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ON GROWTH CHARACTERISTICS OF OPERATOR-VALUED FUNCTIONS

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Abstract. In the work we generalize Liouville theorem and the concept of order and type of entire function to the case of an operator-valued function with values in the space $\mathrm{Lec}(\mathbf{H}_1, \mathbf{H})$ of all linear continuous operators acting from a locally convex space $\mathbf{H}_1$ to a locally convex space $\mathbf{H}$ with an equicontinuous bornology. We find the formulae expressing the order and type of an operator-valued function in terms of the characteristics for the sequence of the coefficients. Some properties of the order and type of an operator-valued function are established.

Keywords: locally convex space, order and type of sequence of operators, order and type of entire function, equicontinuous bornology, convergence by bornology, operator-valued function.

Introduction

It is known [3, 4] that if an entire scalar function $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is not a polynomial, the maximum of its modulus $M_f(r) = \max\limits_{|z|\le r} |f(z)|$ grows faster than any positive power of r as $r \to \infty$ (Liouville theorem). To estimate the growth of such functions, one usually uses the characteristics (order and type),

$$\rho = \overline{\lim_{r\to\infty}} \frac{\ln\ln M_f(r)}{\ln r}, \ \sigma = \overline{\lim_{r\to\infty}} \frac{\ln M_f(r)}{r^\rho}. \tag{1}$$

At that, the formulae expressing these characteristics in terms of the coefficients

$$\rho = \overline{\lim_{n\to\infty}} \frac{n\ln n}{-\ln|a_n|}, \ (\rho e\sigma)^{\frac{1}{\rho}} = \overline{\lim_{n\to\infty}} n^{\frac{1}{\rho}} \sqrt[n]{|a_n|} \tag{2}$$

are known. This work is devoted to the generalization of these formulae and the Liouville theorem for the case of an entire operator-valued function $F(t) = \sum\limits_{n=0}^{\infty} A_n t^n$ with the values in the space $\mathrm{Lec}(\mathbf{H}_1, \mathbf{H})$ of all linear continuous operators acting from a locally convex space $\mathbf{H}_1$ into a locally convex space $\mathbf{H}$. The spaces $\mathbf{H}_1$ and $\mathbf{H}$ are in general not normable.

1. Entire operator-valued functions and analogue to Liouville theorem

$\mathbf{H}_1$ and $\mathbf{H}$ are separable locally convex spaces over the field of complex numbers with the topologies defined respectively by the multinorms $\{\|\cdot\|'_q\}$, $q \in \mathcal{Q}$ and $\{\|\cdot\|_p\}$, $p \in \mathcal{P}$. Without loss of generality one can regard the multinorms in $\mathbf{H}_1$ and $\mathbf{H}$ as majorant [2]. By $\mathcal{A} = \{A_n\}_{n=0}^{\infty}$ we denote a sequence of linear continuous operators acting from the locally convex space $\mathbf{H}_1$

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into the locally convex space $\mathbf{H}$. The sequence $\mathcal{A}$ is called as having an order [1, 5], if there exists a sequence of positive numbers $\{c_n\}_{n=0}^{\infty}$ such that

$$\forall p \in \mathcal{P}\ \exists C_p > 0\ \exists q(p) \in \mathcal{Q}\ \forall x \in \mathbf{H}_1\ \forall n \in \mathbb{N}:\ \|c_n A_n(x)\|_p \le C_p \|x\|'_q, \tag{3}$$

i.e., the family of the operators $\{c_n A_n\}$ is equicontinuous.

Let

$$\theta_{\mathcal{A}}(p,q,n) = \sup_{\|x\|'_q \neq 0} \left\{ \frac{\|A_n(x)\|_p}{\|x\|'_q} \right\},\ n = 0,1,2,\cdots$$

(the case $\theta_{\mathcal{A}}(p,q,n) = +\infty$ is not excluded). We denote

$$\beta_{p,q}(\mathcal{A}) = \overline{\lim_{n\to\infty}} \frac{\ln \theta_{\mathcal{A}}(p,q,n)}{n \ln n}.$$

Definition 1. *The number* $\beta_p(\mathcal{A}) = \inf_{q\in\mathcal{Q}} \beta_{p,q}(\mathcal{A}),\ (p \in \mathcal{P})$ *is called a p-order of the sequence of the operators* $\mathcal{A}$, *and the number* $\beta(\mathcal{A}) = \sup_{p\in\mathcal{P}}\{\beta_p(\mathcal{A})\}$ *is called its order.*

If $\beta(\mathcal{A}) = \pm\infty$ and at that the sequence $\mathcal{A}$ has an order, then it is called a sequence of an *infinite* order.

Remark. Let us note that there is an essential difference between the sequences having an order $\beta(\mathcal{A}) = +\infty$, and that having no order (despite formally $\beta(\mathcal{A}) = +\infty$). If $\beta(\mathcal{A}) = +\infty$, but the sequence $\mathcal{A} = \{A_n\}$ has an order, it is possible to select a sequence of positive numbers $\{c_n\}$ such that condition (3) holds. And one can not select such a sequence for the sequences having no order.

If a sequence of operators $\mathcal{A}$ has a p-order $\beta_p(\mathcal{A}) \neq \pm\infty$, one introduces for it a finer characteristics. Denote

$$\alpha_{p,q}(\mathcal{A}) = \overline{\lim_{n\to\infty}} n^{-\beta_p(\mathcal{A})} \sqrt[n]{\theta_{\mathcal{A}}(p,q,n)}.$$

Definition 2. *The number* $\alpha_p(\mathcal{A}) = \inf_{q\in\mathcal{Q}} \alpha_{p,q}(\mathcal{A}),\ (p \in \mathcal{P})$ *is called a p-type of a sequence of operators* $\mathcal{A}$ *at the p-order* $\beta_p(\mathcal{A})$.

It is obvious that $\beta_p(\mathcal{A}) \le \beta(\mathcal{A})$, $\forall p$. It is possible to show [7] that the case when the identity $\beta_p(\mathcal{A}) = \beta(\mathcal{A})$ is valid not for all p, but just for some p, is reduced to the case $\beta_p(\mathcal{A}) = \beta(\mathcal{A})$, $\forall p$ by replacing the multinorm to an equivalent one. This replacement changes neither the order no the type of a sequence of operators. This is why (without loss of generality) we shall consider two cases, either $\beta_p(\mathcal{A}) = \beta(\mathcal{A})$, $\forall p$, or $\beta_p(\mathcal{A}) < \beta(\mathcal{A})$, $\forall p$.

Definition 3. *Let a sequence of operators* $\mathcal{A}$ *has the p-orders* $\beta_p(\mathcal{A})$ *and the order* $\beta(\mathcal{A}) \neq \pm\infty$. *The number*

$$\alpha(\mathcal{A}) = \begin{cases} \sup_{p\in\mathcal{P}}\{\alpha_p(\mathcal{A})\} & ,\ \beta_p(\mathcal{A}) = \beta(\mathcal{A}),\ \forall p \\ 0 & ,\ \beta_p(\mathcal{A}) < \beta(\mathcal{A}),\ \forall p \end{cases}$$

is called a type of the sequence of operators $\mathcal{A}$ *at the order* $\beta(\mathcal{A})$.

