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REGULARIZATION OF SEQUENCES IN SENSE OF E.M. DYN'KIN

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Abstract. We introduce the notion of strong regularization of positive sequences. We prove an existence criterion of regular in the sense of E.M. Dyn'kin non-quasi-analiticity minorant. The criterion is given in terms on the smallest concave majorant of the logarithm of its trace function. The proof is based on the properties of the Legendre transformation.

Keywords: Carleman class, regular sequences, Legendre transform.

Mathematics Subject Classification: 30D60

1. Introduction

In studying Carleman classes $C_\gamma(M_n)$ on arbitrary continuums γ of the complex plane, a special role is played by regular in some sense sequences $\{M_n\}$; many statements are proven exactly for such sequences. It is happened that if γ is an arc of a bounded slope, as in the case of the segment a $I=[0,1]$, in the Bang type theorems, sequence of numbers $M_n>0$ can be arbitrary [1].

Let $\{M_n\}_{n=0}^{\infty}$ be a sequence of positive numbers. Some of numbers M_n can be equal to $+\infty$ but we assume that there exist infinitely many finite M_n. A Carleman class on an arc $\gamma\subset\mathbb{C}$ is the set

$$C_\gamma(M_n)=\{f\in C^\infty(\gamma):\sup_{z\in\gamma}\left|f^{(n)}(z)\right|\leqslant K_f^n M_n,\quad n=0,1,2,\ldots\}.$$

Here for $a\in\gamma$, derivative $f'(a)$ is understood as the limit

$$f'(a)=\lim_{z\in\gamma,z\to a}\frac{f(z)-f(a)}{z-a}.$$

The higher derivatives $f^{(n)}(a)$ $(n=2,3,\ldots)$ are determined by induction.

Definition 1. *Class $C_\gamma(M_n)$ is called quasi-analytic, if $f\in C_\gamma(M_n)$ and $f^{(n)}(c)=0$ for each $n\geqslant 0$ at some point c of arc γ implies $f(z)\equiv 0$.*

Necessary and sufficient conditions for the quasi-analiticity of (Carleman, Ostrovsky and Mandelbrojt-Bang) class $C_I(M_n)$ are provided in Denjoy-Carleman theorem [2, Ch. IV, 1.III]. This theorem implies that if $\varliminf_{n\to\infty} M_n^{\frac{1}{n}}<\infty$, class $C_I(M_n)$ is quasi-analytic. This is why in such statements one usually assumes $M_n^{\frac{1}{n}}\to\infty$ as $n\to\infty$.

Ostrovsky quasi-analyticity criterion is formulate in terms of trace function

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$$T(r) = \sup_{n\geqslant 0} \frac{r^n}{M_n}, \qquad r > 0.$$

The trace function is defined and finite on the positive semi-axis $\mathbb{R}_+$, while the function $\ln T(e^x)$, as the upper envelope of linear functions, is a convex function on $\mathbb{R}$. Hence, trace function $T(r)$ is continuous on $\mathbb{R}_+$. Sequence

$$M_n^c = \sup_{r>0} \frac{r^n}{T(r)}$$

is called a convex regularization of sequence $\{M_n\}_{n=0}^\infty$ by means of logarithms. It possesses the properties [2]:

$$1)\ M_n^c \leqslant M_n\ (n \geqslant 0); \quad 2)\ (M_n^c)^{\frac{1}{n}} \to \infty, n \to \infty; \quad 3)\ M_n^c \leqslant \sqrt{M_{n-1}^c M_{n+1}^c}\ (n \geqslant 1).$$

Moreover, trace functions $T(r)$, $T_c(r)$ of sequences $\{M_n\}_{n=0}^\infty$, $\{M_n^c\}_{n=0}^\infty$ coincides and

$$M_n^c = \sup_{r>0} \frac{r^n}{T_c(r)}.$$

Sequence $\{M_n\}$ is called regular in the sense of E.M. Dyn'kin if the numbers $m_n = \frac{M_n}{n!}$ possess the properties [3]:

a) $\quad m_n^2 \leqslant m_{n-1}m_{n+1} \qquad (n \geqslant 1)$;

b) $\quad \sup\limits_{n\geqslant 1} \left(\frac{m_{n+1}}{m_n}\right)^{\frac{1}{n}} < \infty$;

c) $\quad m_n^{\frac{1}{n}} \to \infty$ as $n \to \infty$.

