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STATIONARY HARMONIC FUNCTIONS ON HOMOGENEOUS SPACES

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Abstract. Stationary harmonic functions on homogeneous spaces are considered. A relation to double periodic harmonic functions of three variables is showed.

Keywords: harmonic function, multiplicatively periodic function, double periodic function, homogeneous space, Klein space, invariant family, stationary element with respect to a subgroup, punctured Euclidean space.

Mathematics Subject Classification: 31B05

1. Introduction

A function f on $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ is said to be **multiplicatively periodic** if there exists q, $0 < |q| \neq 1$ such that

$$\forall z \in \mathbb{C}^* \quad f(qz) = f(z). \tag{1}$$

Such a q is called multiplicator of f.

The theory of meromorphic functions satisfying (1) is dual to the theory of elliptic functions, which are double periodic meromorphic functions on $\mathbb{C}$ ([1]-[3]).

Holomorphic functions, harmonic functions, subharmonic functions satisfying (1) are constant.

In this connection we try to answer the questions:
(i) do multiplicatively periodic non-constant harmonic functions of several variables exist?
(ii) do double periodic non-constant harmonic functions of three variables exist?
(iii) if yes, what are their representations?

Note that $\mathbb{C}^*$ is a nonlinear homogeneous space on which multiplicative group $\mathbb{C}^*$ acts and that (1) implies

$$\forall n \in \mathbb{Z} \quad \forall z \in \mathbb{C}^* \quad f(q^n z) = f(z).$$

We will say that f is stationary with respect to the cyclic group $\{q^n\}$, $n \in \mathbb{Z}$, generated by q.

Note also that each multiplicatively periodic harmonic functions of multiplicator q, $0 < q \neq 1$, in the punctured Euclidean space $\overset{\circ}{\mathbb{R}}{}^m = \mathbb{R}^m\backslash\{0\}$, $m \geq 3$, is constant due to the extremum principle and the counterpart of the Liouville theorem. Hence, in order to solve problems *(i)-(iii)* we should consider more general homogeneous spaces.

2. Homogeneous spaces

Definition 1. *Let X be a topological space, G be a group of homeomorphic mappings of X onto X. A couple (X, G) is called* **homogeneous space.**

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If G is transitive, that is

$$\forall x_1, x_2 \in X \quad \exists \tau \in G \quad (x_2 = \tau x_1),$$

then (X, G) is said to be *the Klein space* (see [4]).

Example 1. *Let G be a group of the linear transformations of Euclidean space $\mathbb{R}^n$. It is transitive. Then $(\mathbb{R}^n, G)$ is the linear homogeneous space. It is the Klein space.*

Example 2. *Let $X = \mathbb{R}^n \backslash \{0\}$, $G = SO(n)$. X is invariant with respect to homothetic transformations, G is intransitive. Thus, instead of G the compositions of rotations and homothetic transformations G_1 can be taken.Then G_1 is transitive and $(\mathbb{R}^n \backslash \{0\}, G_1)$ is the nonlinear homogeneous space. It is the Klein space, too.*

3. Invariant functional spaces on (X, G). Stationary elements with respect to subgroups

Definition 2. *Let (X, G) be a homogeneous space. A set (family) of functions $\mathcal{F}$ is said to be* **invariant** *if it satisfies the following condition*

$$\forall f \in \mathcal{F} \quad \forall \tau \in G \quad (f \circ \tau \in \mathcal{F}).$$

Definition 3. *Let (X, G) be a homogeneous space, $\mathcal{F}$ be an invariant family, H be a subgroup of group G. An element $f \in \mathcal{F}$ is called* **stationary with respect to** *H if*

$$\forall \tau \in H \quad (f \circ \tau = f).$$

The set of such elements is denoted by $\mathcal{F}_H$.

4. Stationary harmonic functions on homogeneous spaces

The space $\overrightarrow{\mathbb{R}}^3 = \{(x_1, x_2, x_3) = x : x_1^2 + x_2^2 > 0\}$ is called *pierced Euclidean space.* It is nonlinear, invariant with respect to the rotations around axis x_3 and homothetic transformations. The composition of the rotations around axis x_3 and homothetic transformations forms a group, which we denote by G. Hence, we obtain nonlinear homogeneous space $(\overrightarrow{\mathbb{R}}^3, G)$. It is the Klein space.

One of the functional spaces invariant with respect to group G is the linear space of harmonic in $\overrightarrow{\mathbb{R}}^3$ functions (see, for example, [5]). We denote it by $\mathcal{H}$.

The rotation by an angle α around axis x_3 is given by the following matrix

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Fix q, $0 < q < 1$. Let H be a composition of some A and the cyclic group $\{q^n\}$, $n \in \mathbb{Z}$.

