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EXISTENCE OF SOLUTION FOR PARABOLIC EQUATION WITH NON-POWER NONLINEARITIES

E.R. ANDRIYANOVA, F.KH. MUKMINOV

Abstract. We consider the first mixed problem for a class of parabolic equation with double non-power nonlinearities in a cylindrical domain $D = (t > 0) \times \Omega$. By Galerkin's approximations we prove the existence of strong solutions in Sobolev-Orlicz space.

Keywords:parabolic equation, N-functions, existence of solution, Sobolev-Orlicz spaces.

Mathematics Subject Classification: 35D05, 35B50, 35B45, 35K55

1. Introduction

Let Ω be a bounded domain in space $\mathbb{R}_n = \{x = (x_1, x_2, \dots, x_n)\}$, $n \geqslant 2$. In the cylindrical domain $D = \{t > 0\} \times \Omega$ we consider the equation

$$(\beta(x,u))'_t = \sum_{i=1}^{n} (a_{p_i}(x,u,\nabla u))_{x_i} - b(x,u,\nabla u), \text{ where } a(x,u,\nabla u) = a(x,u,p)\Big|_{p=\nabla u}, \tag{1}$$

with a monotonous operator in the right hand side. The boundary conditions are homogeneous:

$$u(t,x)\Big|_S = 0, \quad S = \{t > 0\} \times \partial\Omega; \tag{2}$$

$$u(0,x) = u_0(x). \tag{3}$$

Function $a(x,u,p)$, $p = (p_1, p_2, ..., p_n)$, satisfies Carathéodory condition as $p \in \mathbb{R}_n$ and $x \in \Omega$. Function $\beta(x,u), \beta(x,0) = 0$, is absolutely continuous and grows w.r.t. u, as well as it is measurable w.r.t. $x \in \Omega$ as $u \in \mathbb{R}$.

In the present work we prove the existence of a strong solution to problem (1)–(3) with nonlinearities defined by N-functions. Existence of solution to parabolic equations with double nonlinearities were considered in works [1]–[11] and others. An essential progress was made in work [3], where there was also considered the uniqueness of the solution. Usually the existence of solution was proven by discretization in time and nonlinearities were governed by power estimates. Work [4] was devoted to the existence of weak solutions to quasilinear second order parabolic equations with double nonlinearities in a bounded domain by the Galerkin's approximations methods. In work [5] a solution in unbounded domain was obtained as a limit of solutions in a sequence of bounded domains. The existence of weak solution to parabolic equation with two variable nonlinearities in appropriate Sobolev-Orlicz spaces independent variables for a bounded domain Ω was proven in [6]. The existence of W- and H-solutions to second order parabolic equations with a variable nonlinearity order was proven in work [8].

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Existence of a strong solution to problem (1)–(3) for an isotropic parabolic equation with a double power nonlinearity was established in [9]. There were also obtained exact two-sided estimates for the power decay of norm of solution in time. In [18] there was also considered the case of anisotropic parabolic equations.

In work [10] there was proven the existence of a strong solution to a model parabolic equations with non-power nonlinearities under the boundedness of the derivative $\beta'(u)$ in the vicinity of zero.

In works concerning the uniqueness of solution to problem (1)–(3) there were considered only the equations with power nonlinearities. In work [11] the uniqueness to problem (1)–(3) was proven in the case $\beta = |u|^{\alpha-2}u$, $\alpha \in (1, 2)$ under the assumption $(\beta(u))'_t \in L_1(D^T)$, $u_0 \geqslant 0$. Similar results for equation (1) written in another form were established in [12, 13]. The uniqueness of a renormalized solution to elliptic-parabolic problem with power nonlinearities was established in [14].

2. Notations and conditions for the functions involved in equation

Here we define some functional spaces we employ in the work and we recall some known facts in the theory of Sobolev-Orlicz spaces [21].

We introduce the notations

$$\langle f(t)\rangle = \int\limits_{\Omega} f(t,x)dx, \quad [f] = \int\limits_{D^T} f(t,x)dxdt, \quad f(\phi) = (f,\phi)_{\Omega}.$$

In this identity we write the value of a distribution f on an element ϕ.

For convex functions $B(s)$, $s \geqslant 0$, the function

$$\overline{B}(z) = \sup_{s\geqslant 0}(s|z| - B(s))$$

is called complementary. The following Young inequality

$$|zs| \leqslant B(z) + \overline{B}(s)$$

is obvious.

N-function $B(s)$ is said to satisfy Δ_2-condition if one of three equivalent conditions holds:

1) there exists a number $k > 0$ such that $B(2s) \leqslant kB(s)$, $\forall s \geqslant 0$;

2) there exist numbers $l > 1$, $m > 0$ such that

$$B(ls) \leqslant kl^m B(s), \ \forall s \geqslant 0;$$

3) there exists a number $\alpha_B > 0$ such that $sB'(s) \leqslant \alpha_B B(s)$, $\forall s \geqslant 0$.

