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FINITE-DIFFERENCE SCHEMES FOR DIFFUSION EQUATION OF FRACTIONAL ORDER WITH ROBIN BOUNDARY CONDITIONS IN MULTIDIMENSIONAL DOMAIN

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Abstract. We consider finite-difference schemes for diffusion equation of fractional order in a multidimensional field with Robin boundary conditions. We prove the stability and convergence of difference schemes for considered problem.

Keywords: finite-difference schemes, diffusion equation of fractional order, apriori estimate, maximum principle, Robin boundary conditions, stability and convergence of finite-difference scheme

Introduction

Boundary value problems for fractional order differential equations appear in descriptions of physical processes of stochastic transport [1], in studying filtration of fluids in strongly porous (fractal) media [2]. Fractional derivatives equations describe the evolution of some physical systems with loses, and the order of the derivative indicate the portion of the system conditions being preserved during whole evolution time. Such systems can be classified as systems with "residual" memory having an intermediate position between systems with full memory on one hand, and Markov systems on the other [3].

The work is devoted to considering finite-difference schemes for a fractional order diffusion equation with third type boundary conditions in a multidimensional domain. In work [4] finite-difference methods of solving boundary value problems for fractional order differential equations were considered. Locally-one-dimensional schemes for fractional order diffusion equations with Dirichlet boundary condition were considered in work [5], locally-one-dimensional schemes for Robin boundary value problem for a fractional order diffusion equation in work [6].

1. Formulation of problem

In a cylinder $Q_T = G \times [0 < t \leqslant T]$ whose base is a p-dimensional cuboid $G = \{x = (x_1, x_2, \dots,\ x_p) : 0 < x_\beta < \ell_\beta,\ \beta = 1,\ 2,\ \dots,\ p\}$ with boundary Γ, $\overline{G} = G \cup \Gamma$, we consider the Robin initial boundary value problem,

$$\partial_{0t}^{\alpha} u = Lu + f(x,t),\ (x,t) \in Q_T, \tag{1}$$

$$\begin{cases} k_\beta(x,t)\,\dfrac{\partial u}{\partial x_\beta} = \varkappa_{-\beta}(x,t)u - \mu_{-\beta}(x,t),\ x_\beta = 0,\ 0 \leqslant t \leqslant T, \\ -k_\beta(x,t)\,\dfrac{\partial u}{\partial x_\beta} = \varkappa_{+\beta}(x,t)u - \mu_{+\beta}(x,t),\ x_\beta = \ell_\beta,\ 0 \leqslant t \leqslant T, \end{cases} \tag{2}$$

$$u(x,0) = u_0(x),\ x \in \overline{G}, \tag{3}$$

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where

$$Lu = \sum_{\beta=1}^{p} L_\beta u,\ L_\beta u = \frac{\partial}{\partial x_\beta}\left(k_\beta(x,t)\frac{\partial u}{\partial x_\beta}\right),$$
$$0 < c_0 \leqslant k_\beta \leqslant c_1,\ \varkappa_{\pm\beta} \geqslant \varkappa^* > 0,$$

$\partial_{0t}^{\alpha} u = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t \frac{\dot{u}(x,\eta)}{(t-\eta)^\alpha} d\eta$ is Caputo fractional derivative of order $\alpha,\ 0<\alpha<1$ [7], $\dot{u} = \frac{\partial u}{\partial t},\ c_0,\ c_1$ are positive constants, $\beta = 1,\ 2,\ \ldots,\ p,\ \overline{Q}_T = \overline{G} \times [0 \leqslant t \leqslant T]$.

In what follows we assume that the coefficients of the equation (1) — (3) possess sufficient number of derivatives to ensure a required smoothness of the solution $u(x,t)$ in the cylinder Q_T.