A sequence of operators $\mathcal{A}$ is called belonging to the class $\mathfrak{L}_{\mathbf{H}_1,\mathbf{H}}[b,a]$, (cf. [1, 5]) if its order is less than b or equal to b, but then the type does not exceed a.

Let $\mathbf{H}$ be a complete space. It is known [8] that in this case the space $\mathrm{Lec}(\mathbf{H}_1,\mathbf{H})$ of linear continuous operators acting from $\mathbf{H}_1$ into $\mathbf{H}$ equipped with an equicontinuous bornology is a complete bornological vector convex space.

Definition 4. *An operator-valued function* $F : \mathbb{C} \to \mathrm{Lec}(\mathbf{H}_1,\mathbf{H})$ *is called differentiable at a point* $t_0 \in \mathbb{C}$ *if there exists a limit (w.r.t. the bornology of the space* $\mathrm{Lec}(\mathbf{H}_1,\mathbf{H})$*)*

$$\lim_{t\to t_0} \frac{F(t) - F(t_0)}{t - t_0}. \tag{4}$$

This limit is called a derivative of the operator-valued function F at the point t_0 and is indicated by $F'(t_0)$.

Definition 5. *An operator-valued function $F : \mathbb{C} \to \mathrm{Lec}(\mathbf{H}_1, \mathbf{H})$ is called entire if its defined and differentiable at each point $t \in \mathbb{C}$.*

An entire operator-valued function is obviously continuous everywhere $\big($w.r.t. the bornology of the space $\mathrm{Lec}(\mathbf{H}_1, \mathbf{H})\big)$.

Let
$$\theta_F(p, q, t) = \sup_{\|x\|'_q \neq 0} \left\{ \frac{\|F(t)(x)\|_p}{\|x\|'_q} \right\},\ t \in \mathbb{C}$$
(the case $\theta_F(p, q, t) = +\infty$ is not excluded).

Theorem 1. *An entire operator-valued function $F(t)$ is bounded on each closed disk, i.e., the family of the operators $\{F(t)\}_{|t|\leq r}$ is equicontinuous for each $r > 0$.*

Proof. We fix an arbitrary $r > 0$. Suppose the function $F(t)$ is entire, and the family $\{F(t)\}_{|t|\leq r}$ is not equicontinuous, i.e., there exists $p_0 \in \mathcal{P}$ such that for each $C > 0$ and for each $q \in \mathcal{Q}$ there exists $t_C = t_C(q)$ such that $|t_C| \leq r$ and $\theta_F(p_0, q, t_C) > C$. We fix an arbitrary $q \in \mathcal{Q}$ and take $C = n$, $n \in \mathbb{N}$. We obtain then a sequence of complex numbers $t_n = t_n(q)$ lying within the disk $|t| \leq r$. At that,
$$\theta_F(p_0, q, t_n) > n,\ \forall n. \tag{5}$$
By the boundedness of the sequence $\{t_n\}$ there exists a converging subsequence $\{t_{n_k}\}$. It follows from (5) that $\theta_F(p_0, q, t_{n_k}) > n_k,\ \forall k$, i.e., the sequence $\{F(t_{n_k})\}$ is not equicontinuous and thus diverges. But by the continuity of the function F it must converges. We obtain the contradiction. □

If the function $F(t)$ is entire, then for each fixed $x \in \mathbf{H}_1$, $F(t)(x)$ is an entire function with values in $\mathbf{H}$. Such function is represented as a power series [9]
$$F(t)(x) = \sum_{n=0}^{\infty} x_n t^n,\ x \in \mathbf{H}_1,\ \{x_n\} \subset \mathbf{H}$$
(the sequence $\{x_n\}$ depends on x). We let
$$M_F(p, q, r) = \sup_{|t|\leq r} \theta_F(p, q, t).$$
We define a sequence of operators $A_n : \mathbf{H}_1 \to \mathbf{H}$ as follows, $A_n(x) = x_n,\ \forall x \in \mathbf{H}_1$. We obtain the expansion of the function $F(t)$ as a power series
$$F(t) = \sum_{n=0}^{\infty} A_n t^n. \tag{6}$$
At that, series (6) everywhere pointwise converges to the function $F(t)$ $\Big($for each fixed $x \in \mathbf{H}_1$ the series $\sum\limits_{n=0}^{\infty} A_n(x) t^n$ converges to the function $F(t)(x)$ everywhere$\Big)$. Let us show that $\{A_n\} \subset \mathrm{Lec}(\mathbf{H}_1, \mathbf{H})$ and series (6) converges everywhere to the function $F(t)$ w.r.t. the bornology. First we prove the following theorem.

Theorem 2 (*Analogue of Cauchy inequalty*)**.** *The inequality*
$$\theta_{\mathcal{A}}(p, q, n) \leq \frac{M_F(p, q, r)}{r^n},\ \forall p\ \forall q\ \forall n\ \forall r > 0 \tag{7}$$
holds true.

Proof. Let $p \in \mathcal{P}$, $q \in \mathcal{Q}$, $r > 0$. If $M_F(p,q,r) = \infty$, then inequality (7) holds true. Let $M_F(p,q,r) < \infty$. Since for each fixed x the vector-function $F(t)(x) = \sum_{n=0}^{\infty} A_n(x)t^n$ is entire, then (see, for instance, [9])

$$A_n(x) = \frac{1}{2\pi i} \int\limits_{|\xi|=r} \frac{F(\xi)(x)d\xi}{\xi^{n+1}}, \ n \in \mathbb{N}.$$

Hence, $\forall p \in \mathcal{P}\ \forall x \in \mathbf{H}_1\ \forall r > 0\ \forall n \in \mathbb{N}$ we have

$$\|A_n(x)\|_p \leq \frac{\sup\limits_{|\xi|\leq r} \|F(\xi)(x)\|_p}{r^n} \leq \frac{\sup\limits_{|\xi|\leq r} \theta_F(p,q,\xi)}{r^n} \|x\|'_q = \frac{M_F(p,q,r)}{r^n}\|x\|'_q$$

that yields inequality (7). □

Since the function $F(t)$ is entire, by Theorem 1 for each $r > 0$ the family $\{F(t)\}_{|t|\leq r}$ is equicontinuous, i.e.,

$$\forall p \in \mathcal{P}\ \exists C_p > 0\ \exists q_p \in \mathcal{Q}\ \forall x \in \mathbf{H}_1\ \forall t\ |t| \leq r \Rightarrow \|F(t)(x)\|_p \leq C_p\|x\|'_{q_p}.$$