Hereinafter $B'(s)$ denotes the right derivative of a convex function. We also note that the known identity $uB'(u) = B(u) + \overline{B}(B'(u))$ [21, Ch. 1, Formula (2.7)] implies the estimate

$$\overline{B}(B'(u)) \leqslant cB(u). \tag{4}$$

We shall denote all N-functions by capital Latin letters. All constants in the work are positive.

We assume that there exists an absolutely continuous odd increasing function $\gamma(u)$ satisfying inequalities

$$\gamma(u)/c_\gamma \leqslant u\gamma'(u) \leqslant c_\gamma\gamma(u), \ \gamma'(u) \leqslant \beta'(x,u) \leqslant c_\beta\gamma'(u). \tag{5}$$

as $u \in \mathbb{R}$, $x \in \Omega$. Hereinafter by β', γ', g' we shall denote the derivatives $\beta'_u(x,u)$, $\gamma'_u(x,u)$, $g'_u(x,u)$ of absolutely continuous in u functions and u', v', w' will stand for the derivatives $u'_t(t,x)$, $v'_t(t,x)$, $w'_t(t,x)$. We shall write function $u(t,x)$ as u or $u(t)$ once it produces no ambiguity. The arguments of functions $a = a(x,u,\nabla u)$, $b = b(x,u,\nabla u)$, $a_{p_i} = a_{p_i}(x,u,\nabla u)$ will be also sometimes omitted.

We also suppose that the integral

$$G(u)=\int\limits_0^u s\gamma'(s)ds$$

defines an N-function and in what follows it will be shown that it satisfies Δ_2-condition.

The monotonicity condition is imposed via the inequality

$$(b(x,u,p)-b(x,v,q))(u-v)+\sum_{i=1}^n(a_{p_i}(x,u,p)-a_{p_i}(x,v,q))(p_i-q_i)\geqslant 0 \tag{6}$$

valid for each $u,v\in\mathbb{R}$, $p,q\in\mathbb{R}^n$ and almost each $x\in\Omega$. Suppose also that the condition

$$b(x,u,p)u+\sum_{i=1}^n a_{p_i}(x,u,p)p_i\geqslant S(p)-c(G(u)+f(x)),\quad f\in L_1(\Omega), \tag{7}$$

holds true, where $S(p)=\sum_{i=1}^n B_i(p_i)$, B_i are some N-functions.

By $L_B(Q)$ we denote the Orlicz space associated with N-function $B(s)$ with the Luxembourg norm

$$\|u\|_{L_B(Q)}=\|u\|_{B,Q}=\inf\left\{k>0:\int\limits_Q B\left(\frac{u(x)}{k}\right)dx\leqslant 1\right\}.$$

In what follows as Q there can serve domains Ω, D^T and other, at that, the subscript $Q=\Omega$ can be omitted.

We also define the anisotropic Sobolev-Orlicz space $\mathring{W}^1_{G,B}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{W^1_{G,B}(\Omega)}=\sum_{i=1}^n\|u_{x_i}\|_{B_i,\Omega}+\|u\|_{G,\Omega}.$$

By $V(D^T)$ we indicate the completion of $C_0^\infty(D^T)$ in the norm

$$\|u\|_{V(D^T)}=\sum_{i=1}^n\|u_{x_i}\|_{B_i,D^T}+\|u\|_{G,D^T}.$$

It is known (cf. [21]) that as $\nu\equiv\|u\|_{B,Q}\geqslant 1$, Luxembourg norm satisfies the inequality $\|u\|_{B,Q}\leqslant\int\limits_Q B(u)dx$. As $\nu>1$, we have

$$\int\limits_Q B(u)dx=\int\limits_Q B(\nu u/\nu)dx\leqslant k\nu^m\int\limits_Q B(u/\nu)dx\leqslant k\nu^m.$$

Thus, we always have

$$\int\limits_Q B(u)dx\leqslant f_B(\|u\|_{B,Q}),\quad f_B(s)=ks^m+s. \tag{8}$$

For anisotropic Sobolev-Orlicz spaces the Korolev embedding theorem is known [22]. In order to formulate it, we define function $\Theta(s)$ as follows:

$$\Theta(s)=s^{-\frac{1}{n}}\prod_{i=1}^n\left(B_i^{-1}(s)\right)^{\frac{1}{n}}.$$

The integral

$$\int\limits_0^1\frac{\Theta(s)}{s}ds \tag{9}$$

can happen to diverge at zero, then while calculating $\Theta(s)$ we replace functions B_i by $\widetilde{B}_i$ by the formula

$$\widetilde{B}_i(s) = \begin{cases} B_i(s), & \text{as } |s| \geqslant 1, \\ s^\kappa B_i(1), & \text{as } |s| \leqslant 1. \end{cases}$$

We note that since functions B_i are convex, the inequality $B_i'(1) > B_i(1)$ holds true. We choose $\kappa \in (1, n)$ so that the inequalities

$$B_i'(1) > \kappa B_i(1), \quad i = 1, 2, \ldots, n,$$

are satisfied. The convergence of integral (9) at zero is ensured by the inequality $\kappa < n$.

