ISSN 2304-0122 Ufa Mathematical Journal. Vol. 5. No 2 (2013). P. 131-141.

doi:10.13108/2013-5-2-131 UDC 517.5

COMPLETENESS AND MINIMALITY OF SYSTEMS OF BESSEL FUNCTIONS

B.V. VYNNYTS'KYI, R.V. KHATS'

Abstract. We find the necessary and sufficient conditions for the completeness and minimality in the space $L^2(0;1)$ of system $(\sqrt{x\rho_k}J_\nu(x\rho_k) : k \in \mathbb{N})$ generated by Bessel function of the first kind of index $\nu \geq -1/2$. Moreover, we establish a criterion for the completeness and minimality of system $(x^{-2}\sqrt{x\rho_k}J_{3/2}(x\rho_k) : k \in \mathbb{N})$ in the space $L^2((0;1);x^2dx)$.

Keywords: Paley-Wiener theorem, Bessel function, entire function, complete system, minimal system, biorthogonal system, basis.

Mathematics Subject Classification: 33C10, 30B60, 42A65, 42A38, 30D20, 42B10, 44A15, 30E15.

1. Introduction and preliminaries

Let $p \in [0;+\infty)$, $L^2((0;1);x^p dx)$ be the space of functions $f : (0;1) \to \mathbb{C}$ such that $t^{p/2}f(t) \in L^2(0;1)$ with the inner product $\langle f_1; f_2\rangle = \int_0^1 t^p f_1(t)\overline{f_2(t)}\,dt$ and the norm $\|f\|^2 := \int_0^1 t^p|f(t)|^2\,dt$. Let $J_\nu(x) = \sum_{k=0}^\infty \frac{(-1)^k(x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}$ be Bessel's function of the first kind of index ν. It is known (see [3], [25, p. 345], [32]) that the function J_ν is a solution of the equation $x^2y'' + xy' + (x^2-\nu^2)y = 0$, i.e. the equation $y'' + y'/x + (1-\nu^2/x^2)y = 0$, the function $y(x) = J_\nu(x\rho)$ is a solution of the equation $y'' + y'/x - y\nu^2/x^2 = -\rho^2 y$, and the function $y(x) = \sqrt{x\rho}J_\nu(x\rho)$ satisfies the equation

$$-y'' + \frac{\nu^2 - 1/4}{x^2}y = \rho^2 y.$$

The function J_ν for $\nu > -1$ has (see [3], [25, p. 350], [32]) an infinite set of zeros, among them positive zeros ρ_k, $k \in \mathbb{N}$, and negative zeros $\rho_{-k} := -\rho_k$, $k \in \mathbb{N}$. All zeros are simple, except perhaps, $\rho_0 = 0$.

Theorem A. *(see* [3], [25, p. 357], [32]*) Let $\nu > -1$ and $(\rho_k : k \in \mathbb{N})$ be a sequence of positive zeros of the function J_ν. Then the system $(\sqrt{x}J_\nu(x\rho_k) : k \in \mathbb{N})$ is an orthogonal basis in the space $L^2(0;1)$.*

The system $(\sqrt{x}J_\nu(x\rho_k) : k \in \mathbb{N})$ is also complete in $L^2(0;1)$ if $\rho_k J'_\nu(\rho_k) + \alpha J_\nu(\rho_k) = 0$, $\alpha + \nu > 0$ (see [16, p. 124], [25, pp. 356–357]). From [8] it follows that if $\nu > -1/2$ and $(\rho_k : k \in \mathbb{N})$ is a sequence of distinct positive numbers such that $\rho_k \leq \pi(k + \nu/2)$ for all sufficiently large $k \in \mathbb{N}$, then the system $(\sqrt{x}J_\nu(x\rho_k) : k \in \mathbb{N})$ is complete in the space $L^2(0;1)$.

We say that an entire function G is of formal exponential type $\sigma \in (0;+\infty)$ if

$$|G(z)| \leq c(\varepsilon)\exp((\sigma+\varepsilon)|z|), \quad z \in \mathbb{C},$$

for each $\varepsilon > 0$ and some constant $c(\varepsilon)$.

Submitted January 30, 2012.

Theorem 1. *Let $\nu \geq -1/2$ and $(\rho_k : k \in \mathbb{N})$ be an arbitrary sequence of distinct nonzero complex numbers. For a system $(\sqrt{t\rho_k}J_\nu(t\rho_k) : k \in \mathbb{N})$ to be incomplete in the space $L^2(0;1)$ it is necessary and sufficient that a sequence $(\rho_k : k \in \mathbb{N})$ is a subsequence of zeros of some even entire function G of formal exponential type $\sigma \leq 1$ such that the function $f(z) = z^{\nu+1/2}G(z)$ belongs to the space $L^2(\mathbb{R})$.*

The proof by standard methods (see [20, pp. 131–132], [21]) follows immediately from the following lemmas.

Lemma B. *(see* [2], [13]*) Let $\nu \geq -1/2$. A function f has the representation*

$$f(z) = \int\limits_0^1 \sqrt{zt}J_\nu(zt)\gamma(t)\,dt, \quad \gamma \in L^2(0;1),$$

if and only if $f \in L^2(0;+\infty)$ and $f(z) = z^{\nu+1/2}G(z)$, where G is an even entire function of formal exponential type $\sigma \leq 1$. Moreover, if $f \not\equiv 0$ then G is a transcendental entire function.