For each p we choose $q_0 = q_0(p)$ such that $\|x\|'_{q_0} \geq \|x\|'_{q_p}$, $\forall x \in \mathbf{H}_1$ (it is always possible since the multinorm is majorant). Then

$$\theta_F(p,q_0,t) = \sup_{\|x\|'_{q_0}\neq 0} \left\{\frac{\|F(t)(x)\|_p}{\|x\|'_{q_0}}\right\} \leq \sup_{\|x\|'_{q_0}\neq 0} \left\{\frac{C_p\|x\|'_{q_p}}{\|x\|'_{q_0}}\right\} = \tilde{C}_p(q_0), \ |t| \leq r.$$

Thus, for each $r > 0$ and each $p \in \mathcal{P}$ there exists $q_0 \in \mathcal{Q}$ such that $\theta_F(p,q_0,t)$ (as functions of t) are bounded in the disk $|t| \leq r$. And it means that

$$\forall r\ \forall p\ \exists q_0(p,r): \ M_F(p,q_0,r) < \infty.$$

Hence, by Theorem 2,

$$\varlimsup_{n\to\infty} \sqrt[n]{\theta_{\mathcal{A}}(p,q_0,n)} \leq \frac{1}{r}, \ r > 0. \tag{8}$$

It follows from (8) that either $\beta_p(\mathcal{A}) < 0$ or $\beta_p(\mathcal{A}) = 0$, but then by the arbitrariness of r

$$\alpha_p(\mathcal{A}) = \inf_{q\in\mathcal{Q}} \varlimsup_{n\to\infty} \sqrt[n]{\theta_{\mathcal{A}}(p,q,n)} = 0.$$

Thus, the sequence $\{A_n\}$ belongs to the class $\mathfrak{L}_{\mathbf{H}_1,\mathbf{H}}[0,0]$ and therefore series (6) everywhere converges to the function $F(t)$ w.r.t. bornology (see [1, 5]).

Theorem 3 (*Analogue of Liouville theorem*)**.** *Suppose function (6) is entire and satisfies the condition*

$$\exists k\ \forall p\ \exists K_p > 0\ \exists q(p)\ \forall r > 0: \ M_F(p,q,r) \leq K_p r^k. \tag{9}$$

Then F is an operator-valued polynomial of degree at most k, i.e.,

$$F(t) = \sum_{n=0}^{[k]} A_n t^n.$$

Proof. By inequalities (7), (9) and the definition of the numbers $\theta_{\mathcal{A}}(p,q,n)$ we have

$$\|A_n(x)\|_p \leq \theta_{\mathcal{A}}(p,q,n)\|x\|'_q \leq K_p r^{k-n}\|x\|'_q, \ \forall p\ \forall x \in \mathbf{H}_1\ \forall r > 0\ \forall n, \ q = q(p).$$

By the arbitrariness of r,

$$\|A_n(x)\|_p = 0, \ \forall n > k\ \forall p\ \forall x \in \mathbf{H}_1,$$

thus, $A_n = 0$, $\forall n > k$. □

Theorem 3 shows that if F is an entire transcendental function, then the quantities $M_F(p,q,r)$ grows faster than any positive power as $r \to \infty$.

2. Growth characteristics for entire function and formulae for their calculation

Definition 6. *Let $F : \mathbb{C} \to \mathrm{Lec}(\mathbf{H}_1, \mathbf{H})$ be an entire transcendental function. The number $\rho_p(F) = \inf_{q \in \mathcal{Q}} \rho_{p,q}(F)$, where*

$$\rho_{p,q}(F) = \overline{\lim_{r\to\infty}} \frac{\ln \ln M_F(p,q,r)}{\ln r}$$

will be called a p-order of the function F, and the number $\rho(F) = \sup_{p \in \mathcal{P}} \{\rho_p(F)\}$ will be called its order.

If $0 < \rho_p(F) < \infty$, the number $\sigma_p(F) = \inf_{q \in \mathcal{Q}} \sigma_{p,q}(F)$, where

$$\sigma_{p,q}(F) = \overline{\lim_{r\to\infty}} \frac{\ln M_F(p,q,r)}{r^{\rho_p(F)}},$$

will be called a p-type of the function f at p-order $\rho(F)$.

It can be shown that the case when for some p, $\rho_p(F) < \rho(F)$, while for other $\rho_p(F) = \rho(F)$, is reduced to the case $\rho_p(F) = \rho(F)$, $\forall p$ by the replacement of the multinorm to an equivalent one. This is why (without loss of generality) we shall consider two cases, either $\rho_p(F) < \rho(F)$, $\forall p$, or $\rho_p(F) = \rho(F)$, $\forall p$.

Definition 7. *Suppose a function $F(t)$ has p-orders $\rho_p(F)$ and order $0 < \rho(F) < \infty$. The number*

$$\sigma(F) = \begin{cases} 0 & , \quad \rho_p(F) < \rho(F), \ \forall p \\ \sup_{p\in\mathcal{P}} \{\sigma_p(F)\} & , \quad \rho_p(F) = \rho(F), \ \forall p \end{cases}$$

will be called a type of the function f at the order $\rho(F)$.

Lemma 1. *Suppose*

$$\forall p \ \exists q_p \ \exists a_p, b_p > 0 \ \exists r_0(p) \ \forall r > r_0 : \ M_F(p, q_p, r) < e^{a_p r^{b_p}}. \tag{10}$$

Then

$$\forall p \ \exists n_0(p) \ \forall n > n_0 : \ \sqrt[n]{\theta_{\mathcal{A}}(p, q_p, n)} < \left(\frac{a_p b_p e}{n} \right)^{\frac{1}{b_p}}. \tag{11}$$

Proof. Suppose inequality (10) holds true, then by (7) we have

$$\theta_{\mathcal{A}}(p, q_p, n) < \frac{e^{a_p r^{b_p}}}{r^n}; \ \forall p \ \forall r > r_0(p) \ \forall n. \tag{12}$$

We denote $\mu_p(r) = e^{a_p r^{k_p}} r^{-n}$. It is obvious that

$$\forall p : \ \mu_p(0) = \mu_p(+\infty) = +\infty.$$

Let us find $\min_{r>0} \{\mu_p(r)\}$,

$$\mu_p'(r) = \mu_p(r) \ln' \mu_p(r)$$

$$\mu_p'(r) = \mu_p(r) \left(a_p r^{b_p} - n \ln r \right)'$$

$$\mu_p'(r) = \mu_p(r) \left(a_p b_p r^{b_p - 1} - \frac{n}{r} \right)$$

$\mu_p'(r) = 0$ as $r = r_1 = \left(\frac{n}{a_p b_p} \right)^{\frac{1}{b_p}}$. Substituting r_1 in inequality (12), we obtain (11). □

Lemma 2. *Suppose*

$$\forall p\ \exists q_p\ \exists a_p, b_p > 0\ \exists n_0(p)\ \forall n > n_0 :\ \sqrt[n]{\theta_{\mathcal{A}}(p, q_p, n)} < \left(\frac{a_p b_p e}{n}\right)^{\frac{1}{b_p}}. \tag{13}$$