We introduce an N-function $B^*(z)$ by the formula

$$(B^*)^{-1}(z) = \int\limits_0^{|z|} \frac{\Theta(s)}{s} ds$$

assuming that the integral

$$I(\Theta) = \int\limits_0^\infty \frac{\Theta(s)}{s} ds = \infty \tag{10}$$

converges at infinite. There known the Korolev embedding theorem [22] implied by the inequality

$$\|u\|_{B^*,\Omega} \leqslant C \sum_{i=1}^n \|u_{x_i}\|_{\widetilde{B}_i,\Omega} \tag{11}$$

valid for the functions $u \in C_0^\infty(\Omega)$.

By $\chi(M)$ we denote the characteristic function of a set M. We suppose the following conditions

$$\widetilde{b}^2 \chi(|u| < 1) + G\left(\frac{\widetilde{b}^2}{\beta'(x, u) G'(u)}\right) + \sum_{i=1}^n \overline{B}_i(a_{p_i}(x, u, p)) \leqslant c(\phi(x, u) + S(p)), \tag{12}$$

for each $u \in \mathbb{R}$, $p \in \mathbb{R}^n$ and almost each $x \in \Omega$, where $\widetilde{b}(x, u, p) = b(x, u, p) - a_u(x, u, p)$,

$$\phi(x, u) = f(x) + G(u) + \sum_{i=1}^n B_i(c_\Omega^i u).$$

Here constants c_Ω^i are so that the inequalities

$$\sum_{i=1}^n \langle B_i(c_\Omega^i u)\rangle \leqslant \langle S(\nabla u)\rangle, \ u \in C_0^\infty(\Omega), \tag{13}$$

$$\Gamma_1 a(x, u, p) + c\phi(x, u) \geqslant b(x, u, p)u + \sum_{i=1}^n a_{p_i}(x, u, p)p_i, \ c \in [0, 1/2], \tag{14}$$

$$b(x, u, p)u + \sum_{i=1}^n a_{p_i}(x, u, p)p_i \geqslant \delta_1 a(x, u, p) - c\phi(x, u), \tag{15}$$

$$|b(x, u, p)| \leqslant c\overline{G}^{-1}(S(p) + \phi(x, u)) + c\Lambda(u, p) \tag{16}$$

hold true, cf. Lemma 1, where $\Lambda(u, p) = \overline{B^*}^{-1}(S(p) + B^*(u))$. Hereinafter by c, c_1, c_2, ... we denote constant which can be different even for the same subscript.

Theorem 1. *Suppose that $u_0 \in \overset{\circ}{W}{}^1_{G,B}(\Omega)$ and conditions (5)–(7), (10), (12)–(16) hold true. Then there exists a generalized solution to problem (1)–(3) satisfying the relations*

$$u(t) \in L_{\infty,\mathrm{loc}}([0,\infty); \overset{\circ}{W}{}^1_{G,B}(\Omega)), \quad (\beta'(x,u))^{\frac{1}{2}} u' \in L_2(D^T), \quad \beta(x,u(t,x)) \in C_w([0,\infty); L_{\overline{G}}(\Omega)),$$

where the continuity is understood in the sense of weak topology of space $L_{\overline{G}}(\Omega)$).

The uniqueness of solution to problem (1)–(3) with the properties established in Theorem 1 will be proven in another work.

3. Auxiliary statements

Suppose that for almost each $x \in \Omega$ function $g(x,u)$ is absolutely continuous in $u \in \mathbb{R}$ and is defined by the identity

$$g'(x,u) = u\beta'(x,u), \quad g(x,0) = 0. \tag{17}$$

Let us make sure that it satisfies the inequality

$$ug'(x,u) \leqslant \alpha g(x,u), \quad \forall u \in \mathbb{R}, \quad x \in \Omega, \tag{18}$$

for some $\alpha > 0$. We mention an inequality implied by (5)

$$u\gamma(u) = \int_0^u (s\gamma(s))'ds = \int_0^u \gamma(s)ds + G(u) \leqslant (c_\gamma + 1)\int_0^u \gamma(s)ds, \ u \geqslant 0. \tag{19}$$

Hence,

$$uG'(u) = u^2\gamma'(u) \leqslant c_\gamma u\gamma(u) \leqslant c_\gamma(c_\gamma+1)G(u),$$

i.e., $G(u)$ satisfies Δ_2-condition. By (5) it yields (18).

We observe that if $u_m \to u$ in $L_B(\Omega)$ and B satisfies Δ_2-condition, then there exists C such that $\langle B(u_m)\rangle \leqslant C$. Indeed, a converging sequence in bounded $\|u_m\|_{L_B(\Omega)} \leqslant c$, $m = 1, 2, \ldots$ This is why the desired estimate follows easily from (8).