2. Finite-difference scheme

We choose the spatial grid being uniform in each direction Ox_β with step $h_\beta = \ell_\beta/N_\beta,\ \beta = 1,\ 2,\ \ldots,\ p$,

$$\overline{\omega}_h = \{x_i = (i_1 h_1, \ldots,\ i_p h_p) \in G,\ i_\beta = 0,1,\ldots,N_\beta,\ h_\beta = \ell_\beta/N_\beta,\ \beta = 1,\ 2,\ \ldots,\ p\}.$$

In the segment $[0,T]$ we introduce a uniform grid with the step $\tau = T/j_0$,

$$\overline{\omega}_\tau = \{t_j = j\tau,\ j = 0,1,\ldots,\ j_0\}.$$

In work [4] a discrete analogue of Caputo fractional derivative of order α, $0<\alpha<1$ was suggested,

$$\frac{1}{\Gamma(1-\alpha)} \int\limits_0^{t_j} \frac{\dot{u}(x,\eta)}{(t_j-\eta)^\alpha} d\eta = \Delta_{0t_j}^{\alpha} u + O\left(\tau/p\right), \tag{4}$$

where

$$\Delta_{0t_j}^{\alpha} u = \frac{1}{\Gamma(2-\alpha)} \sum_{s=0}^{j} \left(t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}\right) u_{\bar{t}}^{s/p},\ u_{\bar{t}}^{s} = \frac{u^{s+1}-u^s}{\tau}.$$

Let us pass to constructing a finite-difference scheme for the differential problem (1)–(3). With equation (1) we associate the difference equation

$$\Delta_{0t_{j+1}}^{\alpha} u = \Lambda y + \varphi^{j+1}, \tag{5}$$

$$\Lambda y = \sum_{\beta=1}^{p} \Lambda_\beta y,\ \Lambda_\beta y = \left(a_\beta y_{\overline{x}_\beta}\right)_{x_\beta},\ \beta = 1,2,\ldots,p.$$

To equation (5) we add boundary and initial conditions. We write a difference analogue for boundary conditions (2),

$$\begin{cases} a^{(1_\beta)} y_{x_\beta,0} = \varkappa_{-\beta} y_0 - \mu_{-\beta},\ x_\beta = 0, \\ -a^{(N_\beta)} y_{\overline{x}_\beta,N_\beta} = \varkappa_{+\beta} y_{N_\beta} - \mu_{+\beta},\ x_\beta = \ell_\beta. \end{cases} \tag{6}$$

Conditions (6) has the order of approximation $O(h_\beta)$. Employing a known trick of increasing approximation order up to $O(h_\beta^2)$ on the solutions of equation (1) for some β, we obtain a difference analogue of boundary value conditions

$$\Delta_{0t_{j+1}}^{\alpha} y\Big|_{x_\beta=0} = \frac{\left(a^{(1_\beta)} y_{x_\beta,0} - \varkappa_{-\beta} y_0\right)}{0.5h_\beta} + \frac{\overline{\mu}_{-\beta}}{0.5h_\beta},$$

$$\Delta_{0t_{j+1}}^{\alpha} y\Big|_{x_\beta=N_\beta} = -\frac{\left(a^{(N_\beta)} y_{\overline{x}_\beta,N_\beta} + \varkappa_{+\beta} y_{N_\beta}\right)}{0.5h_\beta} + \frac{\overline{\mu}_{+\beta}}{0.5h_\beta},$$

where

$$\overline{\mu}_{-\beta} = \mu_{-\beta} + 0.5h_\beta f_{\beta,0},\ \overline{\mu}_{+\beta} = \mu_{+\beta} + 0.5h_\beta f_{\beta,N_\beta}.$$