Then

$$\forall p\ \forall \varepsilon > 0\ \exists r_0(p, \varepsilon)\ \forall r > r_0 :\ M_F(p, q_p, r) < e^{(a_p+\varepsilon) r^{b_p}}. \tag{14}$$

Proof. By condition (13) $\mathcal{A} \in \mathfrak{L}_{\mathbf{H}_1,\mathbf{H}}[0,0]$, thus, F is an entire operator-valued function. Let us fix an arbitrary p (and fix by this depending on it q_p, a_p, b_p) and consider the inequality

$$\theta_{\mathcal{A}}(p, q_p, n) r^n < \left(\left(\frac{a_p b_p e}{n}\right)^{\frac{1}{b_p}} r\right)^n.$$

For sufficiently large n

$$\left(\frac{a_p b_p e}{n}\right)^{\frac{1}{b_p}} r < \frac{1}{2}. \tag{15}$$

By $N_p(r)$ we denote the lowest of natural numbers n for which inequality (15) holds true.

Let us find the dependence of $N_p(r)$ on r. We have

$$2r \left(\frac{a_p b_p e}{n}\right)^{\frac{1}{b_p}} < 1, \text{ as } n > (2r)^{b_p}(a_p b_p e).$$

Therefore, we can let $N_p(r) = \left[(2r)^{b_p}(a_p b_p e)\right] + 1$.

Further, for each fixed $p \in \mathcal{P}$, $t \in \mathbb{C}$ and $x \in \mathbf{H}_1$ we have

$$\|F(t)(x)\|_p \le \sum_{n=0}^{\infty} \|A_n(x)\|_p |t|^n \le \sum_{n=0}^{\infty} \theta_{\mathcal{A}}(p, q_p, n)|t|^n \|x\|'_{q_p},$$

hence,

$$\theta_F(p, q_p, t) \le \sum_{n=0}^{\infty} \theta_{\mathcal{A}}(p, q_p, n)|t|^n,$$

i.e.,

$$\forall p\ \forall r > 0 :\ M_F(p, q_p, r) \le \sum_{n=0}^{\infty} \theta_{\mathcal{A}}(p, q_p, n) r^n = \sum_{n=0}^{N_p(r)-1} \theta_{\mathcal{A}}(p, q_p, n) r^n + \sum_{n=N_p(r)}^{\infty} \theta_{\mathcal{A}}(p, q_p, n) r^n.$$

For $n \ge N_p(r)$ the inequality $\theta_{\mathcal{A}}(p, q_p, n) r^n < \left(\frac{1}{2}\right)^n$ holds true and hence

$$\sum_{n=N_p(r)}^{\infty} \theta_{\mathcal{A}}(p, q_p, n) r^n < \sum_{n=N_p(r)}^{\infty} \left(\frac{1}{2}\right)^n < \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2.$$

Since for each fixed p and r

$$\lim_{n\to\infty} \theta_{\mathcal{A}}(p, q_p, n) r^n = 0,$$

the sequence $\{\theta_{\mathcal{A}}(p, q_p, n) r^n\}$ has a maximal term. Let

$$m_p(r) = \max_{n\ge 0}\{\theta_{\mathcal{A}}(p, q_p, n) r^n\},$$

then

$$\sum_{n=0}^{N_p(r)-1} \theta_{\mathcal{A}}(p, q_p, n) r^n \le m_p(r) N_p(r).$$

Let us estimate $m_p(r)$. Let $\theta_{\mathcal{A}}(p, q_p, s) r^s$ be a maximal term. Under an unbound increasing of r the index s of maximal term increases unboundedly as well, i.e., $s \to \infty$ as $r \to \infty$. If r is sufficiently large, then $s > n_0$, where n_0 is a number in (13).

This is why

$$m_p(r) = \theta_{\mathcal{A}}(p, q_p, s) r^s < \left(\frac{a_p b_p e}{s}\right)^{\frac{s}{b_p}} r^s \le \max_{\xi \ge 0} \left\{ \left(\frac{a_p b_p e}{\xi}\right)^{\frac{\xi}{b_p}} r^\xi \right\}.$$

We denote

$$\nu_p(\xi) = \left(\frac{a_p b_p e}{\xi}\right)^{\frac{\xi}{b_p}} r^\xi.$$

Clearly,

$$\forall p: \ \nu_p(0) = 1, \ \nu_p(+\infty) = 0.$$

Let us find $\max\limits_{\xi \ge 0}\{\nu_p(\xi)\}$. We have

$$\nu_p'(\xi) = \nu_p(\xi) \left(\frac{\ln(a_p b_p e)}{b_p} - \frac{\ln \xi}{b_p} - \frac{1}{b_p} + \ln r\right).$$

$\nu_p'(\xi) = 0$ as $\xi = \xi_1 = (a_p b_p) r^{b_p}$.

$$\nu_p(\xi_1) = e^{a_p r^{b_p}}.$$

Therefore (for sufficiently large r), $m_p(r) < e^{a_p r^{b_p}}$.

Thus,

$$M_F(p, q_p, r) \le N_p(r) m_p(r) + 2 \le ((2r)^{b_p}(a_p b_p e) + 1) e^{a_p r^{b_p}} + 2 < e^{(a_p + \varepsilon) r^{b_p}}.$$

□

Theorem 4. *The growth characteristics of function (6) are calculated by the formulae*

$$\rho_p(F) = -\frac{1}{\beta_p(\mathcal{A})}, \ \forall p, \tag{16}$$

$$\sigma_p(F) = -\frac{\beta_p(\mathcal{A})}{e} (\alpha_p(\mathcal{A}))^{-\frac{1}{\beta_p(\mathcal{A})}}, \ \forall p, \tag{17}$$

$$\rho(F) = -\frac{1}{\beta(\mathcal{A})}, \tag{18}$$

$$\sigma(F) = \begin{cases} 0 & , \ \beta_p(\mathcal{A}) < \beta(\mathcal{A}), \ \forall p \\ -\frac{\beta(\mathcal{A})}{e} (\alpha(\mathcal{A}))^{-\frac{1}{\beta(\mathcal{A})}} & , \ \beta_p(\mathcal{A}) = \beta(\mathcal{A}), \ \forall p. \end{cases} \tag{19}$$

Proof. We fix an arbitrary p. Suppose the p-order of the function F equals $\rho_p(F)$. Then

$$\forall p \ \forall \varepsilon > 0 \ \exists q_p(\varepsilon) \ \exists r_0(p, \varepsilon) \ \forall r > r_0: \ M_F(p, q_p, r) \le \exp\left\{r^{\rho_p(F) + \varepsilon}\right\}.$$