In accordance with Denjoy-Carleman, class $C_I(M_n)$ is quasi-analytic if and only if at least of the following equivalent conditions holds true [2], [4]:

$$\text{d)} \quad \int_1^\infty \frac{\ln T(r)}{r^2} dr = \infty; \qquad \text{e)} \quad \sum_{n=0}^\infty \frac{M_n^c}{M_{n+1}^c} = \infty.$$

As E.M. Dyn'kin showed [3], for a regular sequence $\{M_n\}$, Condition e) (and thus, Condition d)) is equivalent to Levinson bi-logarithmic condition

$$\int_0^d \ln \ln h(r) dr = +\infty, \tag{1}$$

where

$$h(r) = \sup_{n\geqslant 0} \frac{1}{m_n r^n} \ (r > 0), \tag{2}$$

and quantity $d > 0$ is chosen so that $h(d) \geqslant e$. It is clear $h(r)$ is a decreasing function $\lim\limits_{r\to 0} h(r) = \infty$ and

$$m_n = \sup_{r>0} \frac{1}{r^n h(r)} \qquad (n \geqslant 0).$$

We observe that introducing of regularized sequences is motivated by the fact "there is no analog of regularization theory for general sets not being a segment ensuring that each Carleman class coincides with a regularized class" [3].

Let γ be an arc described by the equation $y = g(x)$ ($|x| \leqslant a$) and satisfying Lipschitz condition

$$\sup_{x_1\neq x_2} \left| \frac{g(x_2) - g(x_1)}{x_2 - x_1} \right| = q_\gamma < \infty,$$

i.e., γ is an arc of a bounded slope [5].

It was shown in [5] that if $\{M_n\}$ is a regular sequence, then class $C_\gamma(M_n)$ is quasi-analytic if and only if the associated weight h introduced by (2) satisfies condition (1). Thus, in the present case, Denjoy-Carleman theorem remains true for $C_\gamma(M_n)$.

In the general situation, Carleman class $C_\gamma(M_n)$ (γ is a continuum in $\mathbb{C}$) can be non-regular (i.e., sequence $\{M_n\}$ is not regular). Since in the approximation theory non-quasi-analytic Carleman classes are of special interest, in view of said above [3], it is important to find out under which conditions for M_n there exists a regular sequence $\{M_n^*\}$ such that 1) $M_n^* \leqslant M_n$; 2) $\{M_n^*\}$ satisfies non-quasi-analyticity condition for the corresponding Carleman classes. In the present paper we prove a criterion for existence of such sequence $\{M_n^*\}$.

2. Criterion for existence of a regular non-quasi-analyticity minorant

A sequence $\{M_n\}$ ($M_n > 0$) is called weakly regular if numbers $m_n = \frac{M_n}{n!}$ satisfy Conditions a), c) of the definition of the regular sequence [6]. Each its regular minorant (if it exists) will be called regularization in the E.M. Dyn'kin sense. It is obvious that as $m_n^{\frac{1}{n}} \to \infty$, there exists a weakly regular sequence $\{M_n^*\}$ such that $M_n^* \leqslant M_n$ ($n \geqslant 0$), at that, $M_{n_i}^* = M_{n_i}$ for some sequence of indices n_i, $n_i \to \infty$ (n_i are main indices obtained in weak regularization of sequence $\{m_n\}$ by means of logarithms). If sequence $\{M_n^*\}$ is regular, we call it a strong regularization of sequence $\{M_n\}$.

The following theorem was proven in [6].