Lemma C. *(see* [11, p. 67], [24]*) Let $\nu > -1$. Then every function $f \in L^2(0;+\infty)$ can be represented in the form*

$$f(z) = \int\limits_0^{+\infty} \sqrt{zt}J_\nu(zt)h(t)\,dt$$

with some function $h \in L^2(0;+\infty)$. Moreover, $\|f\| = \|h\|$ and

$$h(t) = \int\limits_0^{+\infty} \sqrt{zt}J_\nu(zt)f(z)\,dz.$$

A system $(e_k : k \in \mathbb{N}_0)$ of the Hilbert space is said to be *minimal* (see [20, p. 131], [21, p. 4258], [22]) if for each $n \in \mathbb{N}_0$ the element e_n does not belong to the closure of the linear span of the system $(e_k : k \in \mathbb{N}_0 \setminus \{n\})$. A system is minimal if and only if it has a biorthogonal system. A complete system has, at most, one biorthogonal system (see [21], [22]).

Similarly to [20, Lecture 18], [21], from Lemmas B, C and Theorem 1, we obtain the following result.

Theorem 2. *Let $\nu \geq -1/2$ and $(\rho_k : k \in \mathbb{N})$ be an arbitrary sequence of distinct nonzero complex numbers such that $\rho_k^2 \neq \rho_m^2$ if $k \neq m$. The system $(\sqrt{t\rho_k}J_\nu(t\rho_k) : k \in \mathbb{N})$ is complete and minimal in the space $L^2(0;1)$ if and only if the sequence $(\rho_k : k \in \mathbb{Z} \setminus \{0\})$, $\rho_{-k} := -\rho_k$, is a sequence of zeros of some even entire function G of formal exponential type $\sigma \leq 1$ such that the function $z^{\nu+1/2}G(z)$ does not belongs to the space $L^2(0;+\infty)$ and the function $(z^2 - \rho_1^2)^{-1}z^{\nu+1/2}G(z)$ belongs to $L^2(0;+\infty)$. Moreover, the biorthogonal system $(\gamma_k : k \in \mathbb{N})$ is formed, in particular, by the functions γ_k, defined by the equality*

$$\overline{\gamma_k(t)} = \frac{2}{\rho_k^{\nu-1/2}G'(\rho_k)} \int\limits_0^{+\infty} \frac{\sqrt{zt}J_\nu(zt)z^{\nu+1/2}G(z)}{z^2 - \rho_k^2}\,dz.$$

Using methods of [18], [20] and [21], we can obtain a number of other various necessary and sufficient conditions for the completeness and minimality of system $(\sqrt{t\rho_k}J_\nu(t\rho_k) : k \in \mathbb{N})$ in the space $L^2(0;1)$. In particular, following the arguments of [20, Lecture 18], [21, §§1.7, 3.3], Theorem 1 yields the next statement.

Theorem 3. *Let $\nu \geq -1/2$ and $(\rho_k : k \in \mathbb{N})$ be an arbitrary sequence of distinct nonzero complex numbers such that $|\Im\rho_k| \geq \delta|\rho_k|$ for all $k \in \mathbb{N}$ and some $\delta > 0$. The system $(\sqrt{t\rho_k}J_\nu(t\rho_k) : k \in \mathbb{N})$ is complete in the space $L^2(0;1)$ if and only if*

$$\sum_{k=1}^{\infty} \frac{1}{|\rho_k|} = +\infty.$$

At studying of some non-classical boundary-value problems (see [26]–[31]) and generalized eigenvectors of some linear operators [28], [29] we needed to obtain the analogues of Theorems 1-3 for weighted spaces and establish an approximation properties of the special finite linear combinations of Bessel functions. We don't understand to the end the nature of expected results for an arbitrary $\nu \in \mathbb{R}$. For advance in the given direction it is important to investigate in details the simplest model cases $\nu = -3/2$ and $\nu = 3/2$. The case $\nu = -3/2$ was considered in [27], [30] (see also [26], [28], [31]). Here we consider the case $\nu = 3/2$ more detail. But even in this case we cannot obtain the all necessary facts. In particular, remains an open one for us the problem formulated at the end of this paper. In our view, its solution is very important for the construction of some spectral theory that is based on the notion of a generalized eigenvector (see [28], [29]).

It is well known (see [3], [25, p. 350], [32]) that $\sqrt{z}J_{3/2}(z) = -\sqrt{2/\pi}z^{-1}(z\cos z - \sin z)$. The function $\dfrac{\sqrt{x\rho}J_{3/2}(x\rho)}{x^2\rho^2}$ belongs to the space $L^2((0;1);x^2dx)$ for each $\rho \neq 0$. From Theorem A it follows that if $(\rho_k : k \in \mathbb{N})$ is a sequence of positive zeros of the function $J_{3/2}$ then the system $(\Theta_k : k \in \mathbb{N})$, $\Theta_k(x) := \dfrac{\sqrt{x\rho_k}J_{3/2}(x\rho_k)}{x^2\rho_k^2}$, is complete in the space $L^2((0;1);x^4dx)$. But from this statement it does not follows that the system $(\Theta_k : k \in \mathbb{N})$ is complete in the space $L^2((0;1);x^2dx)$. We investigate some approximation properties of the system $(\Theta_k : k \in \mathbb{N})$ in $L^2((0;1);x^2dx)$ with an arbitrary sequence of nonzero complex numbers $(\rho_k : k \in \mathbb{N})$. The main result of the paper is contained in Theorem 9 where is found a criterion for the completeness and minimality of system $(\Theta_k : k \in \mathbb{N})$ in the space $L^2((0;1);x^2dx)$.

