, 1)$ and $\omega(1) = 0$, then the generalized Riemann-Liouville operator is of the form

$$L_\omega [f] = -\int_0^1 \omega'(\varepsilon) f(\varepsilon z) \, d\varepsilon. \tag{1.2}$$

The current state-of-art of the fractal analysis and its applications is presented in [15-18].

2. Main result

As it was mentioned, the theory of the generalized Riemann-Liouville operator was constructed by M.M. Džrbašjan in work [5]. However, no explicit construction for the inverse operator was proposed. In order to invert the generalized Riemann-Liouville operator in a closed form, we propose to employ the integral Laplace transform.

The idea is to continue the scalar sequence

$$\Delta(k) = k \int_0^1 r^{k-1} \omega(x) \, dx < \infty, k = 1, 2, \ldots$$

into the half-plane $p:\quad \operatorname{Re} p = \sigma > \sigma_0 \geqslant 0$.

The problem on inverting the generalized Riemann-Liouville operator is solved if the function $\Delta(p)$ is a Laplace image, and the function

$$\frac{1}{\Delta(p)}$$

has a power growth, i.e.,

$$\frac{1}{\Delta(p)} = p^\alpha L(p), \alpha > 0,$$

where $L(p)$ is a slowly varying function [14].

Lemma 2.1. *The function*

$$\Delta(p) = p\int_0^1 \varepsilon^{p-1}\omega(\varepsilon)\,d\varepsilon$$

is well-defined for the values p $\operatorname{Re} p = \sigma > \sigma_0 \geqslant 0$ *and serves as the Laplace image of the function*

$$\omega\left(e^{-t}\right), \omega\left(e^{-t}\right) \geqslant 0, t \in [0,\infty).$$

Proof. It is sufficient to make a change of the variable $\varepsilon = e^{-t}$ in the expression for $\Delta(p)$. □

Theorem 2.1. *Suppose that all roots of the polynomial* $Q(p)$ *are located in the left half-plane and* $Q(0) \neq 0$ *and let the function*

$$\frac{1}{Q(p)\Delta(p)}$$

be the image of some original $\omega^*(t)$. *The the inverse operator for* L_ω *is of the form*

$$L_\omega^{-1}[f](z) = Q\left(z\frac{d}{dz}\right)\left[\int_0^\infty \omega^*(\varepsilon) f\left(e^{-\varepsilon}z\right)d\varepsilon\right], \tag{2.1}$$

where

$$Q\left(z\frac{d}{dz}\right) = a_0 + a_1 z\frac{d}{dz} + a_2\left(z\frac{d}{dz}\right)^2 + \ldots + a_n\left(z\frac{d}{dz}\right)^n,$$

numbers a_k *are the coefficients of the polynomial* $Q(p)$.

Proof. The integral component of the operator L_ω^{-1} that is, the operator of the form

$$\int_0^\infty \omega^*(\varepsilon) f\left(e^{-\varepsilon}z\right)d\varepsilon,$$

is continuous in the space of the functions $H(B)\cup C\left(\bar{B}\right)$, and this is why

$$L_\omega^{-1}[f](z) = Q\left(z\frac{d}{dz}\right)\sum_{k=0}^\infty a_k z^k \frac{1}{Q(k)\Delta(k)} = \sum_{k=0}^\infty a_k z^k \frac{Q(k)}{Q(k)\Delta(k)} = \sum_{k=0}^\infty a_k z^k \frac{1}{\Delta(k)}.$$

This completes the proof. □

Corollary 2.1. *The inverse for the Riemann-Liouville operator is either differential or integral-differential.*

Theorem 2.2. *Let the functions* $\frac{1}{\Delta(p)} = Q(z)$ *be a polynomial, then the inverse operator for* L_ω *is a differential operator of the form*

$$L_\omega^{-1}[f](z) = Q\left(z\frac{d}{dz}\right)[f(z)]. \tag{2.2}$$

Proof. We have the following identities

$$Q\left(z\frac{d}{dz}\right)\sum_{k=0}^\infty a_k z^k = \sum_{k=0}^\infty a_k z^k Q(k) = \sum_{k=0}^\infty a_k z^k \frac{1}{\Delta(k)} \equiv L_\omega^{-1}[f](z).$$