Let us show that

$$I_D = \int_0^T \|u(t)\|_{B^*,\Omega}dt < \infty, \ u \in V(D^T). \tag{20}$$

We write the relations

$$\begin{aligned} \|u_{x_i}\|_{\widetilde{B}_i(\Omega)} \leqslant& 1 + \langle \widetilde{B}_i(u_{x_i})\rangle \leqslant 1 + \langle B_i(u_{x_i})\chi(|u_{x_i}|>1)\rangle + \langle B_i(1)\chi(|u_{x_i}| \leqslant 1)\rangle) \\ \leqslant& 1 + \langle B_i(u_{x_i})\rangle + \langle B_i(1)\rangle. \end{aligned} \tag{21}$$

We make use of inequalities (11) and (21)

$$\|u\|_{B^*,\Omega} \leqslant C\sum_{i=1}^n \|u_{x_i}\|_{\widetilde{B}_i,\Omega} \leqslant c_2 + C\sum_{i=1}^n \langle B_i(u_{x_i})\rangle.$$

It implies (20). Moreover, by (8) we obtain

$$\langle B^*(u)\rangle \leqslant c_3(k), \text{if} \langle S(\nabla u)\rangle \leqslant k. \tag{22}$$

Lemma 1. *Suppose that domain Ω is located in the half-space $x_1 > 0$. Then for an arbitrary N-function B the inequality*

$$\int_{\Omega^r} B(u(x))dx \leqslant \int_{\Omega^r} B(ru_{x_1}(x))dx, \quad u \in C_0^\infty(\Omega), \tag{23}$$

holds true.

Proof. Let $f(x_1) \in C^1[0,r]$, $f(0)=0$. By the Newton-Leibnitz formula we obtain

$$|f(x_1)| = \left|\int\limits_0^{x_1} f'(x_1)dx_1\right| \leqslant \int\limits_0^{r} |f'(x_1)|dx_1, \quad x_1 \in [0,r].$$

Now we apply Jensen integral inequality (see [21, Ch. 2, Sect. 8.2, Ineq. (8.2)]):

$$B\left(\frac{f(x_1)}{r}\right) \leqslant B\left(r^{-1}\int\limits_0^{r} |f'(x_1)|dx_1\right) \leqslant \frac{1}{r}\int\limits_0^{r} B(f'(x_1))dx_1.$$

We integrate the latter inequality w.r.t. x_1

$$\int\limits_0^{r} B\left(\frac{f(x_1)}{r}\right)dx_1 \leqslant \int\limits_0^{r} B(f'(x_1))dx_1.$$

After the substitution $f(x_1) = ru(x)$ and integrating w.r.t. $x' = \{x_2,...,x_n\}$ we arrive at (23). □

A corollary of Lemma 1 is, in particular, inequality (13) in the case of a bounded domain Ω. By the passage to a limit we also establish it for functions in $\mathring{W}^1_{G,B}(\Omega)$. By employing (8) it is easy to prove the inequality

$$\langle \phi(x,u)\rangle \leqslant c(\|u\|_{\mathring{W}^1_{G,B}(\Omega)}). \tag{24}$$

The case of a power function $G(u)$, $G(u) = |u|^p$, was considered in work [16]. At that, as $p \geqslant 2$ and $p < 2$, there were employed different techniques of passing to the limit in Galerkin's approximations. In the case of N-functions $G(u)$ it can have different power behavior different p on different intervals. The passages to the limit happen to be made in the sign-definiteness intervals of the function $y(u) = u(\gamma(u) - u\gamma'(u))$. At that, different signs require different techniques of the passage to the limit.

We shall assume that the ray $(0,+\infty)$ is partitioned into disjoint connected intervals I_1, I_2, …without finite accumulation points. In each of these intervals is in turns either $y(u) \leqslant 0$ or $y(u) > 0$. We let $N^+ = \{k | u \in I_k \Rightarrow y(u) > 0\}$, $N^- = \{k | u \in I_k \Rightarrow y(u) \leqslant 0\}$. Sets N^+, N^- can be empty, finite, or countable.

Let $\alpha(t)$ be the inverse function to $\gamma(u)$. We let $j(t) = G(\alpha(\sqrt{|t|}))$, $j(\gamma^2(u)) = G(u)$.

Lemma 2. *Let $\overline{I}_k = [a_k, b_k]$ be the closure of segment I_k. If $k \in N^+$, then function $j(u)$ is convex on $I_k^\gamma = (\gamma^2(a_k), \gamma^2(b_k))$ and function $2j(u) - j(a_k)$ is linearly extended to a convex on $[0,\infty)$ function $j_k(u)$. At that,*

$$\frac{s^2}{\beta'(x,u)} \leqslant c(G(u) + J_k(s)),\ |u| \in I_k,\ s > 0, \tag{25}$$

where $J_k(s) = j_k(s^2)$ is an N-function.

If $k \in N^-$, there exists N-function $H_k(u)$ such that

$$s^2\gamma'(u) \leqslant c(G(u) + H_k(s)),\ |u| \in I_k,\ s > 0. \tag{26}$$

Proof. The graph of convex function $f(s)$, $s > 0$ is located above the tangent $f(s) \geqslant f'(t)(s - t) + f(t)$, for instance, with the right derivative $f'(t)$, and thus

$$sf'(t) \leqslant f(s) - f(t) + tf'(t). \tag{27}$$

Let $k \in N^+$. Then as $u \in I_k$

$$\left(\frac{\gamma(u)}{u}\right)' = \frac{u\gamma'(u) - \gamma(u)}{u^2} = -\frac{y(u)}{u^3} < 0,$$

i.e., function $\gamma(u)/u$ decreases. Then (5) with $1 \in N^+$ implies the inequality

$$\frac{1}{\beta'(x,u)} \leqslant c, u \in (0,b_1), x \in \Omega. \tag{28}$$

Let us check that $j(t)$ is a convex function on the interval I_k^γ

$$j'(t) = u\gamma'(u)\frac{1}{\gamma'(u)}\Big|_{u=\alpha(\sqrt{|t|})}\frac{1}{2\sqrt{|t|}} = \frac{\alpha(\sqrt{|t|})}{2\sqrt{|t|}} = \frac{u}{2\gamma(u)}.$$