2. Main results

Denote by PW_σ^2 the set of all entire functions of formal exponential type $\sigma \in (0;+\infty)$ belonging to the space $L^2(\mathbb{R})$ on the real axis $\mathbb{R}$ in $\mathbb{C}$, and by $PW_{\sigma,-}^2$ we denote the class of odd entire functions from PW_σ^2. According to the Paley-Wiener theorem (see [12], [19]–[21]), the class PW_σ^2 coincides with the class of functions G admitting the representation

$$G(z) = \int_{-\sigma}^{\sigma} e^{itz}g(t)\,dt, \quad g \in L^2(-\sigma;\sigma),$$

and the class $PW_{\sigma,-}^2$ consists of the functions G of the form

$$G(z) = \int_{0}^{\sigma} \sin(tz)g(t)\,dt, \quad g \in L^2(0;\sigma).$$

Moreover, $\|g\|_{L^2(0;\sigma)} = \sqrt{2/\pi}\|G\|_{L^2(0;+\infty)}$ and

$$g(t) = \frac{2}{\pi}\int_{0}^{+\infty} \sin(tz)G(z)\,dz.$$

Theorem 4. *An entire function Ω can be represented in the form*

$$\Omega(z) = \int_0^1 z\sqrt{tz}J_{3/2}(tz)h(t)\,dt, \quad h \in L^2((0;1);x^2dx), \tag{1}$$

if and only if Ω is an odd entire function, $\Omega(0) = \Omega'(0) = \Omega''(0) = 0$ and the function $\Omega'(z)/z$ belongs to the space $PW^2_{1,-}$. If these conditions hold then

$$h(t) = \sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{\Omega'(z)}{tz} \sin(tz)\,dz.$$

Proof. Let the function Ω is representable in the form (1). Since

$$z\sqrt{tz}J_{3/2}(tz) = -\sqrt{\frac{2}{\pi}}\frac{tz\cos(tz) - \sin(tz)}{t},$$

we have

$$\Omega(z) = -\sqrt{\frac{2}{\pi}} \int_0^1 \frac{tz\cos(tz) - \sin(tz)}{t} h(t)\,dt.$$

Therefore,

$$\Omega'(z) = \sqrt{\frac{2}{\pi}} \int_0^1 tz\sin(tz)h(t)\,dt, \quad \frac{\Omega'(z)}{z} = \sqrt{\frac{2}{\pi}} \int_0^1 \sin(tz)q(t)\,dt,$$

where $q(t) = th(t)$. Since $h \in L^2((0;1);x^2dx)$, we have $q \in L^2(0;1)$ and, hence, according to Paley-Wiener theorem, the function $\Omega'(z)/z$ belongs to the space $PW^2_{1,-}$. Conversely, if all the conditions of the theorem hold then the function $q(t) = \sqrt{2/\pi}\int_0^{+\infty} \frac{\Omega'(z)}{z}\sin(tz)\,dz$ belongs to the space $L^2(0;1)$ and $\Omega'(z) = \sqrt{2/\pi}\int_0^1 z\sin(tz)q(t)\,dt$. Using Fubini's theorem, we get

$$\Omega(z) = \Omega(z) - \Omega(0) = \sqrt{\frac{2}{\pi}} \int_0^1 q(t)\,dt \int_0^z w\sin(tw)\,dw$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \frac{\sin(tz) - tz\cos(tz)}{t}\frac{q(t)}{t}\,dt = \int_0^1 z\sqrt{tz}J_{3/2}(tz)h(t)\,dt,$$

where $h(t) = q(t)/t$. Since $q \in L^2(0;1)$, one has that $h \in L^2((0;1);x^2dx)$, and the proof of the theorem is completed. □

Let $\widetilde{E}_{2,-}$ be the class of the entire functions Ω that can be represented in the form (1), and let $E_{2,-}$ be the class of nonzero odd entire functions Ω such that $\Omega(0) = \Omega'(0) = \Omega''(0) = 0$ and the function $\Omega'(z)/z$ belongs to the space $PW^2_{1,-}$.

Corollary 1. $\widetilde{E}_{2,-} = E_{2,-}$.

Corollary 2. *The class $E_{2,-}$ coincides with the set of the entire functions Ω that can be represented in the form*

$$\Omega(z) = \sqrt{\frac{2}{\pi}} \int_0^1 \frac{\sin(tz) - tz\cos(tz)}{t^2} q(t)\,dt, \quad q \in L^2(0;1). \tag{2}$$

Theorem 5. *Let $(\rho_k : k \in \mathbb{N})$ be an arbitrary sequence of distinct nonzero complex numbers such that $\rho_k^2 \neq \rho_n^2$ if $k \neq n$. For a system $(\Theta_k : k \in \mathbb{N})$ to be incomplete in the space $L^2((0;1);x^2dx)$ it is necessary and sufficient that a sequence $(\rho_k : k \in \mathbb{Z} \setminus \{0\})$, $\rho_{-k} := -\rho_k$, is a subsequence of zeros of some nonzero even entire function G such that the function $\Omega(z) = z^3G(z)$ belongs to the space $E_{2,-}$.*