By Lemma 1 ($b_p = \rho_p(F) + \varepsilon, \ a_p = 1$)

$$\sqrt[n]{\theta_{\mathcal{A}}(p, q_p, n)} < \left(\frac{(\rho_p(F) + \varepsilon)\, e}{n}\right)^{\frac{1}{\rho_p(F) + \varepsilon}}, \ \forall n > n_0.$$

By this we successively find

$$\frac{1}{n} \ln \theta_{\mathcal{A}}(p, q_p, n) < \left(\frac{1}{\rho_p(F) + \varepsilon}\right) \ln\big((\rho_p(F) + \varepsilon)\, e\big) - \frac{\ln n}{\rho_p(F) + \varepsilon} = C_p(\varepsilon) - \frac{\ln n}{\rho_p(F) + \varepsilon},$$

$$\ln \theta_{\mathcal{A}}(p, q_p, n) < C_p(\varepsilon) n - \frac{n \ln n}{\rho_p(F) + \varepsilon},$$

$$\ln \frac{1}{\theta_{\mathcal{A}}(p, q_p, n)} > \frac{n \ln n}{\rho_p(F) + \varepsilon} - C_p(\varepsilon) n = n \ln n \left(\frac{1}{\rho_p(F) + \varepsilon} - \frac{C_p(\varepsilon)}{\ln n}\right), \ \forall n > n_0. \tag{20}$$

As $n \to \infty$, the expression in parentheses in (20) tends to $\frac{1}{\rho_p(F)+\varepsilon}$, and for large n

$$\ln \frac{1}{\theta_{\mathcal{A}}(p, q_p, n)} > \frac{n \ln n}{\rho_p(F) + 2\varepsilon},$$

i.e.,

$$\rho_p(F) + 2\varepsilon > \frac{n \ln n}{-\ln \theta_{\mathcal{A}}(p, q_p, n)}.$$

By the arbitrariness of ε,

$$-\frac{1}{\beta_{p,q_p}(\mathcal{A})} = \varlimsup_{n\to\infty} \frac{n \ln n}{-\ln \theta_{\mathcal{A}}(p, q_p, n)} \le \rho_p(F).$$

Since $\beta_p(\mathcal{A}) = \inf\limits_q \{\beta_{p,q}(\mathcal{A})\}$, then

$$-\frac{1}{\beta_p(\mathcal{A})} \le -\frac{1}{\beta_{p,q_p}(\mathcal{A})} \le \rho_p(F).$$

Hence, $\rho_p(F) \ge -\frac{1}{\beta_p(\mathcal{A})}$, $\forall p$.

Vice-versa, since

$$-\frac{1}{\beta_{p,q}(\mathcal{A})} = \varlimsup_{n\to\infty} \frac{n \ln n}{-\ln \theta_{\mathcal{A}}(p, q, n)},$$

then

$$\frac{n \ln n}{-\ln \theta_{\mathcal{A}}(p, q, n)} < -\frac{1}{\beta_{p,q}(\mathcal{A})} + \frac{\varepsilon}{2}, \ \forall p \ \forall \varepsilon > 0 \ \forall q \ \forall n > n_0(p, q, \varepsilon).$$

And since $\beta_p(\mathcal{A}) = \inf\limits_q \{\beta_{p,q}(\mathcal{A})\}$, then

$$\forall p \ \forall \varepsilon > 0 \ \exists q_p(\varepsilon) : \ -\frac{1}{\beta_{p,q_p}(\mathcal{A})} \le -\frac{1}{\beta_p(\mathcal{A})} + \frac{\varepsilon}{2}.$$

Thus,

$$\forall p \ \forall \varepsilon > 0 \ \exists q_p(\varepsilon) \ \exists n_0(p, \varepsilon) \ \forall n > n_0 : \ \frac{n \ln n}{-\ln \theta_{\mathcal{A}}(p, q_p, n)} < -\frac{1}{\beta_p(\mathcal{A})} + \varepsilon,$$

therefore,

$$\forall p \ \forall \varepsilon > 0 \ \exists q_p(\varepsilon) \ \exists n_0(p, \varepsilon) \ \forall n > n_0 : \ \sqrt[n]{\theta_{\mathcal{A}}(p, q_p, n)} < n^{-\frac{1}{-\frac{1}{\beta_p(\mathcal{A})}+\varepsilon}}.$$

By Lemma 2 $\left(b_p = -\frac{1}{\beta_p(\mathcal{A})} + \varepsilon, \ a_p = \frac{1}{e\left(-\frac{1}{\beta_p(\mathcal{A})}+\varepsilon\right)} \right)$

$$\forall p \ \forall \varepsilon > 0 \ \exists q_p(\varepsilon) \ \exists r_0(p, \varepsilon) \ \forall r > r_0 : \ M_F(p, q_p, r) \le \exp\left\{ (a_p + \varepsilon) r^{\left(-\frac{1}{\beta_p(\mathcal{A})}+\varepsilon\right)} \right\}.$$

It means that $\rho_p(F) \le -\frac{1}{\beta_p(\mathcal{A})}$, $\forall p$.

Thus, identity (16) is proven. Identity (18) follows immediately from (16).

Let us prove identity (17).

Suppose the function F has the p-order $0 < \rho_p(F) < \infty$ and the p-type $\sigma_p(F)$. Then

$$\forall p \ \forall \varepsilon > 0 \ \exists q_p(\varepsilon) \ \exists r_0(p, \varepsilon) \ \forall r > r_0 : \ M_F(p, q_p, r) < \exp\left\{ (\sigma_p(F) + \varepsilon) r^{\rho_p(F)} \right\}.$$

By Lemma 1 $\left(a_p = \sigma_p(F) + \varepsilon, \ b_p = \rho_p(F)\right)$ we have

$$\sqrt[n]{\theta_{\mathcal{A}}(p, q_p, n)} < \left(\frac{(\sigma_p(F) + \varepsilon)\, \rho_p(F) e}{n} \right)^{\frac{1}{\rho_p(F)}}, \ \forall n > n_0,$$

$$n^{\frac{1}{\rho_p(F)}} \sqrt[n]{\theta_{\mathcal{A}}(p, q_p, n)} < \left((\sigma_p(F) + \varepsilon)\, \rho_p(F) e \right)^{\frac{1}{\rho_p(F)}}, \ \forall n > n_0.$$

By the arbitrariness of ε

$$\alpha_{p,q_p}(\mathcal{A}) = \overline{\lim_{n\to\infty}}\, n^{-\beta_p(\mathcal{A})} \sqrt[n]{\theta_{\mathcal{A}}(p,q_p,n)} =$$

$$= \overline{\lim_{n\to\infty}}\, n^{\frac{1}{\rho_p(F)}} \sqrt[n]{\theta_{\mathcal{A}}(p,q_p,n)} \le (\sigma_p(F)\rho_p(F)e)^{\frac{1}{\rho_p(F)}}$$

Since $\alpha_p(\mathcal{A}) = \inf\limits_q\{\alpha_{p,q}(\mathcal{A})\}$, then

$$\alpha_p(\mathcal{A}) \le \alpha_{p,q_p}(\mathcal{A}) \le (\sigma_p(F)\rho_p(F)e)^{\frac{1}{\rho_p(F)}}\,,\ \forall p.$$