The convexity is proven since $u/\gamma(u)$ is an increasing function. In view of (19),

$$\frac{j'(t)t}{j(t)}\Big|_{t=\gamma^2(u)} = \frac{u\gamma(u)}{2G(u)} > \frac{1}{2}.$$

Hence, function $j_k(t) = 2j(t) - j(\gamma^2(a_k))$, $t \in [\gamma^2(a_k), \gamma^2(b_k)]$, can be extended from segment I_k^γ to a convex on $[0,\infty)$ function by the formulae $j_k(t) = tj(\gamma^2(a_k))/\gamma^2(a_k)$, $t \in [0, \gamma^2(a_k)]$, $j_k(t) = (t - \gamma^2(b_k))j_k'(\gamma^2(b_k)) + j_k(\gamma^2(b_k))$, $t \in [\gamma^2(b_k), \infty)$. We let $J_k(s) = j_k(s^2)$.

In the case when, for instance, $I_1 = (0,\infty)$, there is no need to extend function $j_1(s)$ and we just assume that $J_1(s)$ satisfies the conditions of N-function at the vicinity of zero and infinity. Here it is sufficient to assume that

$$\lim_{s\to 0}\frac{\gamma(s)}{s} = 0, \quad \lim_{s\to\infty}\frac{\gamma(s)}{s} = \infty.$$

Let us show that inequality (5) ensures Δ_2-condition for function $J_k(u)$. Indeed, in view of (19) we have

$$G(u) \geqslant \frac{1}{c_\gamma}\int_0^u \gamma(s)ds \geqslant \frac{u\gamma(u)}{c_\gamma(c_\gamma+1)}.$$

Then

$$\frac{j_k'(t)t}{j_k(t)}\Big|_{t=\gamma^2(u)} \leqslant \frac{2j'(t)t}{j(t)}\Big|_{t=\gamma^2(u)} = \frac{u\gamma(u)}{G(u)} \leqslant c_2, \quad |u| \in I_k.$$

This inequality implies Δ_2-condition for function $J_k(s)$.

Let us prove (25). Since by (5) we have $c_\gamma\beta'(u) \geqslant c_\gamma\gamma'(u) \geqslant \gamma(u)/u$, $|u| \in I_k$, then

$$\frac{1}{c_\gamma\beta'(u)} \leqslant \frac{u}{\gamma(u)}\Big|_{u=\alpha(\sqrt{|t|})} = j_k'(t).$$

Applying (27), we get

$$\frac{s}{c_\gamma\beta'(u)} \leqslant tj_k'(t) + j_k(s) \leqslant c_2 j_k(t)\Big|_{t=\gamma^2(u)} + j_k(s) \leqslant 2c_2G(u) + j_k(s),$$

that yields (25).

Let $k \in N^-$. We define a function

$$h(u) = \int_0^{\sqrt{|u|}} \gamma(s)ds.$$

Then $h'(u) = \frac{\gamma(\sqrt{|u|})}{2\sqrt{|u|}}$ is an increasing function as $u \in I_k$, i.e., $h(u)$ is a convex function. At that, due to (19),

$$\frac{h'(u)u}{h(u)} = \frac{\sqrt{|u|}\gamma(\sqrt{|u|})}{2\int_0^{\sqrt{|u|}}\gamma(s)ds} > \frac{1}{2}.$$

Hence, function $h_k(u) = 2h(u) - h(a_k)$, $u \in [a_k, b_k]$, can be extended from the segment I_k to a convex on $[0,\infty)$ function by the formulae $h_k(u) = uh(a_k)/a_k$, $u \in [0, a_k]$, $h_k(u) = (u - b_k)h'_k(b_k) + h_k(b_k)$, $u \in [b_k, \infty)$. By inequality (19),

$$\frac{h'_k(u)u}{h_k(u)} \leqslant \frac{2h'(u)u}{h(u)} \leqslant c_\gamma + 1.$$

Inequality $h_k(u^2) \leqslant 2c_\gamma G(u)$, $u^2 \in I_k$ follows from the definition of function h_k and (5). We let $H_k(s) = h_k(s^2)$. Applying (27) to function h_k, we have

$$s\gamma(u)/u = sh'_k(u^2) \leqslant u^2 h'_k(u^2) + h_k(s) \leqslant (c_\gamma + 1)h_k(u^2) + h_k(s) \leqslant c_3(G(u) + H_k(\sqrt{s}))$$

Now (26) follows from (5). □

4. Proof of existence theorem

A generalized solution to problem (1)–(3) is a function $u(t,x)$ belonging to spaces $L_\infty((0,T); \mathring{W}^1_{G,B}(\Omega)) \subset V(D^T)$ for each$T > 0$ and satisfying the identity

$$[-\beta(x,u)\varphi' + b(x,u,\nabla u)\varphi] + \sum_{i=1}^{n}[a_{p_i}(x,u,\nabla u)\varphi_{x_i}] = \langle\beta(x,u_0)\varphi(0)\rangle. \tag{29}$$

for $\varphi(t,x) \in C_0^\infty(D^T_{-1})$.

