

INTEGRATION OF HIGHER KORTEWEG-DE VRIES EQUATION WITH SELF-CONSISTENT SOURCE IN CLASS OF PERIODIC FUNCTIONS

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Abstract. In the present work the inverse spectral problem of Sturm-Liouville operator is applied for integrating higher Korteweg-de Vries equation with a self-consistent source in class of periodic functions

Keywords: Sturm-Liouville operator, inverse spectral problem, eigenvalue, eigenfunction, Korteweg-de Vries equation.

1. INTRODUCTION

In 1967 in work [1] American scientists C.S. Gardner, J.M. Greene, M.D. Kruskal, and R.M. Miura established the integrability of the Korteweg-de Vries equation (KdV) in the class of “fast decaying” w.r.t. x functions by the method of inverse scattering problem for the Sturm-Liouville equation. In work [2] P. Lax showed an universality of the inverse scattering problem method and generalized the KdV equation by introducing a higher (general) KdV equation.

In works [3-10] KdV equation and higher KdV equation were studied in the class of finite-band and periodic functions.

In the present work we study the higher KdV equation with a self-consistent source in the class of periodic functions.

We note that in [11-15] and other papers the KdV equation with a self-consistent source was considered in the class of fast decaying functions, and nonlinear equations with a source in the class of periodic functions in various formulations were studied in works [16-19].

Let

$$H = -\frac{1}{2} \frac{d^3}{dx^3} + 2q \frac{d}{dx} + q',$$

where $q = q(x, t)$, and the prime denotes the derivative w.r.t. x . According to [20], there exists polynomials P_k (of q and the derivatives of q w.r.t. x) such that

$$HP_k = P'_{k+1}.$$

For example,

$$P_0 = 1, \quad P_1 = q, \quad P_2 = -\frac{1}{2}q_{xx} + \frac{3}{2}q^2, \quad P_3 = \frac{1}{4}q_{xxxx} - \frac{5}{2}qq_{xx} - \frac{5}{4}q_x^2 + \frac{5}{2}q^3.$$

It is easy to prove the following properties of the operator H (see [20]).

Lemma 1. *If $y(x, t)$ is a solution to the following Sturm-Liouville equation*

$$L(t)y \equiv -y'' + q(x, t)y = \lambda y, \quad x \in \mathbb{R}^1,$$

the identity

$$H(y^2) = 2\lambda(y^2)'$$

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holds true.

Lemma 2. For any $y(x), z(x) \in C^3[0, \pi]$ the identity

$$\int_0^\pi Hz \cdot y dx = \left(-\frac{1}{2}z''y + 2qzy + \frac{1}{2}z'y' - \frac{1}{2}zy'' \right) \Big|_0^\pi - \int_0^\pi z \cdot Hy dx$$

holds true.

The following equation

$$q_t = HP_N[q], \quad x \in R^1, \quad t > 0,$$

is called a higher KdV equation. Employing the properties of the operator H , we can rewrite this equation as

$$q_t = P'_{N+1}[q], \quad x \in R^1, \quad t > 0.$$

For instance, as $N = 0, 1, 2$ we respectively have

$$q_t = q_x, \quad q_t = -\frac{1}{2}q_{xxx} + 3qq_x, \quad q_t = \frac{1}{4}q_{xxxxx} - 5q_xq_{xx} - \frac{5}{2}qq_{xxx} + \frac{15}{2}q^2q_x.$$

2. FORMULATION OF PROBLEM

In this work we consider the following higher KdV equation with a self-consistent source

$$q_t = P'_{N+1}[q] + 2 \int_0^\infty \beta(\lambda, t) s(\pi, \lambda, t) (\psi_+(x, \lambda, t) \psi_-(x, \lambda, t))_x d\lambda, \quad t > 0, \quad x \in R^1, \quad (1)$$

subject to initial condition

$$q(x, t)|_{t=0} = q_0(x), \quad (2)$$

where $q_0(x) \in C^{2N+1}(R^1)$ is a given real function. It is required to find a real function $q(x, t)$ being π -periodic w.r.t. x ,