It is an open problem to describe stationary elements from $\mathcal{H}_H$ which are harmonic in $\overrightarrow{\mathbb{R}}^3$ functions satisfying the condition

$$\forall x \in \overrightarrow{\mathbb{R}}^3 \qquad h(q^n A x) = h(x).$$

However, we can show that class $\mathcal{H}_H$ is non-trivial, i.e., it contains non-constant harmonic functions.

We consider the series

$$\sum_{n=0}^{+\infty} \Big(\frac{1}{|a|} - \frac{1}{|q^n x - a|} \Big) - \sum_{n=1}^{+\infty} \frac{1}{|q^{-n} x - a|} \tag{2}$$

introduced in ([6]). It was proved there that for any fixed $a \in \overset{\circ}{\mathbb{R}}{}^3$, $q < |a| \leqslant 1$, the remainder of series (2) converges uniformly on the compact subsets from $\overset{\circ}{\mathbb{R}}{}^3$ and that the sum $K(x,a)$ of (2) is subharmonic in $\overset{\circ}{\mathbb{R}}{}^3$.

Denote $X_3 = \{(0,0,x_3) : x_3 \neq 0\}$. If $a \in X_3$, then each term of (2) is a harmonic function in $\overset{\rightarrow}{\mathbb{R}}{}^3$. Indeed, the fundamental solution of the Laplace equation in $\mathbb{R}^3$ is $\frac{1}{|x|}$. Since $q^k x \neq a$, $k \in \mathbb{Z}$, the function $\frac{1}{|q^k x - a|}$ is harmonic in $\overset{\rightarrow}{\mathbb{R}}{}^3$. Therefore, the function $K(x,a)$ is harmonic in $\overset{\rightarrow}{\mathbb{R}}{}^3$ if $a = (0,0,a_3) \in X_3$, $q < |a| \leqslant 1$.

Note that the function $K(x,a)$ is independent of α, namely,

$$\forall A \quad K(Ax,a) = K(x,a). \tag{3}$$

Let $y = Ax$, that is $y_1 = x_1 \cos\alpha - x_2 \sin\alpha$, $y_2 = x_1 \sin\alpha + x_2 \cos\alpha$, $y_3 = x_3$. Consider the absolute value $|q^k y - a|$, $k \in \mathbb{Z}$. We have

$$|q^k y - a| = |(q^k x_1 \cos\alpha - q^k x_2 \sin\alpha,\ q^k x_1 \sin\alpha + q^k x_2 \cos\alpha,\ q^k x_3 - a_3)| =$$

$$= \sqrt{q^{2k} x_1^2 + q^{2k} x_2^2 + (q^k x_3 - a_3)^2} = |q^k x - a|.$$

Thus, identity (3) is valid.

It is easy to check that

$$K(qx,a) = K(x,a) - \frac{1}{|a|}. \tag{4}$$

Let $a = (0,0,1)$ and $b = (0,0,-1)$.

The function

$$h(x) = K(x,a) - K(x,b)$$

is harmonic in $\overset{\rightarrow}{\mathbb{R}}{}^3$.

Using identities (3) and (4), we obtain

$$h(qAx) = K(x,a) - \frac{1}{|a|} - K(x,b) + \frac{1}{|b|}.$$

Since $|a| = |b|$, we get

$$\forall x \in \overset{\rightarrow}{\mathbb{R}}{}^3 \quad h(qAx) = h(x).$$

Thus, $h \in \mathcal{H}_H$.

5. A class of functions in $\mathcal{H}_H$

We denote by $\mathcal{B}$ the class of bounded Borel sets in $\overset{\circ}{\mathbb{R}}{}^3$ whose closures belong to $\overset{\circ}{\mathbb{R}}{}^3$. For $B \in \mathcal{B}$ we let

$$qB = \{qx : x \in B\}, \quad 0 < q < 1.$$

Theorem A ([7]). A measure μ in $\overset{\circ}{\mathbb{R}}{}^3$ is the Riesz measure of a multiplicatively periodic δ-subharmonic functions of multiplicator q if and only if

(i) $\mu(qB) = q\mu(B)$ for each $B \in \mathcal{B}$;

(ii) $\int\limits_{qr<|x|\leqslant r} \frac{d\mu}{|x|} = 0$ for all $r > 0$.

Theorem B ([7]). Each multiplicatively periodic δ-subharmonic in $\overset{\circ}{\mathbb{R}}{}^3$ function u of multiplicator q satisfies the representation

$$u(x) = C + \int\limits_{q<|a|\leqslant 1} K(x,a)\, d\mu_u(a),$$

where C is a constant.

The following theorem describes a class of harmonic functions from $\mathcal{H}_H$.