Vice-versa, since

$$\alpha_{p,q}(\mathcal{A}) = \overline{\lim_{n\to\infty}}\, n^{-\beta_p(\mathcal{A})} \sqrt[n]{\theta_{\mathcal{A}}(p,q,n)} = \overline{\lim_{n\to\infty}}\, n^{\frac{1}{\rho_p(F)}} \sqrt[n]{\theta_{\mathcal{A}}(p,q,n)},\ \forall p,\ \forall q,$$

then

$\forall\varepsilon > 0\ \forall p\ \exists q(p,\varepsilon)\ \exists n_0(p,\varepsilon)\ \forall n > n_0,$

$$\sqrt[n]{\theta_{\mathcal{A}}(p,q,n)} < \left(\frac{(\alpha_{p,q}(\mathcal{A})+\varepsilon)^{\rho_p(F)}}{n}\right)^{\frac{1}{\rho_p(F)}} < \left(\frac{(\alpha_p(\mathcal{A})+2\varepsilon)^{\rho_p(F)}}{n}\right)^{\frac{1}{\rho_p(F)}}.$$

By Lemma 2 $\left(b_p = \rho_p(F),\ a_p = \frac{(\alpha_p(\mathcal{A})+2\varepsilon)^{\rho_p(F)}}{\rho_p(F)e}\right)$ we obtain

$$\forall p\ \forall\varepsilon > 0\ \exists q_p(\varepsilon)\ \exists r_0(p,\varepsilon)\ \forall r > r_0:\ M_F(p,q_p,r) < \exp\left\{(a_p+\varepsilon)r^{\rho_p(F)}\right\}.$$

It implies

$$\sigma_p(F) \le a_p = \frac{(\alpha_p(\mathcal{A})+2\varepsilon)^{\rho_p(F)}}{\rho_p(F)e}.$$

By the arbitrariness of ε

$$\sigma_p(F)\rho_p(F)e \le (\alpha_p(\mathcal{A}))^{\rho_p(F)},$$

therefore,

$$\alpha_p(\mathcal{A}) \ge (\sigma_p(F)\rho_p(F)e)^{\frac{1}{\rho_p(F)}}\,,$$

i.e.,

$$\sigma_p(F) = -\frac{\beta_p(\mathcal{A})}{e}(\alpha_p(\mathcal{A}))^{-\frac{1}{\beta_p(\mathcal{A})}},\ \forall p.$$

Hence, identity (17) is proven.

Let us prove identity (19).

If $\beta_p(\mathcal{A}) < \beta(\mathcal{A}),\ \forall p$, from identity (16) it follows $\rho_p(F) < \rho(F),\ \forall p$ and by the definition $\sigma(F) = 0$.

If $\beta_p(\mathcal{A}) = \beta(\mathcal{A}),\ \forall p$, identity (16) yields $\rho_p(F) = \rho(F),\ \forall p$ and by the definition

$$\sigma(F) = \sup_p\{\sigma_p(F)\} = -\frac{\beta(\mathcal{A})}{e}\sup_p\{(\alpha_p(\mathcal{A}))^{-\frac{1}{\beta(\mathcal{A})}}\} = -\frac{\beta(\mathcal{A})}{e}(\alpha(\mathcal{A}))^{-\frac{1}{\beta(\mathcal{A})}}.$$

□

Remark. We observe that relation (16) is true also for $\rho_p(F) = \infty$. If we suppose $\rho_p(F) = \infty$ and $\beta_p(\mathcal{A}) < 0$, by (above proven) $\rho_p(F) < \infty$ that is false. Similarly, identity (17) holds also for $\sigma_p(F) = \infty$.

Remark. Formulae (16) and (17) show that p-orders and p-type of an entire operator-valued function are completely determined by the characteristics of the sequence of its coefficients.

Examples.

1. Let $\mathbf{H}_1 = \mathbf{H} = \mathbf{H}(\mathbb{C})$ be the space of all entire functions with the topology of uniform convergence on the compacts

$$\|x(z)\|_p = \max_{|z|\le p} |x(z)|, \ p > 0.$$

Let us find the characteristics of the function

$$F(t) = e^{t\frac{d}{dz}} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\frac{d^n}{dz^n} : \mathbb{C} \to \mathrm{Lec}(\mathbf{H}(\mathbb{C})).$$

The sequence $\mathcal{A} = \left\{\frac{1}{n!}\frac{d^n}{dz^n}\right\}$ has the following characteristics [1],

$$\beta_p(\mathcal{A}) = \alpha_p(\mathcal{A}) = 0, \ \forall p.$$

Therefore, $\rho_p(F) = \infty, \ \forall p$.

2. Let $\mathbf{H}_1 = [\rho,\sigma], \ \mathbf{H} = [\rho,\theta], \theta \ge \sigma$. The topologies on these spaces are determined by the multinorms

$$\|x(z)\|_\varepsilon = \sup_{p>0}\left\{\max_{|z|\le p} |x(z)| e^{-(\sigma+\varepsilon)p^\rho}\right\}, \ \varepsilon > 0, \ x \in [\rho,\sigma].$$

$$\|y(z)\|_\varepsilon = \sup_{p>0}\left\{\max_{|z|\le p} |y(z)| e^{-(\theta+\varepsilon)p^\rho}\right\}, \ \varepsilon > 0, \ y \in [\rho,\theta].$$

Let us find the characteristics of the function

$$F(t) = e^{t\frac{d}{dz}} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\frac{d^n}{dz^n} : \mathbb{C} \to \mathrm{Lec}([\rho,\sigma],[\rho,\theta]).$$

The sequence $\mathcal{A} = \left\{\frac{1}{n!}\frac{d^n}{dz^n}\right\}$ has the following characteristics [1]:

$$\beta_\varepsilon(\mathcal{A}) = -\frac{1}{\rho}, \ \alpha_\varepsilon(\mathcal{A}) = (\rho e \sigma \Omega_\varepsilon)^{\frac{1}{\rho}}, \ \forall \varepsilon,$$

where

$$\Omega_\varepsilon = \begin{cases} \left(1 - \left(\frac{\sigma}{\theta+\varepsilon}\right)^{\frac{1}{\rho-1}}\right)^{1-\rho} & , \ \rho > 1 \\ 1 & , \ \rho \le 1 \end{cases}$$

Therefore,

$$\rho_\varepsilon(F) = \rho, \ \sigma_\varepsilon(F) = \sigma\Omega_\varepsilon, \ \forall \varepsilon.$$