Thus, a difference analogue to problem (1) — (3) reads as

$$\begin{gathered}\Delta^\alpha_{0t_{j+1}} y = \overline{\Lambda} y^{j+1} + \Phi,\\ y(x,0) = u_0(x),\ x \in \overline{G},\end{gathered} \tag{7}$$

where

$$\overline{\Lambda} y = \begin{cases} \Lambda y = \sum\limits_{\beta=1}^{p} (a_\beta y_{\overline{x}_\beta})_{x_\beta},\ x_\beta \in \omega_h,\\ \Lambda^- y = \dfrac{a^{(1_\beta)} y_{x_\beta,0} - \varkappa_{-\beta} y_0}{0.5h_\beta},\ x_\beta = 0,\\ \Lambda^+_\beta y = -\dfrac{a^{(N_\beta)} y_{\overline{x}_\beta,N_\beta} + \varkappa_{+\beta} y_{N_\beta}}{0.5h_\beta},\ x_\beta = \ell_\beta, \end{cases}$$

$$\Phi = \begin{cases} \varphi,\ x_\beta \in \omega_h,\\ \overline{\mu}_{-\beta},\ x_\beta = 0,\\ \overline{\mu}_{+\beta},\ x_\beta = \ell_\beta, \end{cases}$$

$$\overline{\mu}_{-\beta} = \mu_{-\beta} + 0.5h_\beta f_{\beta,0},\ \overline{\mu}_{+\beta} = \mu_{+\beta} + 0.5h_\beta f_{\beta,N_\beta}.$$

3. Apriori estimate

Let us obtain an apriori estimate in the grid norm C for the solution of difference problem (7) expressing the stability of finite-difference scheme w.r.t. initial data, right hand side, and boundary data. We shall perform the study of stability of finite-difference scheme (7) on the basis of maximum principle ([8], p. 226). In order to do it, we rewrite difference problem (7) as follows,

$$\Delta^\alpha_{0t_{j+1}} y = \sum_{\beta=1}^{p} (a_\beta y_{\overline{x}_\beta})_{x_\beta} + \varphi(x,t),\ \beta = 1,2,\ldots,p, \tag{8}$$

$$\Delta^\alpha_{0t_{j+1}} y_0 = \frac{(a^{(1_\beta)} y_{x_\beta,0} - \varkappa_{-\beta} y_0)}{0.5h_\beta} + \frac{\overline{\mu}_{-\beta}}{0.5h_\beta},\ x_\beta = 0, \tag{9}$$

$$\Delta^\alpha_{0t_{j+1}} y_{N_\beta} = -\frac{(a^{(N_\beta)} y_{\overline{x}_\beta,N_\beta} + \varkappa_{+\beta} y_{N_\beta})}{0.5h_\beta} + \frac{\overline{\mu}_{+\beta}}{0.5h_\beta},\ x_\beta = \ell_\beta, \tag{10}$$

$$y(x,0) = u_0(x). \tag{11}$$

In ([8], p. 226) the maximum principle was proven and there were obtained apriori estimates for the solution of grid equations of general form

$$A(P)y(P) = \sum_{Q \in \mathrm{U}'(P)} B(P,Q)y(Q) + F(P),$$

where

$$A(P) > 0,\ B(P,Q) > 0,\ D(P) = A(P) - \sum_{Q \in \mathrm{U}'(P)} B(P,Q) \geqslant 0,$$

and P, Q are the nodes of the grid $\overline{\omega}_h$, $\mathrm{U}'(P)$ is a neighborhood of the node P not containing the node P.

By $P(x,t')$, where $x \in \omega_h,\ t' \in \omega'_\tau$, we denote the node of $(p+1)$-dimensional grid $\Omega = \omega_h \times \omega'_\tau$, by S we denote the boundary Ω consisting of nodes $P(x,0)$ as $x \in \overline{\omega}_h$ and nodes $P(x,t_{j+1})$ as $t_{j+1} \in \omega'_\tau$ and $x \in \gamma_{h_\beta}$ for all $\beta = 1,2,\ldots,p;\ j = 0,1,\ldots,j_0$.