Proof. Incompleteness of a system $(\Theta_k : k \in \mathbb{N})$ is equivalent to the incompleteness of the system $(\rho_k^3\Theta_k : k \in \mathbb{N})$. According to the well-known completeness criterion, the last system is incomplete in the space $L^2((0;1);x^2dx)$ if and only if there exists a nonzero function $h \in L^2((0;1);x^2dx)$ such that

$$\int_0^1 \rho_k\sqrt{x\rho_k}J_{3/2}(x\rho_k)h(x)\,dx = 0 \tag{3}$$

for all $k \in \mathbb{N}$. If the system $(\Theta_k : k \in \mathbb{N})$ is incomplete, then the function (1) has zeros at points ρ_k, belongs to the space $E_{2,-}$ and $\Omega(z) \not\equiv 0$. Hence, the function $G(z) = z^{-3}\Omega(z)$ is required. Conversely, if the sequence $(\rho_k : k \in \mathbb{Z} \setminus \{0\})$, $\rho_{-k} := -\rho_k$, is a subsequence of zeros of some even nonzero entire function G such that the function $\Omega(z) = z^3G(z)$ belongs to $E_{2,-}$ then, using (1), we obtain (3). The theorem is proved. □

Lemma 1. *Let an entire function $\Omega \in E_{2,-}$ be defined by the formula (2). Then (here and so on by C_1, C_2, ... we denote arbitrary positive constants) for all $z \in \mathbb{C}$, we have*

$$|\Omega(z)| \le C_1(1+|z|)\frac{e^{|\Im z|}}{\sqrt{1+|\Im z|}} + C_2|z|\left(|\Re z| + \frac{e^{|\Im z|}}{1+|\Im z|}\right)^{1/2}.$$

Proof. Indeed, let

$$I_1(z) = \sqrt{\frac{2}{\pi}}\int_0^{1/2} \frac{\sin(tz) - tz\cos(tz)}{t^2}q(t)\,dt,$$

$$I_2(z) = -\sqrt{\frac{2}{\pi}}\int_{1/2}^1 \frac{q(t)}{t}\cos(tz)\,dt, \quad I_3(z) = \sqrt{\frac{2}{\pi}}\int_{1/2}^1 \frac{q(t)}{t^2}\sin(tz)\,dt.$$

Then $\Omega(z) = I_1(z) + zI_2(z) + I_3(z)$. According to the Paley-Wiener theorem, the functions $I_2(z)$ and $I_3(z)$ belong to the space PW_1^2,

$$I_2(z) = -\sqrt{\frac{2}{\pi}}\int_{1/2}^1 e^{itz}\frac{q(t)}{2t}\,dt + \sqrt{\frac{2}{\pi}}\int_{-1}^{-1/2} e^{itz}\frac{q(-t)}{2t}\,dt,$$

and applying Schwartz's inequality, we get

$$|I_2(z)| \le C_3\frac{e^{|\Im z|}}{\sqrt{1+|\Im z|}}, \quad z \in \mathbb{C}.$$

Similarly,

$$|I_3(z)| \le C_4\frac{e^{|\Im z|}}{\sqrt{1+|\Im z|}}, \quad z \in \mathbb{C}.$$

Finally, since $|\sin(tz) - tz\cos(tz)|^2 = (\sin(tx) - tx\cos(tx))^2 + (\sinh(ty) - ty\cosh(ty))^2 + t^2(x^2\sinh^2(ty) - y^2\sin^2(tx))$ for any $t \in \mathbb{R}$ and $z = x + iy \in \mathbb{C}$, we obtain

$$\int_0^{1/2} \frac{|\sin(tz) - tz\cos(tz)|^2}{t^4}\,dt = \int_0^{1/2} \frac{(\sin(tx) - tx\cos(tx))^2}{t^4}\,dt$$

$$+\int_0^{1/2}\frac{(\sinh(ty)-ty\cosh(ty))^2}{t^4}\,dt+\int_0^{1/2}\frac{x^2\sinh^2(ty)-y^2\sin^2(tx)}{t^2}\,dt$$

$$=x^3\int_0^{x/2}\frac{(\sin t-t\cos t)^2}{t^4}\,dt+y^3\int_0^{y/2}\frac{(\sinh t-t\cosh t)^2}{t^4}\,dt$$

$$+x^2y\int_0^{y/2}\frac{\sinh^2 t}{t^2}\,dt-y^2x\int_0^{x/2}\frac{\sin^2 t}{t^2}\,dt.$$

Therefore, for $z\in\mathbb{C}$

$$|I_1(z)|\le C_5\left(|x|^3+|z|^2\frac{e^{|y|}}{1+|y|}+y^2|x|\right)^{1/2}=C_5|z|\left(|\Re z|+\frac{e^{|\Im z|}}{1+|\Im z|}\right)^{1/2}.$$

This completes the proof of the lemma. □

Theorem 6. *Let $(\rho_k : k\in\mathbb{N})$ be a sequence of distinct nonzero complex numbers such that $\rho_k^2\ne\rho_m^2$ if $k\ne m$, and let a sequence $(\rho_k : k\in\mathbb{Z}\setminus\{0\})$, $\rho_{-k}:=-\rho_k$, be a sequence of zeros of the some even entire function G of finite formal exponential type, for which on the rays $\{z:\arg z=\varphi_j\}$, $j\in\{1;2;3;4\}$, $\varphi_1\in[0;\pi/2)$, $\varphi_2\in[\pi/2;\pi)$, $\varphi_3\in(\pi;3\pi/2]$, $\varphi_4\in(3\pi/2;2\pi)$, we have*

$$|G(z)|\ge C_6(1+|z|)^{-2}\exp(|\Im z|).$$

Then the system $(\Theta_k : k\in\mathbb{N})$ is complete in the space $L^2((0;1);x^2dx)$.