Let us show that the functional

$$\widetilde{a}(u) = b(x,u,\nabla u) - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}a_{p_i}(x,u,\nabla u)$$

is bounded on the unit ball in space $V(D^T)$

$$(\widetilde{a}(u), v)_{D^T} = [b(x,u,\nabla u)v] + \sum_{i=1}^{n}[a_{p_i}(x,u,\nabla u)v_{x_i}] = I_1 + I_2.$$

Let us estimate integral I_2 by means (12), (24), and by of Young inequality for $\|v\|_{V(D^T)} \leqslant 1$:

$$|I_2| \leqslant \left[S(\nabla v) + \sum_{i=1}^{n}\overline{B}_i(a_{p_i}(x,u,\nabla u))\right] \leqslant c_1 + c\,[S(\nabla u) + \phi(x,u)] \leqslant c_2.$$

We estimate integral I_1 employing (16)

$$\begin{aligned}|I_1| \leqslant &\left[c\overline{G}^{-1}\left(S(\nabla u) + \phi(x,u)\right)|v|\right] + c\,[\Lambda(u,\nabla u)|v|]\\ \leqslant &c[2S(\nabla u) + G(u) + 2G(v)] + [\Lambda(u,\nabla u)|v|] + c_3 \leqslant c_4.\end{aligned}$$

In the latter inequality we have employed relations (20), (22), and

$$[\Lambda(u,\nabla u)|v|] \leqslant \int_0^T \|v(t)\|_{B^*,\Omega}\|\Lambda(u,\nabla u)\|_{\overline{B}^*,\Omega}dt \leqslant c_5\int_0^T \|v(t)\|_{B^*,\Omega}dt \leqslant c_6.$$

Thus, we have proven the boundedness of functional $\widetilde{a}(u)$.

We proceed to constructing Galerkin's approximations, We choose a sequence $\omega_k \in C_0^\infty(\Omega)$ of linearly independent functions whose linear span is dense in $\mathring{W}^1_{G,B}(\Omega)$. We let $\beta_m(x,u) = \int\limits_0^u \beta'_m(x,s)ds$, where

$$\beta'_m(x,u) = \varepsilon_m + \beta'(x,u) + c_+(\beta'(x,\varepsilon_m) - \beta'(x,u))\chi(0 < |u| < \varepsilon_m),$$

$c_+ = 1$ if $1 \in N^+$, and $c_+ = 0$ if $1 \in N^-$.

We introduce the functions $g_m(x,u) = \int\limits_0^u s\beta'_m(x,s)ds$. Numbers $\varepsilon_m > 0$ will be chosen later. As $u > 0$, we mention the inequality

$$g(x,u) - g_m(x,u) = \int\limits_0^u s(\beta'(x,s) - \beta'_m(x,s))ds \leqslant \int\limits_0^{\varepsilon_m} s\beta'(x,s)ds = g(x,\varepsilon_m) \leqslant c_\beta G(\varepsilon_m). \quad (30)$$

A similar inequality is valid also for $u < 0$. In the same way, by (18) we establish that

$$g_m(x,u) - g(x,u) \leqslant \varepsilon_m u^2/2 + \int\limits_0^{\varepsilon_m} s\beta'(x,\varepsilon_m)ds \leqslant \varepsilon_m u^2/2 + c_\beta G(\varepsilon_m). \quad (31)$$

We shall seek Galerkin's approximation for the solution as

$$u_m(t,x) = \sum_{k=1}^m c_{mk}(t)\omega_k(x),$$

where functions $c_{mk}(t)$ are determined by the equations

$$\Big\langle \omega_j \frac{\partial}{\partial t}\left(\beta_m(x,u_m)\right) + \sum_{i=1}^n a_{p_i}(x,u_m,\nabla u_m)(\omega_j)_{x_i} + b(x,u_m,\nabla u_m)\omega_j \Big\rangle = 0, \quad (32)$$

where $j = 1, 2, \dots, m$.

Let us make sure that equations (32) are solvable w.r.t. the derivatives c'_{mk}. It is obvious that they read as

$$\sum_{k=1}^m A_{jk}(t)c'_{mk} = F_j(c_{m1}, c_{m2}, ..., c_{mm}).$$

For each t, the matrix of the coefficients $A_{jk}(t) = \langle \beta'_m(x,u_m)\omega_j\omega_k\rangle$ is the Gram matrix of the system of linearly independent vectors ω_k, $k = 1, 2, \dots, m$ and thus it has an inverse. By equations (32) and initial conditions $c_{mk}(0)$ chosen so that $u_m(0,x) \to u_0(x)$ in $\mathring{W}^1_{G,B}(\Omega)$ we find functions $c_{mk}(t)$. First we find these functions on a small time interval, but the boundedness of Galerkin's approximations to be established below will allow to define these approximations on the infinite time interval.