3. Let $\mathbf{H}_1 = \mathbf{H} = \mathbf{H}(\mathbb{C})$ be the space of all entire functions with the topology of uniform convergence on the compacts

$$\|x(z)\|_p = \max_{|z|\le p} |x(z)|, \ p > 0.$$

Let us find the characteristics of the function

$$F(t)(x) = x(z) + t\int_0^z e^{(z-\xi)t} x(\xi)d\xi =$$

$$= x(z) + t\sum_{n=0}^{\infty}\int_0^z \frac{(z-\xi)^n t^n}{n!} x(\xi)d\xi,$$

$$F(t) : \mathbb{C} \to \mathrm{Lec}(\mathbf{H}(\mathbb{C})).$$

Here

$$A_n(x) = \int_0^z \frac{(z-\xi)^{n-1}}{(n-1)!} x(\xi)d\xi, \ n = 1, 2, \ldots, \ A_0 = E.$$

The sequence $\mathcal{A} = \{A_n\}$ has the following characteristics [1],

$$\beta_p(\mathcal{A}) = -1,\ \alpha_p(\mathcal{A}) = p,\ \forall p.$$

Therefore, $\rho_p(F) = 1,\ \sigma_p(F) = \frac{p}{e},\ \forall p.$

3. Properties of growth characteristics for operator-valued functions

Let us note certain properties of the growth characteristics for operator-valued functions implied by Theorem 4.

1^0. *Entire function F and its kth derivative $F^{(k)}$ has the same p-orders and p-types of growth.*

The validity follows from the fact the sequences $\{A_n\}$ and $\left\{\frac{(n+k)!}{n!}A_{n+k}\right\}$ has the same characteristics for each fixed k.

2^0. *If a function F_1 has the p-orders $\rho_p(F_1)$ and the p-types $\sigma_p(F_1)$, and a function F_2 has the p-orders $\rho_p(F_2) > \rho_p(F_1),\ \forall p$ and the p-types $\sigma_p(F_2)$, the function $F = F_1 + F_2$ has the p-orders $\rho_p(F) = \rho_p(F_2),\ \forall p$ and the p-types $\sigma_p(F) = \sigma_p(F_2),\ \forall p$.*

The validity is implied by the fact that the characteristics of the sum of operators are equal to the characteristics of the term of the greater order.

3^0. *If a function F_1 has the p-orders $\rho_p(F_1)$ and the p-types $\sigma_p(F_1)$, and a function F_2 has the p-orders $\rho_p(F_2) = \rho_p(F_1),\ \forall p$ and the p-types $\sigma_p(F_2) > \sigma_p(F_1),\ \forall p$, then the function $F = F_1 + F_2$ has the p-orders $\rho_p(F) = \rho_p(F_2),\ \forall p$ and the p-types $\sigma_p(F) = \sigma_p(F_2),\ \forall p$.*

The validity follows from the fact that the characteristics of the sum of operators of same orders are equal to the characteristics of the term of the greater type.

4^0. (The case $\mathbf{H}_1 = \mathbf{H}$.) *Suppose a function F_1 has the order $\rho(F_1)$ and the type $\sigma(F_1)$, and a function F_2 has the order $\rho(F_2) > \rho(F_1)$ and the type $\sigma(F_2)$. Then the function $F = F_1F_2$ has the order $\rho(F) \le \rho(F_2)$ and the type $\sigma(F)$. If $\rho(F) = \rho(F_2)$, then $\sigma(F) \le \sigma(F_2)$. A similar statement holds for the function $\tilde{F} = F_2F_1$.*

The proof is based on the following lemma.

Lemma 3. *Suppose a sequence of operators $\mathcal{A} = \{A_n\}$ has the order $\beta(\mathcal{A})$ and the type $\alpha(\mathcal{A})$, and a sequence of the operators $\mathcal{B} = \{B_n\}$ has the order $\beta(\mathcal{B}) > \beta(\mathcal{A})$ and the type $\alpha(\mathcal{B})$. Then the sequence of the operators $\mathcal{C} = \{C_n\}$, where $C_n = \sum\limits_{k=0}^{n} A_kB_{n-k}$ has the order $\beta(\mathcal{C}) \le \beta(\mathcal{B})$ and the type $\alpha(\mathcal{C})$. If $\beta(\mathcal{C}) = \beta(\mathcal{B})$, then $\alpha(\mathcal{C}) \le \alpha(\mathcal{B})$.*

Proof. Denote $a = \alpha(\mathcal{A})e^{\beta(\mathcal{A})},\ b = \alpha(\mathcal{B})e^{\beta(\mathcal{B})}$.

The definition of the order and type of a sequence of operators implies [1]

$$\forall \varepsilon, \varepsilon_1 > 0,\ \forall p,\ \exists M_p,\ \exists q,\ \forall n,\ \forall x \in \mathbf{H}:$$

$$\|C_n(x)\|_p \le M_p\Big((b+\varepsilon)^n n!^{\beta(\mathcal{B})} + (a+\varepsilon_1)(b+\varepsilon)^{n-1}1!^{\beta(\mathcal{A})}(n-1)!^{\beta(\mathcal{B})} + \dots +$$

$$+ (a+\varepsilon_1)^{n-1}(b+\varepsilon)(n-1)!^{\beta(\mathcal{A})}1!^{\beta(\mathcal{B})} + (a+\varepsilon_1)^n n!^{\beta(\mathcal{A})}\Big)\|x\|_q \le$$

$$\le M_p(b+\varepsilon)^n n!^{\beta(\mathcal{B})}\left[1 + \binom{n}{1}^{-\beta(\mathcal{B})}\left(\frac{a+\varepsilon_1}{b+\varepsilon}\right)1!^{\nu} + \binom{n}{2}^{-\beta(\mathcal{B})}\left(\frac{a+\varepsilon_1}{b+\varepsilon}\right)^2 2!^{\nu} + \dots + \right.$$

$$\left. + \binom{n}{n}^{-\beta(\mathcal{B})}\left(\frac{a+\varepsilon_1}{b+\varepsilon}\right)^n n!^{\nu}\right]\|x\|_q, \quad (21)$$

where $\nu = \beta(\mathcal{A}) - \beta(\mathcal{B})$.

If $\beta(\mathcal{B}) > \beta(\mathcal{A})$ $(\nu < 0)$, for large n the expression in brackets in (21) does not exceed $(1+\varepsilon_2)^n$, $\forall\varepsilon_2 > 0$ and thus $\beta(\mathcal{C}) \le \beta(\mathcal{B})$, and if $\beta(\mathcal{C}) = \beta(\mathcal{B})$, then $\alpha(\mathcal{C}) \le \alpha(\mathcal{B})$. □

5^0. (The case $\mathbf{H}_1 = \mathbf{H}$.) *Suppose a function* F_1 *has the order* $\rho(F_1)$ *and the type* $\sigma(F_1)$, *and a function* F_2 *has the order* $\rho(F_2) = \rho(F_1)$ *and the type* $\sigma(F_2) \ge \sigma(F_1)$. *Then the function* $F = F_1F_2$ *has the order* $\rho(F) \le \rho(F_2)$ *and the type* $\sigma(F)$. *If* $\rho(F) = \rho(F_2)$, *then* $\sigma(F) \le 2\sigma(F_2)$. *A similar statement holds true for the function* $\tilde{F} = F_2F_1$.