Proof. Assume the converse. Then, according to Theorem 5, there exists an entire function $\Omega\in E_{2,-}$ for which the sequence $(\rho_k : k\in\mathbb{Z}\setminus\{0\})$ is a subsequence of zeros. Let $V(z)=\Omega(z)/(z^3G(z))$. Then V is an even entire function of finite exponential type, for which (see Lemma 1)

$$|V(z)|\le C_7\frac{1}{\sqrt{1+|\Im z|}},\quad \arg z=\varphi_j,\quad j\in\{1;2;3;4\}.$$

Hence, according to the Phragmén-Lindelöf theorem (see [20], [21]), $V(z)\equiv 0$. Therefore, $\Omega(z)\equiv 0$. This contradiction proves the theorem. □

Corollary 3. *Let $(\rho_k : k\in\mathbb{Z})$, $\rho_{-k}:=-\rho_k$, be a sequence of zeros of the function $J_{3/2}$. Then the system $(\Theta_k : k\in\mathbb{N})$ is complete in the space $L^2((0;1);x^2dx)$.*

Proof. Indeed, the sequence $(\rho_k : k\in\mathbb{Z}\setminus\{0\})$ is a sequence of zeros of the entire function $G(z)=z^{-3}(z\cos z-\sin z)$, and this function satisfies the conditions of Theorem 6. Therefore, the system $(\Theta_k : k\in\mathbb{N})$ is complete in the space $L^2((0;1);x^2dx)$. □

Theorem 7. *Let $(\rho_k : k\in\mathbb{N})$ be a sequence of distinct nonzero complex numbers such that $\rho_k^2\ne\rho_m^2$ if $k\ne m$, and let a sequence $(\rho_k : k\in\mathbb{Z}\setminus\{0\})$, $\rho_{-k}:=-\rho_k$, be a sequence of zeros of the some even entire function G of finite formal exponential type such that the function $z^3G(z)$ does not belongs to the space $E_{2,-}$ and for which on the rays $\{z:\arg z=\varphi_j\}$, $j\in\{1;2;3;4\}$, $\varphi_1\in[0;\pi/2)$, $\varphi_2\in[\pi/2;\pi)$, $\varphi_3\in(\pi;3\pi/2]$, $\varphi_4\in(3\pi/2;2\pi)$, the inequality*

$$|G(z)|\ge C_8(1+|z|)^{-2-\alpha}\exp(|\Im z|)$$

holds, where $\alpha<5/2$ is a some constant. Then the system $(\Theta_k : k\in\mathbb{N})$ is complete in the space $L^2((0;1);x^2dx)$.

Proof. Assume the converse. Then, according to Theorem 5, there exists an entire function $\Omega \in E_{2,-}$ for which the sequence $(\rho_k : k \in \mathbb{Z} \setminus \{0\})$ is a subsequence of zeros. Let $V(z) = \Omega(z)/(z^3 G(z))$. Then V is an even entire function of finite formal exponential type, for which (see Lemma 1)

$$|V(z)| \le C_9(1+|z|)^{\alpha-1/2}, \quad \arg z = \varphi_j, \quad j \in \{1;2;3;4\}.$$

Since $\alpha - 1/2 < 2$ and V is an even entire function, then, according to the Phragmén-Lindelöf theorem, the function V is a constant. Hence, $\Omega(z) = C_{10} z^3 G(z)$. Therefore, $\Omega \notin E_{2,-}$. Thus, we have a contradiction and the proof of the theorem is completed. □

Lemma 2. *If an odd entire function L belongs to the space $E_{2,-}$ and has a root at a point $\rho \neq 0$, then the function $\widetilde{L}(z) = L(z)/(z^2-\rho^2)$ also belongs to $E_{2,-}$.*

Proof. Indeed, the function $\widetilde{L}$ is an odd entire function of formal exponential type $\sigma \le 1$,

$$\widetilde{L}'(z) = \frac{L'(z)(z^2-\rho^2) - 2zL(z)}{(z^2-\rho^2)^2},$$

and $\widetilde{L}(0) = \widetilde{L}'(0) = \widetilde{L}''(0) = 0$. Besides,

$$\frac{\widetilde{L}'(z)}{z} = \frac{L'(z)}{z(z^2-\rho^2)} - \frac{2L(z)}{(z^2-\rho^2)^2},$$

$$\int\limits_{1+\Re\rho}^{+\infty} \left| \frac{L'(x)}{x(x^2-\rho^2)} \right|^2 dx \le C_{11} \int\limits_{1+\Re\rho}^{+\infty} \left| \frac{L'(x)}{x} \right|^2 dx < +\infty,$$

and according to Lemma 1

$$\int\limits_{1+\Re\rho}^{+\infty} \left| \frac{L(x)}{(x^2-\rho^2)^2} \right|^2 dx \le C_{12} \int\limits_{1+\Re\rho}^{+\infty} \left| \frac{(1+|x|)^3}{(x^2-\rho^2)^4} \right| dx < +\infty.$$

Hence, the function $\widetilde{L}'(z)/z$ belongs to $L^2(\mathbb{R})$. This concludes the proof of the lemma. □

Lemma 3. *If an odd entire function L has zeros at points $\rho_k \neq 0$, $k \in \mathbb{N}$, and the function $L(z)/(z^2-\rho_1^2)$ belongs to the space $E_{2,-}$, then the functions $L_k(z) = L(z)/(z^2-\rho_k^2)$ also belong to $E_{2,-}$ for every $k \in \mathbb{N} \setminus \{1\}$.*

Proof. In fact, let $Q_k(z) = (\rho_k^2 - \rho_1^2)\dfrac{L(z)}{(z^2-\rho_k^2)(z^2-\rho_1^2)}$. Then $Q_k(z) = (\rho_k^2-\rho_1^2)\dfrac{L_1(z)}{z^2-\rho_k^2}$ and $L_k = Q_k + L_1$. Therefore, taking into account the previous lemma, we obtain the required proposition. □