We multiply equations (32) by $c_{mj}(t)$, sum up, and employ the definition of function g_m:

$$\langle g'_m(x,u_m)u'_m + b(x,u_m,\nabla u_m)u_m\rangle + \sum_{i=1}^n \langle a_{p_i}(x,u_m,\nabla u_m)u_{mx_i}\rangle = 0.$$

Employing inequality (7), we get

$$\big\langle (g_m(x,u_m))'_t + S(\nabla u_m(t)\big\rangle \leqslant c\langle G(u_m(t)) + f(x)\rangle.$$

Integrating w.r.t. t and using (13), we obtain

$$\langle g_m(x,u_m(t))\rangle + \int\limits_{D_0^t} S(\nabla u_m)dxdt \leqslant \langle g_m(x,u_m(0))\rangle + c\int\limits_{D_0^t} G(u_m)dxdt + c_1 t.$$

We choose ε_m so that

$$G(\varepsilon_m)c_\beta \mathrm{mes}(\Omega) \leqslant \frac{1}{2m}, \quad (33)$$

and

$$\langle \varepsilon_m u_m^2(0)\rangle \leqslant \frac{1}{m},\ m = 1, 2, \dots$$

Then by (30) we obtain the inequality

$$\langle g(x,u_m(t)) - g_m(x,u_m(t))\rangle \leqslant \int\limits_\Omega c_\beta G(\varepsilon_m)dx \leqslant c_\beta G(\varepsilon_m)\mathrm{mes}(\Omega) \leqslant \frac{1}{2m}.$$

It follows from (31) that the integrals

$$\langle g_m(x,u_m(0))\rangle \leqslant c,\ m=1,2,\dots$$

are bounded. Then by means of (5) we establish the inequality

$$\langle G(u_m(t))\rangle + \int\limits_{D_0^t} S(\nabla u_m)dxdt \leqslant C(1+t) + c\int\limits_{D_0^t} G(u_m)dxdt.$$

By Gröwnwall's lemma we obtain the following estimate

$$\langle G(u_m(t))\rangle + [S(\nabla u_m)] \leqslant \overline{C}(T),\ \forall t\in[0,T]. \tag{34}$$

By (34) we arrive at the boundedness of sequence u_m in space $V(D^T)$ for each $T>0$ and in space $L_\infty([0,T];L_G(\Omega))$.

We multiply equations (32) by $c'_{mj}(t)$ and sum up to obtain

$$\langle \beta'_m(x,u_m(t,x))(u'_m)^2 + b(x,u_m,\nabla u_m)u'_m\rangle + \sum_{i=1}^n \langle a_{p_i}(x,u_m,\nabla u_m)u'_{mx_i}\rangle = 0,$$

or

$$\left\langle \beta'_m(x,u_m(t,x))(u'_m)^2 + (a(x,u_m,\nabla u_m))'_t\right\rangle = \langle (a_u-b)(x,u_m,\nabla u_m)u'_m\rangle.$$

We integrate the latter inequality w.r.t. t:

$$\begin{aligned}&[\beta'_m(x,u_m)(u'_m)^2] + \langle a(x,u_m(T,x),\nabla u_m(T,x))\rangle\\ &\qquad = \langle a(x,u_m(0),\nabla u_m(0))\rangle + [(a_u-b)(x,u_m,\nabla u_m)u'_m] = I_\Omega + I_D.\end{aligned} \tag{35}$$

We employ conditions (15), (16), (12), (13):

$$I_\Omega = \langle a(x,u_m(0),\nabla u_m(0))\rangle \leqslant c(\langle \phi(x,u_m(0)) + S(\nabla u_m(0))\rangle \leqslant c_1.$$

In what follows we shall show the boundedness of the sequence fo integrals

$$I_\beta = [\beta'_m(x,u_m)(u'_m)^2] \leqslant c. \tag{36}$$

To estimate integral I_D, we write the inequalities

$$[|u'_m\varphi|] \leqslant \left[(\beta'_m(x,u_m))^{\frac12}|u'_m|\frac{|\varphi|}{(\beta'_m(x,u_m))^{\frac12}}\right] \leqslant I_\beta/2 + \left[\frac{\varphi^2\chi(|u_m|>0)}{2\beta'_m(x,u_m)}\right]. \tag{37}$$

We estimate the second integral in the right hand side of the latter inequality by means of Young inequality, relation (28) and the formula for β_m

$$\begin{aligned}\left[\frac{\varphi^2\chi(|u_m|>0)}{\beta'_m(x,u_m)}\right] &\leqslant \left[\frac{\varphi^2\chi(|u_m|>0)}{\beta'(x,u_m)}\right] + c_+\left[\frac{\varphi^2\chi(\varepsilon_m\geqslant|u_m|>0)}{\beta'(x,\varepsilon_m)}\right]\\ &\leqslant \left[\overline{G}(u_m\gamma'(u_m)) + G\left(\frac{\varphi^2}{u_m\beta'(x,u_m)\gamma'(u_m)}\right)\right] + c[\varphi^2\chi(\varepsilon_m\geqslant|u_m|>0)].\end{aligned}$$