$$q(x + \pi, t) \equiv q(x, t), \quad t \geq 0, \quad x \in R^1, \quad (3)$$

and satisfying the smoothness conditions

$$q(x, t) \in C_x^{2N+1}(t > 0) \cap C_t^1(t > 0) \cap C(t \geq 0). \quad (4)$$

Here $\beta(\lambda, t) \in C([0, \infty) \times [0, \infty))$ is a given real function having the uniform asymptotics $\beta(\lambda, t) = O\left(\frac{1}{\lambda}\right)$, $\lambda \rightarrow \infty$, $\psi_\pm(x, \lambda, t)$ are the Floquet solutions (normalized by the condition $\psi_\pm(0, \lambda, t) = 1$) to the Sturm-Liouville equation

$$L(t)y \equiv -y'' + q(x, t)y = \lambda y, \quad x \in R^1, \quad (5)$$

$s(x, \lambda, t)$ is the solution to equation (5) satisfying the initial conditions $s(0, \lambda, t) = 0$, $s'(0, \lambda, t) = 1$.

Remark 1. Let us show the uniform convergence of the integral involved in (1). In order to do it, we employ the identity

$$s(\pi, \lambda, t) \psi_+(\tau, \lambda, t) \psi_-(\tau, \lambda, t) = s(\pi, \lambda, t, \tau), \quad (6)$$

where $s(x, \lambda, t, \tau)$ solves the equation

$$-y'' + q(x + \tau, t)y = \lambda y, \quad x \in R^1,$$

and obeys the initial conditions $s(0, \lambda, t, \tau) = 0$, $s'(0, \lambda, t, \tau) = 1$.

The asymptotic formulae

$$c(x, \lambda, t) = \cos \sqrt{\lambda}x + O\left(\frac{1}{\sqrt{\lambda}}\right), \quad s(x, \lambda, t) = \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right),$$

$$c'(x, \lambda, t) = -\sqrt{\lambda} \sin \sqrt{\lambda}x + O(1), \quad s'(x, \lambda, t) = \cos \sqrt{\lambda}x + O\left(\frac{1}{\sqrt{\lambda}}\right), \quad (\lambda \rightarrow \infty)$$

and identities

$$s(\pi, \lambda, t, \tau) = c(\tau, \lambda, t)s(\pi + \tau, \lambda, t) - s(\tau, \lambda, t)c(\pi + \tau, \lambda, t)$$

imply the estimates

$$s(\pi, \lambda, t, \tau) = O\left(\frac{1}{\sqrt{\lambda}}\right), \quad \frac{\partial s(\pi, \lambda, t, \tau)}{\partial \tau} = O\left(\frac{1}{\sqrt{\lambda}}\right), \quad (\lambda \rightarrow \infty).$$

These estimates and identity (6) ensures the uniform convergence of the integral involved in equation (1).

The aim of the present work is to provide the procedure of constructing a solution $q(x, t)$, $\psi_{\pm}(x, \lambda, t)$ to problem (1)-(5) in the framework of the inverse spectral problem for the Sturm-Liouville operator with a periodic coefficient.

3. PRELIMINARIES

In this section, for the completeness of the content, we present some basic information concerning the inverse spectral problem for the Sturm-Liouville operator with a periodic coefficient (see [21–26]).

Consider the following Sturm-Liouville operator on the axis

$$Ly \equiv -y'' + q(x)y = \lambda y, \quad x \in R^1, \tag{7}$$

where $q(x)$ is a real continuous π -periodic function.

By $c(x, \lambda)$ and $s(x, \lambda)$ we denote the solutions to equation (7) satisfying the initial conditions $c(0, \lambda) = 1$, $c'(0, \lambda) = 0$ and $s(0, \lambda) = 0$, $s'(0, \lambda) = 1$. The function $\Delta(\lambda) = c(\pi, \lambda) + s'(\pi, \lambda)$ is called Lyapunov function or Hill discriminant.