In order to obtain an apriori estimate for the solution to difference problem (8)–(11), we represent its solution as the sum

$$y = \overset{\circ}{y} + \overset{*}{y},$$

where $\overset{\circ}{y}$ is the solution of homogeneous equations (8) with inhomogeneous boundary condition (9)–(10) and homogeneous initial conditions (11), while $\overset{*}{y}$ is the solution to inhomogeneous equations (8) with homogeneous boundary conditions (9)–(10) and inhomogeneous initial conditions (11).

First we estimate $\overset{\circ}{y}$. In order to do it, we write the equation for $\overset{\circ}{y}$ in the canonical form

$$\left[\frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau^\alpha} + \sum_{\beta=1}^{p} \frac{a_{\beta,i_\beta+1} + a_{\beta,i_\beta}}{h_\beta^2}\right] \overset{\circ}{y}{}^{j+1}_{i_\beta} = \sum_{\beta=1}^{p} \left(\frac{a_{\beta,i_\beta+1}}{h_\beta^2} \overset{\circ}{y}{}^{j+1}_{i_\beta+1} + \right.$$
$$\left. + \frac{a_{\beta,i_\beta}}{h_\beta^2} \overset{\circ}{y}{}^{j+1}_{i_\beta-1}\right) + \frac{2-2^{1-\alpha}}{\Gamma(2-\alpha)\tau^\alpha} \overset{\circ}{y}{}^{j}_{i_\beta} +$$
$$+ \frac{1}{\tau}\frac{1}{\Gamma(2-\alpha)}\Big[\left(t_{j+1}^{1-\alpha} - t_j^{1-\alpha}\right) \overset{\circ}{y}{}^{0}_{i_\beta} + \left(- t_{j+1}^{1-\alpha} + 2t_j^{1-\alpha} - t_{j-1}^{1-\alpha}\right) \overset{\circ}{y}{}^{1}_{i_\beta} +$$
$$+ \ldots + \left(-t_3^{1-\alpha} + 2t_2^{1-\alpha} - t_1^{1-\alpha}\right) \overset{\circ}{y}{}^{j-1}_{i_\beta} \Big]. \tag{12}$$

The boundary conditions should be also transform to the canonical form. At the point $P = P(x_0, t_{j+1})$ we have

$$\left[\frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau^\alpha} + \frac{a^{(1_\beta)}}{0.5h_\beta^2} + \frac{\varkappa_{-\beta}}{0.5h_\beta}\right] \overset{\circ}{y}{}^{j+1}_{0} = \frac{a^{(1_\beta)}}{0.5h_\beta^2} \overset{\circ}{y}{}^{j+1}_{0} + \frac{2-2^{1-\alpha}}{\Gamma(2-\alpha)\tau^\alpha} \overset{\circ}{y}{}^{j}_{0} +$$
$$+ \frac{1}{\tau}\frac{1}{\Gamma(2-\alpha)}\left[\left(t_{j+1}^{1-\alpha} - t_j^{1-\alpha}\right) \overset{\circ}{y}{}^{0}_{0} + \left(-t_{j+1}^{1-\alpha} + 2t_j^{1-\alpha} - t_{j-1}^{1-\alpha}\right) \overset{\circ}{y}{}^{1}_{0} + \right.$$
$$\left. + \ \ldots \ + \left(-t_3^{1-\alpha} + 2t_2^{1-\alpha} - t_1^{1-\alpha}\right) \overset{\circ}{y}{}^{j-1}_{0} \right] + \frac{\overline{\mu}_{-\beta}}{0.5h_\beta}. \tag{13}$$