Theorem 1. *If a Borel measure μ on $\overset{\circ}{\mathbb{R}}{}^3$ satisfies the conditions*
1) $\mu(qB) = q\mu(B)$ for each $B \in \mathcal{B}$;
2) $\int\limits_{q<|a|\leqslant 1} \frac{d\mu}{|a|} = 0$;
3) $\mu(B) = \mu(B \bigcap X_3), \quad \mu(\emptyset) = 0$;
then the function

$$h(x) = \int\limits_{q<|a|\leqslant 1} K(x,a)d\mu(a) \tag{5}$$

belongs to $\mathcal{H}_H$ and vice versa each $h \in \mathcal{H}_H$ which admits a δ-subharmonic continuation on $\overset{\circ}{\mathbb{R}}{}^3$ satisfies the representation

$$h(x) = C + \int\limits_{q<|a|\leqslant 1} K(x,a)d\mu(a), \tag{6}$$

where C is a constant and μ satisfies 1)-3).

Proof. Let μ satisfy Conditions 1), 2). According to Theorem B, function h defined by (5) is multiplicatively periodic δ-subharmonic in $\overset{\circ}{\mathbb{R}}{}^3$ of multiplicator q. In virtue of condition 3) h is harmonic in $\overset{\rightarrow}{\mathbb{R}}{}^3$. Taking into account that $K(x,a)$ is independent of A we have $h \in \mathcal{H}_H$.

Now let h be a function from $\mathcal{H}_H$ admitting a δ-subharmonic continuation on $\overset{\circ}{\mathbb{R}}{}^3$. According to Theorems A and B it has representation (6), where μ satisfies Conditions 1), 2). Since $\triangle h = 0$ in $\overset{\rightarrow}{\mathbb{R}}{}^3$, μ satisfies also Condition 3). This completes the proof. □

6. Double periodic harmonic functions in a layer

Let $h(x)$ be a multiplicatively periodic harmonic in $\overset{\rightarrow}{\mathbb{R}}{}^3$ function of multiplicator q, $0 < q \neq 1$. Consider the mapping

$$x_1 = e^{\xi}\cos\eta, \quad x_2 = e^{\xi}\sin\eta, \quad x_3 = e^{\xi}\cot\zeta,$$

where $\xi, \eta \in \mathbb{R}$, $0 < \zeta < \pi$. We have $x_1^2 + x_2^2 = e^{2\xi} > 0$. Hence, it maps the layer $\{(\xi,\eta,\zeta) : \xi,\eta \in \mathbb{R},\ 0 < \zeta < \pi\}$ onto $\overset{\rightarrow}{\mathbb{R}}{}^3$ with the Jacobians $\mathcal{J} = \frac{-e^{3\xi}}{\sin^2\zeta}$. Laplacian $\triangle$ becomes

$$\triangle = e^{-2\xi}\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} + \sin^2\zeta\frac{\partial^2}{\partial\zeta^2} + \sin 2\zeta\left(\frac{1}{2}\frac{\partial}{\partial\zeta} + \frac{\partial^2}{\partial\xi\partial\zeta}\right)\right).$$

Denote

$$g(\xi,\eta,\zeta) = h\big(e^{\xi}\cos\eta, e^{\xi}\sin\eta, e^{\xi}\cot\zeta\big).$$

The function g is defined in the layer $\{(\xi,\eta,\zeta) : \xi,\eta \in \mathbb{R},\ 0 < \zeta < \pi\}$.

Since $h(qx) = h(x)$, we have

$$g(\xi + \log q, \eta + 2\pi, \zeta) = g(\xi,\eta,\zeta).$$

Indeed,

$$g(\xi + \log q, \eta + 2\pi, \zeta) = h\big(e^{\xi+\log q}\cos(\eta+2\pi), e^{\xi+\log q}\sin(\eta+2\pi), e^{\xi+\log q}\cot\zeta\big) =$$
$$= h\big(qe^{\xi}\cos\eta, qe^{\xi}\sin\eta, qe^{\xi}\cot\zeta\big) = h\big(e^{\xi}\cos\eta, e^{\xi}\sin\eta, e^{\xi}\cot\zeta\big) = g(\xi,\eta,\zeta).$$

Denoting $\omega_1 = \log q$, $\omega_2 = 2\pi$, we obtain double periodic harmonic function g of period $\Lambda = (\mathbb{Z}\omega_1, \mathbb{Z}\omega_2, \zeta)$. That is, such a function is stationary with respect to a group of the translations indicated above.

Remark. *The connection between the local spherical coordinates and the new substitution is as follows*

$$e^{\xi} = r \sin\theta, \quad \eta = \phi, \quad \zeta = \theta.$$

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