The spectrum of operator (7) is pure continuous and coincides with the set

$$E = \{\lambda \in R^1 : -2 \leq \Delta(\lambda) \leq 2\} = [\lambda_0, \lambda_1] \cup [\lambda_2, \lambda_3] \cup \dots \cup [\lambda_{2n}, \lambda_{2n+1}] \cup \dots$$

The intervals $(-\infty, \lambda_0)$, $(\lambda_{2n-1}, \lambda_{2n})$, $n = 1, 2, \dots$ are called gaps. Here $\lambda_0, \lambda_{4k-1}, \lambda_{4k}$ are the eigenvalues of the periodic problem ($y(0) = y(\pi)$, $y'(0) = y'(\pi)$), and $\lambda_{4k+1}, \lambda_{4k+2}$ are that of the antiperiodic problem ($y(0) = -y(\pi)$, $y'(0) = -y'(\pi)$) for equation (7).

Let $\xi_n, n = 1, 2, \dots$, be the roots to the equation $s(\pi, \lambda) = 0$. We observe that $\xi_n, n = 1, 2, \dots$, coincide with the eigenvalues of the Dirichlet problem ($y(0) = y(\pi) = 0$) for equation (7), and moreover, the belongings $\xi_n \in [\lambda_{2n-1}, \lambda_{2n}]$, $n = 1, 2, \dots$ are fulfilled.

The numbers $\xi_n, n = 1, 2, \dots$ with the signs $\sigma_n = \text{sign}\{s'(\pi, \xi_n) - c(\pi, \xi_n)\}$, $n = 1, 2, \dots$ are called spectral parameters of problem (7). The spectral parameters $\xi_n, \sigma_n, n = 1, 2, \dots$ and the edges $\lambda_n, n = 0, 1, 2, \dots$ of the spectrum are called spectral data of operator (7). Recovering of the coefficient $q(x)$ by the spectral data is called the inverse spectral problem for operator (7).

The spectrum of the Sturm-Liouville operator with the coefficient $q(x + \tau)$ is independent of the real parameter τ , and the spectral parameters depend on τ ; $\xi_n(\tau), \sigma_n(\tau), n = 1, 2, \dots$. The spectral parameters satisfy the following Dubrovin system of equations

$$\begin{aligned} \frac{d\xi_n}{d\tau} &= 2(-1)^{n-1} \sigma_n(\tau) \sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)} \times \\ &\times \sqrt{(\xi_n - \lambda_0) \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{(\lambda_{2k-1} - \xi_n)(\lambda_{2k} - \xi_n)}{(\xi_k - \xi_n)^2}}, \quad n \geq 1. \end{aligned} \tag{8}$$

The Dubrovin system of equations and the following trace formula

$$q(\tau, t) = \lambda_0 + \sum_{k=1}^{\infty} (\lambda_{2k-1} + \lambda_{2k} - 2\xi_k(\tau, t))$$

give the method for solving the inverse spectral problem.

There are also other trace formulae, for instance, the second and third trace formulae read as

$$\begin{aligned} q^2(\tau) - \frac{1}{2}q_{\tau\tau}(\tau) &= \lambda_0^2 + \sum_{k=1}^{\infty} (\lambda_{2k-1}^2 + \lambda_{2k}^2 - 2\xi_k^2(\tau)), \\ \frac{3}{16}q_{\tau\tau\tau\tau}(\tau) - \frac{3}{2}q(\tau)q_{\tau\tau}(\tau) - \frac{15}{16}q_{\tau}^2(\tau) + q^3(\tau) &= \\ &= \lambda_0^3 + \sum_{k=1}^{\infty} (\lambda_{2k-1}^3 + \lambda_{2k}^3 - 2\xi_k^3(\tau)). \end{aligned}$$

Employing Dubrovin system of equations and the first trace formula, E. Trubowitz [25] succeeded to prove theorems relating the analyticity of the potential and the decay of the gaps lengths for the periodic potential of the Sturm-Liouville operator (7); if $q(x)$ is a real analytic π -periodic function, the lengths $\lambda_{2n} - \lambda_{2n-1}$ of the gaps exponentially tend to zero, i.e., there exist the constants $a > 0$, $b > 0$ such that $\lambda_{2n} - \lambda_{2n-1} < ae^{-bn}$, $n \geq 1$; and vice-versa, if $q(x) \in C^2(R^1)$ is a real π -periodic function and the lengths $\lambda_{2n} - \lambda_{2n-1}$ of the gaps exponentially tend to zero, then $q(x)$ is an analytic function.