Theorem 1. *Let $M_n > 0$, $\left(\frac{M_n}{n!}\right)^{\frac{1}{n}} \to \infty$ as $n \to \infty$. There exists a regular sequence $\{M_n^*\}$, such that*

$$M_n^* \leqslant M_n, \qquad \sum_{n=1}^{\infty} \frac{M_n^*}{M_{n+1}^*} < \infty$$

if and only if there exists a positive continuous on $\mathbb{R}_+$ function $r = r(t)$, $tr(t) \downarrow 0$, $t^2 r(t) \uparrow$ as $t \to \infty$, such that

$$1) \frac{1}{M_n^{\frac{1}{n}}} \leqslant r(n) \ (n \geqslant 1); \qquad 2) \int_1^{\infty} r(t)dt < \infty.$$

Theorem 1 happens to be possible to reformulate in another way, namely, in terms of trace function of sequence $\{M_n\}$.

The following theorem is true.

Theorem 2. *Let $M_n > 0$, $\left(\frac{M_n}{n!}\right)^{\frac{1}{n}} \to \infty$ as $n \to \infty$. There exists a regular sequence $\{M_n^*\}$ such that*

$$M_n^* \leqslant M_n, \qquad \sum_{n=1}^{\infty} \frac{M_n^*}{M_{n+1}^*} < \infty$$

if and only if

$$\int_1^{\infty} \frac{\omega_T(r)}{r^2} dr < \infty.$$

Here $\omega_T = \omega_T(r)$ is the smallest concave majorant of function $\ln T(r)$, where

$$T(r) = \max_{n \geqslant 0} \frac{r^n}{M_n}.$$

The proof of the theorem is based on properties of Legendre transformation. This is why we briefly dwell on them.

Let $M(x)$ be a continuous increasing on $[0,\infty)$ function, $M(x) = o(x)$ as $x \to \infty$. Then function

$$m(y) = \sup_{x>0}(M(x) - xy)$$

defined for $y > 0$ is called Legendre transformation of function $M(x)$. If $M(x) \to \infty$ as $x \to \infty$, then $m(y) \to \infty$ as $y \to 0$. As the upper envelope of decreasing in $y > 0$ functions, function $m(y)$ decreases, too. We let

$$M^*(x) = \inf_{y>0}(m(y) + yx).$$

It is clear that M^* is the smallest concave increasing majorant of function M: $M(x) \leqslant M^*(x)$. We note that if function M is concave, then $M(x)/x \downarrow$ as $x \geqslant a$. On the other hand, if $0 < M(x) \uparrow$, $M(x)/x \downarrow$ as $x > 0$, then $M^*(x) < 2M(x)$, as M^* is the smallest concave majorant M [7; Ch. VII, Sect. D].

Theorem 3 (7; Ch. VII, Sect. D)**.** *Let $M(x)$ be an increasing concave on $[0,\infty)$ function, $m(y)$ be the Legendre transformation of function $M(x)$, $a > 0$ be such that $m(a) = 1$. Then the integrals*

$$\int_0^a \ln m(y)dy, \qquad \int_1^\infty \frac{M(x)}{x^2}dx$$

converge or diverge simultaneously.

We proceed to the proof of Theorem 2.

Proof. Necessity. The identities

$$\frac{z^n}{M_n} = \frac{n!}{M_n}\,\frac{1}{2\pi i}\int_{|t|=\delta}\frac{e^{zt}}{t^{n+1}}dt \qquad (\delta > 0)$$

and conditions $M_n^* \leqslant M_n$, $\left(\frac{M_n^*}{n!}\right)^{\frac{1}{n}} \to \infty$ as $n \to \infty$ imply

$$\frac{r^n}{M_n} \leqslant \frac{n!}{M_n^*\delta^n}\; e^{\delta r} \leqslant H^*(\delta)\; e^{\delta r} \qquad (|z| = r), \tag{3}$$

where

$$H^*(\delta) = \sup_{n\geqslant 0}\frac{n!}{M_n^*\delta^n}.$$

It follows from (3) that

$$T(r) \leqslant \exp\left[\inf_{\delta>0}(\ln H^*(\delta) + \delta r)\right] = \exp(\omega^*(r)).$$