Theorem 8. *Let $(\rho_k : k \in \mathbb{N})$ be an arbitrary sequence of distinct complex numbers such that $\rho_k^2 \neq \rho_m^2$ if $k \neq m$. If the sequence $(\rho_k : k \in \mathbb{N})$ is a subsequence of zeros of some even entire function G which has simple roots at all points ρ_k and the function $z^3(z^2-\rho_1^2)^{-1}G(z)$ belongs to $E_{2,-}$, then the system $(\Theta_k : k \in \mathbb{N})$ has a biorthogonal system $(\gamma_k : k \in \mathbb{N})$ in the space $L^2((0;1); x^2dx)$. The biorthogonal system $(\gamma_k : k \in \mathbb{N})$ is formed, in particular, by the functions γ_k, defined by the equality*

$$\overline{\gamma_k(t)} = \sqrt{\frac{2}{\pi}} \int\limits_0^{+\infty} \frac{V_k'(z)}{tz} \sin(tz)\, dz, \quad V_k(z) := \frac{2\rho_k z^3 G(z)}{G'(\rho_k)(z^2-\rho_k^2)}. \tag{4}$$

Proof. In fact, according to Lemma 3, the functions V_k belong to the space $E_{2,-}$. Therefore, there exist nonzero elements γ_k of the space $L^2((0;1);x^2dx)$ such that

$$V_k(z)=\int\limits_0^1 z\sqrt{tz}J_{3/2}(tz)\gamma_k(t)\,dt,$$

and by Theorem 4 the functions γ_k can be found by (4). Moreover,

$$\frac{V_k(\rho_n)}{\rho_n^3}=\begin{cases}1, & n=k,\\ 0, & n\neq k,\end{cases}$$

and we obtain the required proposition. □

Theorem 9. *Let $(\rho_k : k\in\mathbb{N})$ be an arbitrary sequence of nonzero complex numbers such that $\rho_k^2\neq\rho_m^2$ as $k\neq m$. The system $(\Theta_k : k\in\mathbb{N})$ is complete and minimal in the space $L^2((0;1);x^2dx)$ if and only if the sequence $(\rho_k : k\in\mathbb{Z}\setminus\{0\})$, $\rho_{-k}:=-\rho_k$, is a sequence of zeros of some even entire function G such that the function $z^3(z^2-\rho_1^2)^{-1}G(z)$ belongs to the space $E_{2,-}$ and the function $z^3G(z)$ does not belongs to this space.*

Proof. If the considered system is minimal then there exists a nonzero function $\gamma_1\in L^2((0;1);x^2dx)$ such that

$$\int\limits_0^1\rho_k\sqrt{t\rho_k}J_{3/2}(t\rho_k)\gamma_1(t)\,dt=\begin{cases}1, & k=1,\\ 0, & k\neq 1.\end{cases}$$

Let $T(z)=\int_0^1 z\sqrt{tz}J_{3/2}(tz)\gamma_1(t)\,dt$. The function $G(z)=z^{-3}(z^2-\rho_1^2)T(z)$ is the required, because the function $T(z)=z^3(z^2-\rho_1^2)^{-1}G(z)$ belongs to the space $E_{2,-}$ and has zeros at all points ρ_k, all its zeros are simple and it has no other zeros. Indeed, if ρ is another root of the function G, then the function $V(z)=G(z)/(z^2-\rho^2)$ which has roots at all points ρ_k, would belongs to the space $E_{2,-}$ that, according to Theorem 5, contradicts the completeness of the considered system. Besides, the function $z^3G(z)$ does not belongs to $E_{2,-}$, because otherwise the system would be incomplete. Conversely, if all the conditions of the theorem hold then, basing on Theorem 8, we obtain the required proposition. The proof of theorem is thus completed. □

Corollary 4. *Let $(\rho_k : k\in\mathbb{Z})$, $\rho_{-k}:=-\rho_k$, be a sequence of zeros of the function $J_{3/2}$. Then the system $(\Theta_k : k\in\mathbb{N})$ has in the space $L^2((0;1);x^2dx)$ a biorthogonal system $(\gamma_k : k\in\mathbb{N})$ which formed by the functions γ_k, defined by the formula*

$$\overline{\gamma_k(t)}=\pi(1+\rho_k^2)\sqrt{t\rho_k}J_{3/2}(t\rho_k).$$

This corollary can be proved by standard methods of the theory of Bessel functions (see [3], [25, p. 347], [32]). However, it can be proved by Theorem 8. In fact, the sequence $(\rho_k : k\in\mathbb{Z}\setminus\{0\})$, $\rho_{-k}:=-\rho_k$, is a sequence of zeros of even entire function $G(z)=z^{-3}(z\cos z-\sin z)$. Further, the function $z^3G(z)$ dose not belongs to the space $E_{2,-}$ and the function $z^3(z^2-\rho_1^2)^{-1}G(z)$ belongs to this space. Furthermore, according to Theorem 8, the system $(\Theta_k : k\in\mathbb{N})$ has in the space $L^2((0;1);x^2dx)$ a biorthogonal system $(\gamma_k : k\in\mathbb{N})$ which formed by the functions γ_k, defined by the equality (4), where

$$V_k(z):=\frac{2\rho_k(z\cos z-\sin z)}{G'(\rho_k)(z^2-\rho_k^2)},\quad G'(\rho_k)=-\frac{\sin\rho_k}{\rho_k^2}.$$