In 1946 G. Borg proved a unique theorem (Borg's inverse theorem) on the period of the potential of the Hill equation (see [27]): the number $\pi/2$ is a period of the potential $q(x)$ of the Sturm-Liouville equation (7), if and only if all the roots to the equation $\Delta(\lambda) + 2 = 0$ are double, i.e., if and only if all the gaps with odd indices disappear.

In 1977 (see [28]) H. Hochstadt gave a short proof, and in 1984 a generalization of the Borg's theorem (see [29]). Let $q(x) \in C^1(R^1)$ be a real π -periodic function. The number π/n is the period of the potential $q(x)$ of Sturm-Liouville equation (7), if and only if all the gaps whose indices are not divisible by n disappear. Here $n \geq 2$ is a natural number.

4. MAIN THEOREM

The main result of the present work is the following theorem.

Theorem 1. *Let $q(x, t)$, $\psi_{\pm}(x, \lambda, t)$ be a solution to problem (1)-(5). Then the spectrum of operator (5) is independent of the parameter t , and the spectral parameters $\xi_n(t)$, $n = 1, 2, \dots$ satisfies an analogue of Dubrovin system of equations,*

$$\begin{aligned} \dot{\xi}_n &= 2(-1)^{n-1}\sigma_n(t) \left\{ \sum_{k=0}^N (2\xi_n)^{N-k} \cdot P_k[q(0, t)] + \int_0^{\infty} \frac{s(\pi, \lambda, t)\beta(\lambda, t)}{\lambda - \xi_n} d\lambda \right\} \times \\ &\times \sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)} \times \sqrt{(\xi_n - \lambda_0) \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{(\lambda_{2k-1} - \xi_n)(\lambda_{2k} - \xi_n)}{(\xi_k - \xi_n)^2}}, \quad n \geq 1, \end{aligned} \quad (9)$$

where the sign of $\sigma_n(t)$ changes to the opposite under each collision of the point $\xi_n(t)$ and the edges of the gap $[\lambda_{2n-1}, \lambda_{2n}]$. Moreover, the initial conditions

$$\xi_n(t)|_{t=0} = \xi_n^0, \quad \sigma_n(t)|_{t=0} = \sigma_n^0, \quad n \geq 1,$$

hold true, where ξ_n^0 , σ_n^0 , $n \geq 1$ are spectral parameters to the Sturm-Liouville operator with the coefficient $q_0(x)$.

Proof. We introduce the notation

$$G(x, t) = 2 \int_0^{\infty} \beta(\lambda, t) s(\pi, \lambda, t) (\psi_+(x, \lambda, t) \cdot \psi_-(x, \lambda, t))_x d\lambda,$$

and rewrite equation (1) as

$$q_t = P'_{N+1}[q] + G(x, t). \quad (10)$$

By $y_n(x, t)$, $n = 1, 2, \dots$ we denote orthonormalized eigenfunctions to the Dirichlet problem ($y(0) = 0$, $y(\pi) = 0$) for equation (5) with the π -periodic potential $q(x, t)$ being a solution to equation (10); these eigenfunctions are associated with the eigenvalues $\xi_n(t)$, $n = 1, 2, \dots$.

Differentiating the identity $(L(t)y_n, y_n) = \xi_n$ w.r.t. t and employing the symmetricity of the operator $L(t)$, we have

$$\begin{aligned} \dot{\xi}_n &= (L\dot{y}_n + q_t y_n, y_n) + (Ly_n, \dot{y}_n) = (\dot{y}_n, Ly_n) + (Ly_n, \dot{y}_n) + (q_t y_n, y_n) = \\ &= \xi_n((y_n, y_n)') + (q_t y_n, y_n) = \int_0^\pi q_t(x, t) y_n^2(x, t) dx. \end{aligned} \quad (11)$$

Here (\cdot, \cdot) is a scalar product in the space $L_2(0, \pi)$.