It is clear that ω^* is non-negative, unboundedly increasing and concave on $[0,\infty)$ function. It is obvious that $\ln H^*(\delta) + \delta r \geqslant \omega^*(r)$. Therefore,

$$\ln H^*(\delta) \geqslant \sup_{r>0}(\omega^*(r) - \delta r) \equiv m(\delta).$$

Since

$$\sum_{n=1}^{\infty}\frac{M_n^*}{M_{n+1}^*} < \infty,$$

then [3]

$$\int_0^d \ln\ln H^*(\delta)d\delta < \infty,$$

and moreover,

$$\int_0^d \ln m(\delta)d\delta < \infty,$$

where $d > 0$ is such that $m(d) = 1$. Then by Theorem 3, increasing function ω^* belongs to convergence class Ω, i.e.,

$$\tfrac{\omega^*(r)}{r} \downarrow 0 \text{ as } r \to \infty, \qquad \int_1^\infty \tfrac{\omega^*(r)}{r^2}dr < \infty.$$

Since $\omega_T(r) \leqslant \omega^*(r)$, function ω_T belongs to Ω as well.

Sufficiency. Let

$$\int_1^\infty \frac{\omega_T(r)}{r^2}dr < \infty.$$

Then

$$I(\delta) = \delta\int_0^\infty T(r)e^{-\delta r}dr \leqslant \delta\int_0^\infty \exp(\omega_T(r) - \delta r)dr = \delta M(\delta) = H(\delta).$$

Since

$$\frac{r^n}{M_n} \leqslant T(r) \qquad (n \geqslant 0),$$

then

$$\delta\,\frac{1}{M_n}\int_0^\infty r^n e^{-\delta r}dr \leqslant I(\delta) \leqslant H(\delta).$$

It implies that for each $n \geqslant 0$

$$\frac{n!}{M_n\delta^n} \leqslant H(\delta) \qquad (\delta > 0), \tag{4}$$

in particular, $M_n' \leqslant M_n$, where

$$M_n' = \sup_{\delta>0}\frac{n!}{H(\delta)\delta^n} \qquad (n \geqslant 0).$$

Since estimates (4) are valid for each $n \geqslant 0$, as $\delta \to 0$ we have

$$\lim_{\delta\to 0}\frac{\ln\frac{1}{\delta}}{\ln H(\delta)} = 0.$$

Therefore, $\frac{\ln H(e^{-x})}{x} \to +\infty$ as $x \to +\infty$ and hence $M_n' = n!\, e^{c_n}$, where

$$c_n = \sup_{x>0}(nx - \ln H(e^{-x})).$$

Let us employ the properties of Yonug-Fenchel-Legendre transformation [8, P. II, Ch. 1, Sect. 5, Prop. 1]: *if function φ is continuous on $\mathbb{R}_+$ and $\frac{\varphi(y)}{y} \to +\infty$ as $y \to +\infty$, then function*

$$\psi(x) = \sup_{y>0}(xy - \varphi(y))$$

adjoint to φ in the Young sense is convex on $\mathbb{R}_+$ and satisfies the condition

$$\frac{\psi(x)}{x} \to +\infty \quad as \quad x \to +\infty.$$

Thus, we see that sequence $\{c_n\}$ is convex and $\frac{c_n}{n} \to \infty$ as $n \to \infty$. It means that $\frac{c_n}{n} \uparrow$ as $n \to \infty$. Hence,

$$\left(\frac{M'_n}{n!}\right)^{\frac{1}{n}} \uparrow \infty$$

as $n \to \infty$ and sequence $\{M'_n\}$, $M'_n \leqslant M_n$ is weakly regular.

Since $H(\delta) = \delta M(\delta)$, then

$$M'_n \geqslant \frac{n!}{M(\delta)\delta^{n+1}} \qquad (n \geqslant 0).$$

It implies

$$h(\delta) = \sup_{n\geqslant 0} \frac{n!}{M'_n \delta^{n+1}} \leqslant M(\delta). \tag{5}$$

Recalling the definition of function M, we have

$$M(\delta) \leqslant \exp\left(\sup_{r>0}(\omega_T(r) - \frac{\delta}{2}r)\right) \int_0^\infty e^{-\frac{\delta}{2}r} dr = \frac{2}{\delta} e^{m(\frac{\delta}{2})},$$

where $m(\delta)$ is the Legendre tranform of function $\omega_T(r)$.