At the point $P = P(x_{N_\beta}, t_{j+1})$ we have

$$\left[\frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau^\alpha} + \frac{a^{(N_\beta)}}{0.5h_\beta^2} + \frac{\varkappa_{+\beta}}{0.5h_\beta}\right] \overset{\circ}{y}{}^{j+1}_{N_\beta} = \frac{a^{(1_\beta)}}{0.5h_\beta^2} \overset{\circ}{y}{}^{j+1}_{N_\beta} + \frac{2-2^{1-\alpha}}{\Gamma(2-\alpha)\tau^\alpha} \overset{\circ}{y}{}^{j}_{N_\beta} +$$
$$+ \frac{1}{\tau}\frac{1}{\Gamma(2-\alpha)}\left[\left(t_{j+1}^{1-\alpha} - t_j^{1-\alpha}\right) \overset{\circ}{y}{}^{0}_{N_\beta} + \left(-t_{j+1}^{1-\alpha} + 2t_j^{1-\alpha} - t_{j-1}^{1-\alpha}\right) \overset{\circ}{y}{}^{1}_{N_\beta} + \right.$$
$$\left. + \ \ldots \ + \left(-t_3^{1-\alpha} + 2t_2^{1-\alpha} - t_1^{1-\alpha}\right) \overset{\circ}{y}{}^{j-1}_{N_\beta} \right] + \frac{\overline{\mu}_{+\beta}}{0.5h_\beta}. \tag{14}$$

Taking into consideration the positivity of the expressions in the parentheses (according to Lemma in [5]), let us check the assumption of Theorem 3 ([9], Ch. V., App. §2, Eq. (16)).

At the point $P = P(x_{i_\beta}, t_{j+1})$ we have

$$A(P) > 0, \ B(P, Q) > 0, \ D(P) = 0,$$

at the point $P = P(x_0, t_{j+1})$ we have

$$A(P) > 0, \ B(P, Q) > 0, \ D(P) = \frac{\varkappa_{-\beta}}{0.5h_\beta} \geqslant \frac{\varkappa^*}{0.5h_\beta} > 0,$$

at the point $P = P(x_{N_\beta}, t_{j+1})$ we have

$$A(P) > 0,\ B(P,Q) > 0,\ D(P) = \frac{\varkappa_{+\beta}}{0.5h_\beta} \geqslant \frac{\varkappa^*}{0.5h_\beta} > 0.$$

Thus, all the conditions 3 ([9], Ch. V. App. §2, Eq. (16)) hold and

$$D(x_\beta, t_{j+1}) = 0,\ D(0, t_{j+1}) = \frac{\varkappa^*}{0.5h_\beta} > 0,\ D(\ell_\beta, t_{j+1}) = \frac{\varkappa^*}{0.5h_\beta} > 0.$$

On the basis of aforementioned Theorem 3 we obtain the estimate for $\overset{\circ}{y}$,

$$\|\overset{\circ}{y}{}^{j+1}\| \leqslant \frac{1}{\varkappa^*} \max_{x\in\gamma_h,\ t\in\overline{\omega}_\tau} \left(|\overline{\mu}_{-\beta}(x,t')| + |\overline{\mu}_{+\beta}(x,t')|\right),\ \varkappa_{\pm\beta} \geqslant \varkappa^* > 0. \tag{15}$$

We pass to estimating the function $\overset{*}{y}$. We rewrite the equation for $\overset{*}{y}$ as

$$\left[\frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau^\alpha} + \sum_{\beta=1}^{p} \frac{a_{\beta,i_\beta+1} + a_{\beta,i_\beta}}{h_\beta^2}\right] \overset{*}{y}{}^{j+1}_{i_\beta} =$$
$$= \sum_{\beta=1}^{p} \frac{1}{h_\beta^2}\left(a_{\beta,i_\beta+1}\overset{*}{y}{}^{j+1}_{i_\beta+1} + a_{\beta,i_\beta}\overset{*}{y}{}^{j+1}_{i_\beta-1}\right) + \Phi(P_{j+1}), \tag{16}$$

where

$$\Phi(P_{j+1}) = \frac{2-2^{1-\alpha}}{\Gamma(2-\alpha)\tau^\alpha}\overset{*}{y}{}^{j}_{i_\beta} + \frac{1}{\Gamma(2-\alpha)}\frac{1}{\tau}\left(t_2^{1-\alpha} - t_1^{1-\alpha}\right)\overset{*}{y}{}^{j-1}_{i_\beta} -$$
$$-\frac{1}{\tau}\frac{1}{\Gamma(2-\alpha)}\sum_{s=1}^{j-1}\left(t_{j-s+1}^{1-\alpha} - t_{j-s}^{1-\alpha}\right)\left(\overset{*}{y}{}^{s}_{i_\beta} - \overset{*}{y}{}^{s-1}_{i_\beta}\right) + \varphi^{j+1}.$$