Employing (10) and identity $HP_k = P'_{k+1}$, we rewrite identity (11) as

$$\dot{\xi}_n = \int_0^\pi y_n^2(x, t) HP_N dx + \int_0^\pi y_n^2(x, t) G(x, t) dx. \quad (12)$$

Employing Lemmata 1 and 2, we convert the following integral

$$\begin{aligned} J_k &= \int_0^\pi y_n^2(x, t) HP_k dx = \left(-\frac{1}{2} P_k'' \cdot y_n^2 + 2q P_k \cdot y_n^2 + \frac{1}{2} P_k' \cdot (y_n^2)' - \frac{1}{2} P_k \cdot (y_n^2)'' \right) \Big|_0^\pi - \\ &- \int_0^\pi P_k \cdot H(y_n^2) dx = -P_k[q(0, t)] \cdot [y_n'^2(\pi, t) - y_n'^2(0, t)] - \int_0^\pi P_k \cdot 2\xi_n (y_n^2)' dx = \\ &= -P_k[q(0, t)] \cdot [y_n'^2(\pi, t) - y_n'^2(0, t)] + 2\xi_n \int_0^\pi P_k' \cdot y_n^2 dx, \end{aligned}$$

i.e.,

$$J_k - 2\xi_n \cdot J_{k-1} = -P_k[q(0, t)] \cdot [y_n'^2(\pi, t) - y_n'^2(0, t)].$$

Calculating the following sum

$$\begin{aligned} J_N - (2\xi_n)^N \cdot J_0 &= \sum_{k=1}^N (2\xi_n)^{N-k} \cdot (J_k - 2\xi_n \cdot J_{k-1}) = \\ &= -[y_n'^2(\pi, t) - y_n'^2(0, t)] \cdot \sum_{k=1}^N (2\xi_n)^{N-k} \cdot P_k[q(0, t)] \end{aligned}$$

and the integral

$$J_0 = \int_0^\pi y_n^2(x, t) HP_0 dx = \int_0^\pi y_n^2(x, t) q_x dx = -[y_n'^2(\pi, t) - y_n'^2(0, t)],$$

we deduce the identity

$$J_N = -[y_n'^2(\pi, t) - y_n'^2(0, t)] \cdot \sum_{k=0}^N (2\xi_n)^{N-k} \cdot P_k[q(0, t)]. \quad (13)$$

Now we proceed to calculating the second integral in identity (12),

$$\int_0^\pi G \cdot y_n^2 dx = \int_0^\infty s(\pi, \lambda, t) \beta(\lambda, t) \left\{ 2 \int_0^\pi y_n^2 \cdot (\psi_+ \psi_-)' dx \right\} d\lambda.$$

Integrating by parts, it is easy to see that

$$\begin{aligned} I &= 2 \int_0^\pi y_n^2 \cdot (\psi_+ \psi_-)' dx = \int_0^\pi \{ y_n^2 \cdot (\psi_+ \psi_-)' - (y_n^2)' \cdot (\psi_+ \psi_-) \} dx = \\ &= \int_0^\pi \{ y_n \psi_- (y_n \psi_+' - y_n' \psi_+) + y_n \psi_+ (y_n \psi_-' - y_n' \psi_-) \} dx. \end{aligned}$$

It yields

$$I = \frac{1}{\xi_n - \lambda} \cdot [y_n'^2(\pi, t) - y_n'^2(0, t)].$$

Hence,

$$\int_0^\pi G \cdot y_n^2 dx = [y_n'^2(\pi, t) - y_n'^2(0, t)] \cdot \int_0^\infty \frac{s(\pi, \lambda, t) \beta(\lambda, t)}{\xi_n - \lambda} d\lambda. \quad (14)$$

Substituting expressions (13) and (14) into (12), we obtain the identity

$$\begin{aligned} \dot{\xi}_n &= [y_n'^2(\pi, t) - y_n'^2(0, t)] \times \\ &\times \left\{ - \sum_{k=0}^N (2\xi_n)^{N-k} \cdot P_k[q(0, t)] + \int_0^\infty \frac{s(\pi, \lambda, t) \beta(\lambda, t)}{\xi_n - \lambda} d\lambda \right\}. \end{aligned} \quad (15)$$