□

3. Various cases of generalized Riemann-Liouville operators

3.1. Let $\omega(x) = 1 - x^h$, $h > 0$, then

$$L_\omega[f] = f(0) + z\int_0^1 \left(1-\varepsilon^h\right) f'(\varepsilon z)\, d\varepsilon = h\int_0^1 \varepsilon^{h-1} f(\varepsilon z)\, d\varepsilon,$$

and we obtain the operator which differs by a factor from the operator

$$L_h^{-1}[f] = \int_0^1 \varepsilon^{h-1} f(\varepsilon z)\, d\varepsilon$$

studied in the works by I.I. Bavrin [1, 2]. In order to find the inverse operator we note that

$$\Delta(p) = p\int_0^1 \varepsilon^{p-1}\left(1-\varepsilon^h\right) d\varepsilon = \frac{h}{h+p}.$$

Then, taking into consideration that the expression

$$\frac{1}{\Delta(p)} = \frac{h+p}{h}$$

is a polynomial, by Theorem 2.1 we obtain

$$L_\omega^{-1}[f] = \frac{hf + zf'(z)}{h} = \frac{L_h[f(z)]}{h}. \tag{3.1}$$

3.2. Let $\Delta(p) = \prod_{j=1}^m \frac{h_j}{h_j+p}$, $h_j > 0$. The original of the function $\Delta(p)$ is of the form

$$\sum_{k=1}^m e^{-h_k t} \prod_{j\neq k} \frac{h_j}{h_j - h_k} \to \Delta(p).$$

By formula (1.2) we establish the following expression for the operator L_ω

$$L_\omega[f](z) = \prod_{j\neq k} \frac{h_j}{h_j - h_k} \int_0^1 \varepsilon^{h_k - 1} f(\varepsilon z)\, d\varepsilon.$$

At that, the inverse operator L_ω^{-1} is a differential operator of the form

$$L_\omega^{-1}[f](z) = \prod_{j=1}^m \frac{h_j + L_0}{h_j}.$$

The theory of such operators was exposed in the monographs by I.I. Bavrin [1, 2].

3.3. Let us discuss a fractional power of the operator $h + z\frac{d}{dz}$. We choose the function $\Delta(p)$ as

$$\Delta(p) = \frac{h^\alpha}{(h+p)^\alpha}.$$

Taking into consideration the formula

$$\frac{h^\alpha}{\Gamma(\alpha)} e^{-ht} t^{\alpha-1} \to \frac{h^\alpha}{(h+p)^\alpha}$$

in the table of Laplace transformations, according to formula (1.2), we establish the expression for the operator L_ω:

$$L_\omega[f](z) = \frac{h^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-h\varepsilon} \varepsilon^{\alpha-1} f\left(e^{-\varepsilon} z\right) d\varepsilon.$$

After the change of the variable $e^{-\varepsilon} = \tau$, we arrive at the following expression for the operator L_ω:

$$L_\omega[f](z) = \frac{h^\alpha}{\Gamma(\alpha)} \int_0^1 \varepsilon^{h-1} \left(\ln\frac{1}{\varepsilon}\right)^{\alpha-1} f(\varepsilon z)\, d\varepsilon, \quad h > 0, \quad \alpha > 0.$$

Thus, in the definition of the generalized Riemann-Liouville operator we should let

$$\omega(x) = \frac{h^{\alpha}}{\Gamma(\alpha)} \int_x^1 \varepsilon^{h-1} \left(\ln \frac{1}{\varepsilon}\right)^{\alpha-1} d\varepsilon.$$

a) If $\alpha = n \in N$, then the inverse operator L_{ω}^{-1} is calculated by the formula

$$L_{\omega}^{-1} = \left(h + z\frac{d}{dz}\right)^{n},$$

i.e., is differential.

b) If $\alpha \notin N$, the inverse operator L_{ω}^{-1} is of the form:

$$L_{\omega}^{-1}[f](z) \equiv \left(h + z\frac{d}{dz}\right)^{\alpha}[f](z)$$
$$= \frac{1}{\Gamma(n+1-\alpha)} \left(h + z\frac{d}{dz}\right)^{n+1} \int_0^1 \varepsilon^{h-1} \left(\ln \frac{1}{\varepsilon}\right)^{n-\alpha} f(\varepsilon z)\, d\varepsilon.$$