Therefore,

$$\overline{\gamma_k(t)}=-\sqrt{\frac{2}{\pi}}\int\limits_0^{+\infty}V_k(z)\frac{tz\cos(tz)-\sin(tz)}{tz^2}\,dz$$

$$= -\sqrt{\frac{2}{\pi}}\frac{2\rho_k}{tG'(\rho_k)}\int\limits_0^{+\infty}\frac{(z\cos z-\sin z)(tz\cos(tz)-\sin(tz))}{z^2(z^2-\rho_k^2)}\,dz.$$

Let $\eta(z;t)=tz^2e^{i(1+t)z}+tz^2e^{i(1-t)z}+ize^{i(1+t)z}-ize^{i(1-t)z}+itze^{i(1+t)z}+itze^{i(1-t)z}-e^{i(1+t)z}+e^{i(1-t)z}$. Then $(z\cos z-\sin z)(tz\cos(tz)-\sin(tz))=\frac{1}{4}(\eta(z;t)+\eta(-z;t))$. Hence,

$$\overline{\gamma_k(t)} = -\frac{1}{\sqrt{2\pi}}\frac{\rho_k}{tG'(\rho_k)}\int\limits_{-\infty}^{+\infty}\frac{\eta(z;t)}{z^2(z^2-\rho_k^2)}\,dz$$

$$= -\frac{\sqrt{2\pi}}{t\sin\rho_k}(t\rho_k\cos(t\rho_k)-\sin(t\rho_k))(\rho_k\sin\rho_k+\cos\rho_k) = \pi(1+\rho_k^2)\sqrt{t\rho_k}J_{3/2}(t\rho_k).$$

Problem. Let $(\rho_k : k\in\mathbb{Z})$, $\rho_{-k}:=-\rho_k$, be a sequence of zeros of the function $J_{3/2}$. Since (see [32], [25, p. 352]) $\rho_k\sim\pi k$ as $k\to\infty$ and

$$\|\Theta_k\|^2\|\gamma_k\|^2 = \frac{\pi^2(1+\rho_k^2)^2}{\rho_k^6}\int\limits_0^{\rho_k}|t\sqrt{t}J_{3/2}(t)|^2\,dt\int\limits_0^{\rho_k}\frac{|\sqrt{t}J_{3/2}(t)|^2}{t^2}\,dt$$

$$= \frac{\pi(1+\rho_k^2)^2}{3\rho_k^3}(1+o(1))\left(\int\limits_0^{+\infty}\frac{|\sqrt{t}J_{3/2}(t)|^2}{t^2}\,dt+o(1)\right)\longrightarrow+\infty,\quad k\to\infty,$$

then the system $(\Theta_k : k\in\mathbb{N})$ is not uniformly minimal (see [21, p. 4258], [22, p. 62]) in the space $L^2((0;1);x^2dx)$ and therefore is not a basis in this space (see [21, p. 4258], [22, p. 62]). However, it is easy to show that the biorthogonal system $(\gamma_k : k\in\mathbb{N})$ is complete in $L^2((0;1);x^2dx)$. Therefore, the numbers $d_k=\int_0^1 t^2f(t)\overline{\gamma_k(t)}\,dt$ determine the function $f\in L^2((0;1);x^2dx)$ uniquely. But the series $\sum_{k=1}^{\infty}d_k\Theta_k(x)$ not for each function $f\in L^2((0;1);x^2dx)$ converges in $L^2((0;1);x^2dx)$ to the function f. We do not know the methods of restoration of the function $f\in L^2((0;1);x^2dx)$ by numbers d_k and, in particular, whether the given series converges in $L^2((0;1);x^2dx)$ to f in the sense of Cesàro.

Similar problems are studied in [1], [4]–[7], [9], [10], [14], [15], [23], [32, Ch. XVIII], [33] and for exponential systems in [17], [18], [21], but we cannot use these results.

BIBLIOGRAPHY

1. V.A. Abilov, F.V. Abilova. *Approximation of functions by Fourier-Bessel sums* // Izv. Vyssh. Uchebn. Zaved. Mat. 2001. No. 8. P. 3-9. [Russ. Math. (Izvestiya Vuzov. Matem.) 2001. V. 45, No. 8. P. 1-7.]
2. N.I. Akhiezer. *To the theory of coupled integral equations* // Zapiski matem. otdelenia fiziko-matem. fakulteta i Kharkovskogo matem. obschestva. 1957. V. 25. P. 5-31. (in Russian.)
3. H. Bateman, A. Erdélyi. *Higher transcendental functions.* Vol. 2. McGraw-Hill, New York-Toronto-London. 1953.
4. A. Benedek, R. Panzone. *Mean convergence of series of Bessel functions* // Rev. Un. Mat. Argentina. 1972. V. 26, No. 1. P. 42-61.
5. A. Benedek, R. Panzone. *On mean convergence of Fourier-Bessel series of negative order* // Stud. Appl. Math. 1971. V. 4, No. 3. P. 281-292.
6. A. Benedek, R. Panzone. *Pointwise convergence of series of Bessel functions* // Rev. Un. Mat. Argentina. 1972. V. 26, No. 3. P. 167-186.
7. J.J. Betancor, K. Stempak. *Relating multipliers and transplantation for Fourier-Bessel expansions and Hankel transform* // Tohoku Math. J. 2001. V. 53, No. 1. P. 109-129.
8. R.P. Boas, H. Pollard. *Complete sets of Bessel and Legendre functions* // Ann. of Math. 1947. V. 48, No. 2. P. 366-384.