Employing the identities

$$\begin{aligned} y_n(x, t) &= \frac{1}{c_n(t)} s(x, \xi_n(t), t), \\ c_n^2(t) &\equiv \int_0^\pi s^2(x, \xi_n(t), t) dx = s'(\pi, \xi_n(t), t) \frac{\partial s(\pi, \xi_n(t), t)}{\partial \lambda}, \end{aligned}$$

we have

$$y_n'^2(\pi, t) - y_n'^2(0, t) = \frac{1}{\frac{\partial s(\pi, \xi_n(t), t)}{\partial \lambda}} \left(s'(\pi, \xi_n(t), t) - \frac{1}{s'(\pi, \xi_n(t), t)} \right).$$

Substituting here the expression

$$s'(\pi, \xi_n, t) - \frac{1}{s'(\pi, \xi_n, t)} = \sigma_n(t) \sqrt{\Delta^2(\xi_n(t)) - 4},$$

we obtain

$$y_n'^2(\pi, t) - y_n'^2(0, t) = \frac{\sigma_n(t) \sqrt{\Delta^2(\xi_n(t)) - 4}}{\frac{\partial s(\pi, \xi_n(t), t)}{\partial \lambda}}.$$

Here $\sigma_n(t) = \text{sign}\{s'(\pi, \xi_n(t), t) - c(\pi, \xi_n(t), t)\}$.

The expansions

$$\begin{aligned} \Delta^2(\lambda) - 4 &= 4\pi^2(\lambda_0 - \lambda) \prod_{k=1}^{\infty} \frac{(\lambda_{2k-1} - \lambda)(\lambda_{2k} - \lambda)}{k^4}, \\ s(\pi, \lambda, t) &= \pi \prod_{k=1}^{\infty} \frac{\xi_k(t) - \lambda}{k^2} \end{aligned}$$

imply

$$y_n'^2(\pi, t) - y_n'^2(0, t) = 2(-1)^n \sigma_n(t) \sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)} \times \sqrt{(\xi_n - \lambda_0) \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{(\lambda_{2k-1} - \xi_n)(\lambda_{2k} - \xi_n)}{(\xi_k - \xi_n)^2}}. \tag{16}$$

By (15) and (16) we get (9).

Let us prove the independence of t for the eigenvalues $\lambda_n, n = 0, 1, 2, \dots$ of the periodic and anti-periodic problems for Sturm-Liouville equation (5). By analogy with formula (15) one can show that

$$\dot{\lambda}_n(t) = \int_0^\pi G(x, t) v_n^2(x, t) dx,$$

where $v_n(x, t)$ is a normalized eigenfunction of either periodic or antiperiodic problem for Sturm-Liouville equation (5). Taking into consideration for structure of the function $G(x, t)$ and proceeding as above, we obtain $\dot{\lambda}_n(t) = 0$. The proof is complete. \square

5. COROLLARIES OF MAIN THEOREM

Corollary 1. *If instead of $q(x, t)$ we consider $q(x + \tau, t)$, the eigenvalues to periodic and antiperiodic problem are independent of the parameters τ and t , and the eigenvalues ξ_n of the Dirichlet problem and the signs σ_n depend on τ and t ; $\xi_n = \xi_n(\tau, t), \sigma_n = \sigma_n(\tau, t) = \pm 1, n \geq 1$. In this case system (9) casts into the form*

$$\frac{\partial \xi_n}{\partial t} = 2(-1)^{n-1} \sigma_n(\tau, t) \left\{ \sum_{k=0}^N (2\xi_n)^{N-k} \cdot P_k[q(\tau, t)] + \int_0^\infty \frac{s(\pi, \lambda, t, \tau) \beta(\lambda, t)}{\lambda - \xi_n} d\lambda \right\} \times \sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)} \times \sqrt{(\xi_n - \lambda_0) \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{(\lambda_{2k-1} - \xi_n)(\lambda_{2k} - \xi_n)}{(\xi_k - \xi_n)^2}}, \quad n \geq 1. \tag{17}$$