For instance, as $\alpha = \frac{1}{2}$, we obtain the following expression:

$$\left(h + z\frac{d}{dz}\right)^{\frac{1}{2}}[f](z) = \frac{1}{\sqrt{\pi}} \left(h + z\frac{d}{dz}\right) \int_0^1 \varepsilon^{h-1} \left(\ln \frac{1}{\varepsilon}\right)^{-\frac{1}{2}} [f](\varepsilon z)\, d\varepsilon.$$

3.4. Hadamard operator. We consider the class of functions $w = f(z)$ analytic in the unit circle $B = \{z : |z| < 1\}$ such that $f(0) = 0$. We choose the function $\Delta(p)$ as

$$\Delta(p) = \frac{1}{p^{\alpha}}.$$

Taking into consideration the formula

$$\frac{t^{\alpha-1}}{\Gamma(\alpha)} \to \frac{1}{p^{\alpha}}$$

from the table of the Laplace transformations, according to formula (2.1), we get the expression for the operator L_{ω}

$$L_{\omega}[f](z) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \varepsilon^{\alpha-1} f\left(e^{-\varepsilon} z\right) d\varepsilon, \quad \alpha > 0.$$

After the change of the variable $e^{-\varepsilon} = \tau$, we arrive at the following expression for operator L_{ω}

$$L_{\omega}[f](z) = \frac{1}{\Gamma(\alpha)} \int_0^1 \left(\ln \frac{1}{\varepsilon}\right)^{\alpha-1} \frac{f(\varepsilon z)}{\varepsilon} d\varepsilon. \tag{3.2}$$

The right hand side of formula (3.2) defines Hadamard operator [12].

a) If $\alpha = n \in N$, the inverse operator L_{ω}^{-1} is calculated by the formula

$$L_{\omega}^{-1} = L_0^{n},$$

i.e., is differential.

b) If $\alpha \notin N$, then inverse operator L_{ω}^{-1} is of the form:

$$L_{\omega}^{-1}[f](z) \equiv L^{\alpha}[f](z) = \frac{1}{\Gamma(n+1-\alpha)} L^{n+1} \left[\int_0^1 \left(\ln \frac{1}{\varepsilon}\right)^{n-\alpha} \frac{f(\varepsilon z)}{\varepsilon} d\varepsilon\right].$$

For instance, as $\alpha = \frac{1}{2}$, we obtain the following expression:

$$L^{\frac{1}{2}}[f](z) = \frac{1}{\sqrt{\pi}} L \left[\int_0^1 \left(\ln \frac{1}{\varepsilon}\right)^{-\frac{1}{2}} \frac{f(\varepsilon z)}{\varepsilon} d\varepsilon\right].$$

3.5. If $\Delta(p) = \frac{h^2}{h^2+p^2}$, we obtain $Q(z) = \frac{h^2+z^2}{h^2}$, and, hence, the inverse operator is a differential operator of the form

$$L_\omega^{-1} = \frac{h^2 + L_0^2}{h^2}, L_0 = z\frac{d}{dz}.$$

Let us find the direct operator L_ω. By a formula in the table of the Laplace transformations,

$$h \sin ht \to \frac{h^2}{h^2 + p^2}.$$

According to Theorem 2.1, let us find the expression for the operator L_ω in formula (2.1)

$$L_\omega [f](z) = \int_0^\infty h \sin (h\varepsilon) f\left(e^{-\varepsilon} z\right) d\varepsilon.$$

As a result, after a change of the variable, we have

$$L_\omega [f](z) = \int_0^1 \frac{h}{\varepsilon} \sin \left(h \ln \frac{1}{\varepsilon}\right) f(\varepsilon z) d\varepsilon.$$

Remark 1. *Here we have* $\omega'(x) = -\frac{h}{x} \sin \left(h \ln \frac{1}{x}\right)$ *and hence,* $\omega(x) \notin \Omega$.