Let us check the assumption of Theorem 4 (cf. [9], p. 347)

$$D'(P_{(j+1)}) = A(P_{(j+1)}) - \sum_{Q\in \mathrm{U}'_{j+1}(P_{(j+1)})} B(P_{(j+1)}, Q) = \frac{1}{\Gamma(2-\alpha)\tau^\alpha} > 0,$$

$$A(P_{(j+1)}) > 0,\ B(P_{(j+1)}, Q) > 0, P_{(j+1)} = P(x, t_{j+1})$$

for all $Q \in \mathrm{U}''_j, Q \in \mathrm{U}'_{j+1}$ due to Lemma (cf. [9], p. 347)

$$\sum_{Q\in\mathrm{U}''_j} B(P_{j+1}, Q) = \frac{1}{\Gamma(2-\alpha)\tau^\alpha} > 0,$$

$$\frac{1}{D'(P_{(j+1)})}\sum_{Q\in\mathrm{U}''_j} B(P_{(j+1)}, Q) = 1, \tag{17}$$

where

$$\mathrm{U}'_{(P(x,t_{j+1}))} = \mathrm{U}'_{j+1} + \mathrm{U}''_j,$$
$\mathrm{U}'_{j+1}$is set of nodes $Q = Q(\xi, t_{j+1}) \in \mathrm{U}'_{(P(x,t_{j+1}))}$,
$\mathrm{U}''_j$is set of nodes $Q = Q(\xi, t_j) \in \mathrm{U}'_{(P(x,t_j))}$.

By aforementioned Theorem 4 (cf. [9], p. 347) and by (17) we obtain the estimate

$$\|\overset{*}{y}{}^{j+1}\|_C \leqslant \|\overset{*}{y}{}^{0}\|_C + \Gamma(2-\alpha)\sum_{j'=0}^{j}\tau^\alpha \max_{0\leqslant s\leqslant j'}\|\varphi^s\|. \tag{18}$$

Estimates (15) and (18) imply final inequality

$$\|y^{j+1}\|_C \leqslant \|y^0\|_C + \frac{1}{\varkappa^*}\max_{0<t'\leqslant j\tau}\left(|\overline{\mu}_{-\beta}(x,t')| + |\overline{\mu}_{+\beta}(x,t')|\right) +$$

$$+\Gamma(2-\alpha)\sum_{j'=0}^{j}\tau^{\alpha}\max_{0\leqslant s\leqslant j'}\|\varphi^{s}\|. \tag{19}$$

Thus, the following theorem holds.

Theorem 1. *Finite-difference scheme (7) is stable w.r.t. initial data and right hand side so that for the solution of problem (7) the estimate (19) is valid.*

4. Convergence of finite-difference scheme

The error $z = y - u$ satisfies the estimate

$$\|z^{j+1}\|_C \leqslant \Gamma(2-\alpha)\sum_{j'=0}^{j}\tau^{\alpha}\max_{0\leqslant s\leqslant j'}\|\psi^{s}\|. \tag{20}$$

Since $\psi = O(|h|^2+\tau),\ |h|^2 = h_1^2 + h_2^2 + \ldots + h_p^2$, it follows from (20) that

$$\|z^{j+1}\|_C = O\left(\frac{|h^2|}{\tau^{1-\alpha}} + \tau^{\alpha}\right).$$

For $\alpha \to 1$, as in [4], we obtain a known result

$$\|z^{j+1}\|_C = O\left(|h|^2 + \tau\right).$$

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