Here

$$s(\pi, \lambda, t, \tau) = \pi \prod_{k=1}^{\infty} \frac{\xi_k(t, \tau) - \lambda}{k^2}. \tag{18}$$

Corollary 2. *Consider the case $N = 2$. In this case differential equation (1) becomes*

$$q_t = \frac{1}{4} q_{xxxx} - 5q_x q_{xx} - \frac{5}{2} q q_{xxx} + \frac{15}{2} q^2 q_x + G(x, t), \tag{19}$$

and Dubrovin system of differential equations (17) is written as

$$\frac{\partial \xi_n}{\partial t} = 2(-1)^{n-1} \sigma_n(\tau, t) \left\{ 4\xi_n^2 + 2\xi_n q - \frac{1}{2} q_{\tau\tau} + \frac{3}{2} q^2 + \int_0^\infty \frac{s(\pi, \lambda, t, \tau) \beta(\lambda, t)}{\lambda - \xi_n} d\lambda \right\} \times \sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)} \times \sqrt{(\xi_n - \lambda_0) \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{(\lambda_{2k-1} - \xi_n)(\lambda_{2k} - \xi_n)}{(\xi_k - \xi_n)^2}}, \quad n \geq 1. \tag{20}$$

Employing the following trace formulae

$$q(\tau, t) = \lambda_0 + \sum_{k=1}^{\infty} (\lambda_{2k-1} + \lambda_{2k} - 2\xi_k(\tau, t)), \tag{21}$$

$$q^2(\tau, t) - \frac{1}{2}q_{\tau\tau}(\tau, t) = \lambda_0^2 + \sum_{k=1}^{\infty} (\lambda_{2k-1}^2 + \lambda_{2k}^2 - 2\xi_k^2(\tau, t)), \quad (22)$$

system (20) can be rewritten in a closed form.

Corollary 3. *This theorem provides a method for solving problem (19), (2)-(5).*

Indeed, denote by λ_n , $n = 0, 1, 2, \dots$, $\xi_n(\tau, t)$, $\sigma_n(\tau, t)$, $n = 1, 2, \dots$, the spectral data of the problem

$$-y'' + q(x + \tau, t)y = \lambda y, \quad x \in R^1.$$

Let us find spectral data λ_n , $n = 0, 1, 2, \dots$, $\xi_n^0(\tau)$, $\sigma_n^0(\tau)$, $n = 1, 2, \dots$ for the equation

$$-y'' + q_0(x + \tau)y = \lambda y, \quad x \in R^1.$$

We solve then the Cauchy problem $\xi_n(\tau, t)|_{t=0} = \xi_n^0(\tau)$, $\sigma_n(\tau, t)|_{t=0} = \sigma_n^0(\tau)$, $n = 1, 2, \dots$ for Dubrovin system of equations (20). By trace formula (21) we find the solution to problem (19), (2)-(5). Then it is easy to find the Floquet solutions $\psi_{\pm}(x, \lambda, t)$.

Remark 2. Let us show that the constructed function $q(\tau, t)$ satisfies equation (19). In order to do it, we employ the following Dubrovin system of equations

$$\begin{aligned} \frac{\partial \xi_n}{\partial \tau} &= 2(-1)^{n-1} \sigma_n(\tau, t) \sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)} \times \\ &\times \sqrt{(\xi_n - \lambda_0) \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{(\lambda_{2k-1} - \xi_n)(\lambda_{2k} - \xi_n)}{(\xi_k - \xi_n)^2}}, \quad n = 1, 2, \dots, \end{aligned} \quad (23)$$

and trace formulae (21), (22), as well as (see [26])

$$\begin{aligned} \frac{3}{16}q_{\tau\tau\tau\tau}(\tau, t) - \frac{3}{2}q(\tau, t)q_{\tau\tau}(\tau, t) - \frac{15}{16}q_{\tau}^2(\tau, t) + q^3(\tau, t) = \\ = \lambda_0^3 + \sum_{k=1}^{\infty} (\lambda_{2k-1}^3 + \lambda_{2k}^3 - 2\xi_k^3(\tau, t)). \end{aligned} \quad (24)$$