3.6. Fractional power of the operator $h^2 + L_0^2$.

We let

$$\Delta(p) = \frac{\left(h^2 + p^2\right)^\alpha}{h^{2\alpha}}, \quad h \neq 0, \quad \alpha > 0.$$

By a formula in the table of the Laplace transformations,

$$\frac{\sqrt{\pi} \cdot h}{\Gamma(\alpha)} \left(\frac{th}{2}\right)^{\alpha - \frac{1}{2}} J_{\alpha - \frac{1}{2}}(ht) \to \frac{h^{2\alpha}}{\left(h^2 + p^2\right)^\alpha},$$

where $J_{\alpha-\frac{1}{2}}(z)$ is the Bessel function of order $\alpha - \frac{1}{2}$. By Theorem 2.1 and formula (2.1) we find the expression for operator L_ω

$$L_\omega [f](z) = \frac{\sqrt{\pi}}{\Gamma(\alpha)} \int_0^1 \frac{h}{\varepsilon} \left(\frac{h \ln \frac{1}{\varepsilon}}{2}\right)^{\alpha - \frac{1}{2}} J_{\alpha - \frac{1}{2}} \left(h \ln \frac{1}{\varepsilon}\right) f(\varepsilon z) d\varepsilon.$$

a) If $\alpha = n \in N$, then the inverse Riemann-Liouville operator is a differential operator of the form

$$L_\omega^{-1} = \frac{\left(h^2 + L_0^2\right)^n}{h^{2n}}.$$

b) If $\alpha \notin N$, then the inverse operator L_ω^{-1} is of the form:

$$L_\omega^{-1} [f](z) \equiv \left(h^2 + L_0^2\right)^\alpha [f](z) / h^{2\alpha}$$

$$= \frac{\sqrt{\pi} h^{-2\alpha}}{\Gamma(n + 1 - \alpha)} \int_0^1 \varepsilon^{-1} \left(\frac{\ln \frac{1}{\varepsilon}}{2h}\right)^{n - \alpha + \frac{1}{2}} J_{n - \alpha + \frac{1}{2}} \left(h \ln \frac{1}{\varepsilon}\right) \left(h^2 + L_0^2\right)^{n+1} [f](\varepsilon z) d\varepsilon,$$

i.e., is an integral-differential operator.

3.7. We let

$$\Delta(p) = \sqrt{h + p} e^{\frac{k}{h+p}}; \quad k, h > 0.$$

In the table of the Laplace transformations we choose the formula

$$e^{-ht} \frac{1}{\sqrt{\pi t}} \cos 2\sqrt{kt} \to \frac{1}{\sqrt{h + x} e^{\frac{k}{h+x}}}.$$

Due to Theorem 2.1 and formula (2.1) we find the expression for the operator L_ω:

$$L_\omega [f] = \frac{1}{\sqrt{\pi}} \int_0^1 \frac{\varepsilon^{h-1}}{\sqrt{\ln \frac{1}{\varepsilon}}} \cos 2\sqrt{k \ln \frac{1}{\varepsilon}} f(\varepsilon z) d\varepsilon.$$

Taking into consideration the formula

$$e^{-ht}\frac{1}{\sqrt{\pi t}}ch2\sqrt{kt} \to \frac{1}{\sqrt{h+x}}e^{\frac{k}{h+x}},$$

we find the inverse operator L_ω^{-1}:

$$L_\omega^{-1}[f] = \frac{1}{\sqrt{\pi}}(h+L_0)\int_0^1 \frac{\varepsilon^{h-1}}{\sqrt{\ln\frac{1}{\varepsilon}}}ch2\sqrt{k\ln\frac{1}{\varepsilon}}\cdot f(\varepsilon z)\,d\varepsilon.$$

4. Conclusion

In the work we presented the technique of applying integral Laplace transform in the theory of the generalized Riemann-Liouville operators. We found the closed formulae for the inverse of the generalized Riemann-Liouville operators. The main result of the work is the extension of the scalar sequence

$$\Delta(k) = k\int_0^1 r^{k-1}\omega(x)\,dx < \infty, k = 1, 2, \ldots$$

in the half-plane $p:\quad \mathrm{Re}\,p = \sigma > \sigma_0 \geqslant 0$. The problem on inversion of the generalized Riemann-Liouville operator is solved in the case when the function $\Delta(p)$ is the Laplace image, while the function $\Delta^{-1}(p)$ has a power growth, i.e.,

$$\frac{1}{\Delta(p)} = p^\alpha L(p), \quad \alpha > 0,$$

where $L(p)$ is a slowly varying function [14].

In particular, in the work we presented the theory of fractional powers of the Hadamard operator; in our notations, this is the operator L_0. We considered the theory of the fractional powers of the generalized Hadamard operators such as the operator $h + L_0$, $h > 0$, and the operator $h^2 + L_0^2$, $h > 0$.

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