The proof is based on the following lemma.

Lemma 4. *Let a sequence of operators* $\mathcal{A} = \{A_n\}$ *has the order* $\beta(\mathcal{A})$ *and the type* $\alpha(\mathcal{A})$, *and a sequence of operators* $\mathcal{B} = \{B_n\}$ *does the order* $\beta(\mathcal{B}) = \beta(\mathcal{A})$ *and the type* $\alpha(\mathcal{B}) \ge \alpha(\mathcal{A})$. *The the sequence of the operators* $\mathcal{C} = \{C_n\}$, *where* $C_n = \sum_{k=0}^{n} A_kB_{n-k}$, *has the order* $\beta(\mathcal{C}) \le \beta(\mathcal{B})$ *and the type* $\alpha(\mathcal{C})$. *If* $\beta(\mathcal{C}) = \beta(\mathcal{B})$, *then* $\alpha(\mathcal{C}) \le 2^{-\beta(\mathcal{B})}\alpha(\mathcal{B})$.

Proof. Under the hypothesis of the lemma the expression in the brackets in (21) does not exceed $2^{-\beta(\mathcal{B})n}n$ and thus $\beta(\mathcal{C}) \le \beta(\mathcal{B})$, and if $\beta(\mathcal{C}) = \beta(\mathcal{B})$, then $\alpha(\mathcal{C}) \le 2^{-\beta(\mathcal{B})}\alpha(\mathcal{B})$. □

Remark. As it is known, in the scalar case the theorem on categories [3, Th. 12] is valid. In the case of operator-valued function this question is still open.

6^0. (Invariance). Suppose $\mathbf{H}_1$, $\widetilde{\mathbf{H}}_1$, $\mathbf{H}$ and $\widetilde{\mathbf{H}}$ are four locally convex spaces with the topologies induced respectively by the multinorms $\|\cdot\|'_q$, $q \in \mathcal{Q}$, $\|\cdot\|'_{\widetilde{q}}$, $\widetilde{q} \in \widetilde{\mathcal{Q}}$, $\|\cdot\|_p$, $p \in \mathcal{P}$, $\|\cdot\|_{\widetilde{p}}$, $\widetilde{p} \in \widetilde{\mathcal{P}}$ and let $T_1 : \mathbf{H}_1 \to \widetilde{\mathbf{H}}_1$, $T : \mathbf{H} \to \widetilde{\mathbf{H}}$ are two topological isomorphisms. Then

1) for each operator-valued function

$$F(t) = \sum_{n=0}^{\infty} A_nt^n : \mathbb{C} \to \mathrm{Lec}(\mathbf{H}_1, \mathbf{H})$$

its order and type coincide with the order and type of the function

$$\tilde{F}(t) = \sum_{n=0}^{\infty} TA_nT_1^{-1}t^n : \mathbb{C} \to \mathrm{Lec}(\widetilde{\mathbf{H}}_1, \widetilde{\mathbf{H}});$$

2) if all the p-orders of the function F are strictly less than its order, then all the $\widetilde{p}$-orders of the function $\tilde{F}$ are strictly less than its order;

3) if at least one p-order of the function F equals to its order, then at least one $\widetilde{p}$-order of the function $\tilde{F}$ equals to its order;

4) if the function F has the p-orders $\rho_p(F)$, the order $\rho(F)$, the p-types $\sigma_p(F)$ and the type $\sigma(F)$, at that the set

$$\mathcal{P}_F = \{p \in \mathcal{P} : \ \rho_p(F) = \rho(F)\}$$

is non-empty and $\forall p \in \mathcal{P}_F : \ \sigma_p(F) < \sigma(F)$, then the function $\tilde{F}$ has the $\widetilde{p}$-orders $\rho_{\widetilde{p}}(\tilde{F})$, the order $\rho(\tilde{F})$, the $\widetilde{p}$-types $\sigma_{\widetilde{p}}(\tilde{F})$ and the type $\sigma(\tilde{F})$, at that the set

$$\widetilde{\mathcal{P}}_{\tilde{F}} = \{\widetilde{p} \in \widetilde{\mathcal{P}} : \ \rho_{\widetilde{p}}(\tilde{F}) = \rho(\tilde{F})\}$$

is non-empty and $\forall\widetilde{p} \in \widetilde{\mathcal{P}}_{\tilde{F}} : \ \sigma_{\widetilde{p}}(\tilde{F}) < \sigma(\tilde{F})$;

5) if under hypothesis of Item 4) $\exists p \in \mathcal{P}_F : \ \sigma_p(F) = \sigma(F)$, then $\exists\widetilde{p} \in \widetilde{\mathcal{P}}_{\tilde{F}} : \ \sigma_{\widetilde{p}}(\tilde{F}) = \sigma(\tilde{F})$.

The validity of property 6^0 follows from analogous properties for the characteristics of a sequence of operators [1, 6].

The invariance property implies that under any replacements of the multinorms in $\mathbf{H}_1$ and $\mathbf{H}$ to equivalent ones (T_1 and T are identity operators)

1) the order and type of the operator-valued function F remain the same;

2) if all the p-orders of the function F were strictly less than its order before the replacement of the multinorms, after the replacement of the multinorms all its $\widetilde{p}$-orders are also strictly less than the order;

3) if at least one p-order of the function F equals its order before the replacement of the multinorms, after the replacement at least one its $\widetilde{p}$-order (not necessarily the same) is also equal to its order;

4) if the function F has the p-order $\rho_p(F)$ before the replacement of the multinorms, the order $\rho(F)$, the p-types $\sigma_p(F)$, and the type $\sigma(F)$, at that the set

$$\mathcal{P}_F = \{p \in \mathcal{P} : \ \rho_p(F) = \rho(F)\}$$

is non-empty and $\forall p \in \mathcal{P}_F : \ \sigma_p(F) < \sigma(F)$, then after the replacement of the multinorms this function has the $\widetilde{p}$-orders $\rho_{\widetilde{p}}(F)$, the order $\rho(F)$, the $\widetilde{p}$-types $\sigma_{\widetilde{p}}(F)$, and the type $\sigma(F)$, at that the set

$$\widetilde{\mathcal{P}}_F = \{\widetilde{p} \in \widetilde{\mathcal{P}} : \ \rho_{\widetilde{p}}(F) = \rho(F)\}$$

is non-empty and $\forall \widetilde{p} \in \widetilde{\mathcal{P}}_F : \ \sigma_{\widetilde{p}}(F) < \sigma(F)$;

5) if under the hypothesis of Item 4) $\exists p \in \mathcal{P}_F : \ \sigma_p(F) = \sigma(F)$, then $\exists \widetilde{p} \in \widetilde{\mathcal{P}}_F : \ \sigma_{\widetilde{p}}(F) = \sigma(F)$.

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