Dubrovin system (20) and (23) imply

$$\frac{\partial \xi_k}{\partial t} = \left\{ 4\xi_k^2 + 2\xi_k q - \frac{1}{2}q_{\tau\tau} + \frac{3}{2}q^2 + \int_0^{\infty} \frac{s(\pi, \lambda, t, \tau)\beta(\lambda, t)}{\lambda - \xi_k} d\lambda \right\} \frac{\partial \xi_k}{\partial \tau}, \quad k \geq 1. \quad (25)$$

First trace formula (21) and (25) yield

$$\begin{aligned} q_t = -2 \sum_{k=1}^{\infty} \frac{\partial \xi_k}{\partial t} = (q_{\tau\tau} - 3q^2) \cdot \sum_{k=1}^{\infty} \frac{\partial \xi_k}{\partial \tau} - 4q \sum_{k=1}^{\infty} \xi_k \frac{\partial \xi_k}{\partial \tau} - 8 \sum_{k=1}^{\infty} \xi_k^2 \frac{\partial \xi_k}{\partial \tau} + \\ + 2 \int_0^{\infty} \beta(\lambda, t) \left\{ \sum_{k=1}^{\infty} \frac{s(\pi, \lambda, t, \tau) \partial \xi_k}{\xi_k - \lambda} \frac{\partial \xi_k}{\partial \tau} \right\} d\lambda. \end{aligned} \quad (26)$$

Differentiating trace formulae (21), (22), and (24) w.r.t. τ , we obtain

$$\begin{aligned} 2 \sum_{k=1}^{\infty} \frac{\partial \xi_k}{\partial \tau} = -q_{\tau}, \quad 4 \sum_{k=1}^{\infty} \xi_k \frac{\partial \xi_k}{\partial \tau} = \frac{1}{2}q_{\tau\tau\tau} - 2qq_{\tau}, \\ -2 \sum_{k=1}^{\infty} \xi_k^2 \frac{\partial \xi_k}{\partial \tau} = \frac{1}{16}q_{\tau\tau\tau\tau} - \frac{1}{2}qq_{\tau\tau\tau} - \frac{9}{8}q_{\tau}q_{\tau\tau} + q^2q_{\tau}. \end{aligned}$$

Employing these identities and expansion (18), by (26) we deduce

$$q_t = \frac{1}{4}q_{\tau\tau\tau\tau} - 5q_{\tau}q_{\tau\tau} - \frac{5}{2}qq_{\tau\tau} + \frac{15}{2}q^2q_{\tau} + 2 \int_0^{\infty} \beta(\lambda, t) \frac{\partial s(\pi, \lambda, t, \tau)}{\partial \tau} d\lambda.$$

It follows from identity (6) that

$$q_t = \frac{1}{4}q_{\tau\tau\tau\tau} - 5q_{\tau}q_{\tau\tau} - \frac{5}{2}qq_{\tau\tau} + \frac{15}{2}q^2q_{\tau} + 2 \int_0^{\infty} \beta(\lambda, t) s(\pi, \lambda, t) \frac{\partial}{\partial \tau} (\psi_+(\tau, \lambda, t) \cdot \psi_-(\tau, \lambda, t)) d\lambda.$$

Corollary 4. *From the results of work [25] we deduce that if the initial function $q_0(x)$ is real and analytic, then the lengths $\lambda_{2n} - \lambda_{2n-1}$ of the gaps corresponding to this coefficients decay exponentially. Since the lengths of the gaps corresponding to the coefficients $q(x, t)$ are independent of t , the function $q(x, t)$ is analytic w.r.t. x .*

Corollary 5. *The generalized Borg's inverse theorem (see [29]) follows that if $q_0(x)$ has the period $\frac{\pi}{n}$, the solution $q(x, t)$ to problem (19), (2)-(5) is $\frac{\pi}{n}$ -periodic w.r.t. x .*

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