Let $c > 0$ be such that $h(c) = e$. Since $h(\delta) \leqslant M(\delta)$, then $M(\delta) \geqslant e$ as $0 < \delta \leqslant c$. We have $\ln M(\delta) \leqslant \ln \frac{2}{\delta} + m(\frac{\delta}{2}) \quad (0 < \delta \leqslant c)$. Using estimate

$$\ln^+(a + b) \leqslant \ln^+ a + \ln^+ b + \ln 2,$$

where $\ln^+ x = \max(0, \ln x)$, we obtain that for $0 < \delta \leqslant q \leqslant c$

$$0 \leqslant \ln\ln M(\delta) < \ln m(\frac{\delta}{2}) + \ln\ln \frac{2}{\delta} + \ln 2.$$

But by Theorem 3, integral $\int_0^q \ln m(\delta) d\delta$ converges simultaneously with the integral $\int_1^\infty \frac{\omega_T(r)}{r^2} dr$. And since $\int_0^q \ln\ln \frac{2}{\delta} d\delta < \infty$, then

$$\int_0^q \ln\ln h(\delta) d\delta \leqslant \int_0^q \ln\ln M(\delta) d\delta < \infty.$$

Therefore,

$$\sum_{n=1}^\infty \frac{M'_n}{M'_{n+1}} < \infty. \tag{6}$$

Indeed, since sequence $\left\{\frac{M'_n}{n!}\right\}$ i logarithmically convex, by (5) we conclude that

$$\sup_{\delta>0} \frac{n!}{h(\delta)\delta^{n+1}} = M'_n < \infty \qquad (n \geqslant 0).$$

Now convergence of series (6) follows from Theorem 7 in work [5]. Then there exists a function $R = R(t)$, $0 < R(t) \downarrow 0$, $tR(t) \downarrow 0$, $t^2R(t) \uparrow$ as $t \to \infty$ such that [6]

$$1)\frac{1}{(M_n')^{\frac{1}{n}}} < R(n); \qquad 2)\int\limits_1^\infty R(t)dt < \infty.$$

Therefore, in accordance with Theorem 1, there exists a regular sequence $\{M_n^*\}$, $M_n^* \leqslant M_n'$,

$$\sum_{n=1}^\infty \frac{M_n^*}{M_{n+1}^*} < \infty.$$

The proof is complete. □

BIBLIOGRAPHY

1. R.A. Gaisin. *Estimation of intermediate derivatives and theorems of Bang type. I* // Alg. Anal. **27**:1, 23-48 (2015). [St. Petersburg Math. J. **27**:1, (2016), to appear.]
2. S. Mandelbrojt. *Séries adhŕentes, régularisations des suites, applications*. Gauthier-Villars, Paris (1952).
3. E.M. Dyn'kin. *Pseudo-analytic continuation of smooth functions. Uniform scale* // Mathematical programming and related question. Theory of functions and functional analysis. Proc. of VII Winter School, Drogobych, 40-73 (1976). (in Russian).
4. S. Mandelbrojt. *Quasi-analytic classes of functions*. Glav. red. nauch.-teor. liter. Leningrad-Moscow (1937). (in Russian).
5. A.M. Gaisin, I.G. Kinzyabulatov. *A Levinson-Sjöberg type theorem. Applications* // Matem. Sborn. **199**:7, 41-62 (2008). [Sb. Math. **199**:7, 985-1007 (2008).]
6. R.A. Gaisin. *Quasi-analyticity criteria of Salinas-Korenblum type for general domains* // Ufimskij Matem. Zhurn. **5**:3, 28-40 (2013). [Ufa Math. J. **5**:3, 28-39 (2013).]
7. P. Koosis. *The logarithmic integral I*. Cambridge University Press, Cambridge (1998).
8. G.G. Braichev. *Introduction to growth theory of convex and entire functions*. Prometej, Moscow (2005). (in Russian).

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