9. Ó. Ciaurri, K. Stempak, J.L. Varona. *Mean Cesàro-type summability of Fourier-Neumann series* // Studia Sci. Math. Hungar. 2005. V. 42, No. 4. P. 413-430.
10. L. Colzani, A. Crespi, G. Travaglini, M. Vignati. *Equiconvergence theorems for Fourier-Bessel expansions with applications to the harmonic analysis of radial functions in euclidean and non euclidean spaces* // Trans. Amer. Math. Soc. 1993. V. 338, No. 1. P. 43-55.
11. M.M. Dzhrbashyan. *Integral transforms and representations of functions in the complex domain.* Nauka, Moscow, 1966. (in Russian.)
12. John B. Garnett. *Bounded analytic functions.* Academic Press, New York. 1981.
13. J.L. Griffith. *Hankel transforms of functions zero outside a finite interval* // J. Proc. Roy. Soc. New South Wales. 1955. V. 89, No. 2. P. 109-115.
14. J.J. Guadalupe, M. Pérez, F.J. Ruiz. *Mean and weak convergence of Fourier-Bessel series* // J. Math. Anal. Appl. 1993. V. 173, No. 2. P. 370-389.
15. J.J. Guadalupe, M. Pérez, F.J. Ruiz, J.L. Varona. *Two notes on convergence and divergence a.e. of Fourier series with respect to some orthogonal systems* // Proc. Amer. Math. Soc. 1992. V. 116, No. 2. P. 457-464.
16. J.R. Higgins. *Completeness and basis properties of sets of special functions.* Cambridge University Press, Cambridge. 1977.
17. K.P. Isaev, R.S. Yulmukhametov. *Unconditional exponential bases in Hilbert spaces* // Ufa Math. J. 2011. V. 3, No. 1. P. 3-15.
18. B.N. Khabibullin. *Completeness of exponential systems and uniqueness sets*, Bashkir State University Press, Ufa, 2008. (in Russian)
19. P. Koosis. *Introduction to H_p spaces.* Cambridge University Press, Cambridge. 1998.
20. B.Ya. Levin. *Lectures on entire functions.* Transl. Math. Monogr. V. 150. Amer. Math. Soc., Providence, RI. 1996.
21. A.M. Sedletskii. *Analytic Fourier transforms and exponential approximations. I, II* // J. Math. Sci. 2005. V. 129, No. 6. P. 4251-4408; 2005. V. 130, No. 6. P. 5083-5255.
22. I. Singer. *Bases in Banach Spaces.* V. 1. Springer-Verlag, Berlin. 1970.
23. K. Stempak. *On convergence and divergence of Fourier-Bessel series.* Electron. Trans. Numer. Anal. 2002. V. 14. P. 223-235.
24. E.C. Titchmarsh. *Introduction to the theory of Fourier integrals, Second edition.* Clarendon Press, Oxford. 1948.
25. V.S. Vladimirov. *Equations of mathematical physics.* Nauka, Moscow. 1981. [Marcel Dekker, Inc., New York. 1971.]
26. B.V. Vynnyts'kyi, V.M. Dilnyi. *On some analogues of Paley-Wiener theorem and one boundary value problem for Bessel operator.* in Proc. of Int. Conf. on complex analysis in memory of A.A. Gol'dberg, Lviv, Ukraine, May 31 – June 5, 2010, pp. 63-64.
27. B.V. Vynnyts'kyi, R.V. Khats'. *Some approximation properties of the systems of Bessel functions of index $-3/2$* // Mat. Stud. 2010. V. 34, No. 2. P. 152-159.
28. B.V. Vynnyts'kyi, R.V. Khats'. *Approximation properties of Bessel functions and a generalization of the notion of eigenvector.* in: Proc. of Int. Conf. on complex analysis in memory of A.A. Gol'dberg, Lviv, Ukraine, May 31 – June 5, 2010, pp. 64-65.
29. B.V. Vynnyts'kyi, R.V. Khats'. *A generalization of the notion of eigenvector.* in: Proc. of Int. Conf. "Modern problems of analysis", Chernivtsi, Ukraine, September 30 – October 3, 2010, pp. 51-52. (in Ukrainian)
30. B.V. Vynnyts'kyi, O.V. Shavala. *Boundedness of solutions of a second-order linear differential equation and a boundary value problem for Bessel's equation* // Mat. Stud. 2008. V. 30, No. 1. P. 31-41. (in Ukrainian)
31. B.V. Vynnyts'kyi, O.V. Shavala. *On completeness of the system* $(\cos(\rho_n x) + \rho_n x \sin(\rho_n x))$ *and a boundary value problem for Bessel operator*, in: Proc. of Int. Conf. "Analysis and topology", Lviv, Ukraine, May 26 – June 7, 2008, pp. 54-55.
32. G.N. Watson. *A treatise on the theory of Bessel functions.* Cambridge University Press, Cambridge. 1944.
33. G.M. Wing. *The mean convergence of orthogonal series* // Amer. J. Math. 1950. V. 72, No. 4. P. 792-808.

Bohdan V. Vynnyts'kyi,
Institute of Physics, Mathematics and Informatics,
Ivan Franko Drohobych State Pedagogical University,
3 Stryiska Str.,
82100 Drohobych, Ukraine
E-mail: vynnytskyi@ukr.net

Ruslan V. Khats',
Institute of Physics, Mathematics and Informatics,
Ivan Franko Drohobych State Pedagogical University,
3 Stryiska Str.,
82100 Drohobych, Ukraine
E-mail: khats@ukr.net