Hence, substituting $\varphi=(b-a_u)(x,u_m,\nabla u_m)$, $u_m\gamma'(u_m)=G'(u_m)$ and employing (4),(12), we have

$$[|u'_m\varphi|] \leqslant c_1(T) + I_\beta/2 + c_2[\phi(x,u_m) + S(\nabla u_m)].$$

Applying (13) as well as (34), we obtain

$$I_D \leqslant c_3(T) + I_\beta/2.$$

Combining estimates for the integrals and identity (35) and employing (14),(7), we have

$$\frac{1}{2}[\beta_m'(x,u_m)(u_m')^2] + \langle S(\nabla u_m(T))/\Gamma_1\rangle \leqslant c_4 + \left\langle \frac{1}{2\Gamma_1}\phi(x,u_m(T,x))\right\rangle \leqslant c_5 + \frac{1}{2\Gamma_1}\langle S(\nabla u_m(T))\rangle.$$

The latter inequality and (34) follow the boundedness of sequence $(\beta_m')^{\frac{1}{2}}u_m'$ in $L_2(D^T)$. Sequence u_m is also bounded in space $L_\infty([0,T];\overset{\circ}{W}{}^1_{G,B}(\Omega))$ for each $T>0$.

By means of the diagonal process we choose a subsequence u_{m_k} weakly converging in the below mentioned spaces. For the sake of simplicity of notations we shall omit the subscript k in the subsequences. We have

$$u_m \to u \quad \text{weakly in} \quad V(D^T), \quad (\beta_m'(x,u_m))^{\frac{1}{2}}\,u_m' \to \tilde{u} \quad \text{weakly in} \quad L_2(D^T), \quad \forall T>0.$$

Since u_m is a bounded sequence in $L_\infty((0,T);\mathring{W}^1_{G,B}(\Omega))$, $\widetilde{a}(u_m)$ is a bounded sequence in space $(V(D^T))'$ and it contains a weakly convergent subsequence: $\tilde{a}(u_m)\to\chi$ weakly in $(V(D^T))'$. The convergence holds for each $T=1,2,\dots$, at that, the limiting functions coincide in common domain. Then, in fact, the convergence holds true for each $T>0$.

In what follows we shall show that $\tilde{u}=(\beta'(x,u))^{\frac{1}{2}}\,u'$, $\chi=\tilde{a}(u)$, and function u is a generalized solution to problem (1)–(3). We split our arguments into three steps.

Step 1. Sequence $u_m(t)$ is bounded in space $\overset{\circ}{W}{}^1_{G,B}(\Omega)$ on each finite segment $t\in[0,T]$:

$$\|u_m(t)\|_{W^1_{G,B}(\Omega)} \leqslant C(T), \quad m=1,2,\dots$$

We fix a countable dense subset $\{t_s\}\subset[0,\infty]$. We can assume that $t_0=0$. For a bounded domain Ω we know that the embedding $\mathring{W}^1_1(\Omega)\subset L_1(\Omega)$ is compact. Since $\mathring{W}^1_{G,B}(\Omega)\subset\mathring{W}^1_1(\Omega)$, by the diagonal process we can choose a subsequence such that $u_{m_k}(t_s)\to h_s$ strongly in $L_1(\Omega)$ for each natural s. Choosing a subsequence once again, we can suppose that (omitting subscripts), that $u_m(t_s,x)\to h_s(x)$ almost everywhere in Ω for each t_s. In particular, as $t_0=0$, we have $u_m(0,x)\to u_0(x)$ almost everywhere in Ω.

In the next step we employ a lemma proven in [16].

Lemma 3. *Suppose that a sequence $v_m(t)\in C([0,T];L_2(\Omega))$ possesses the properties*
1) $v_m(t_s,x)$ converges almost everywhere in Ω for each t_s;
2) sequence v_m' is bounded in $L_2(D^T)$.
Then we can choose a subsequence v_{m_k} converging to a function v in space $C([0,T];L_1(\Omega))$ and $v_{m_k}\to v$ almost everywhere in $(0,T)\times\Omega$.

Step 2. We apply Lemma 3 to the sequence $v_m=f_m(x,u_m)=\int\limits_0^{u_m}(\beta_m'(x,\tau)-\varepsilon_m)^{\frac{1}{2}}d\tau$. At that, the sequence $v_m'=(\beta_m'-\varepsilon_m)^{\frac{1}{2}}u_m'$ is bounded due to (36). In what follows we shall establish the uniform convergence $f_m(x,u)\to f(x,u)=\int\limits_0^{u}(\beta'(x,\tau))^{\frac{1}{2}}d\tau, m\to\infty$ w.r.t. u for almost each fixed $x\in\Omega$ that will imply the first statement of the lemma. The belonging of $v_m(t)$ to $L_2(\Omega)$ follows from its smoothness and the boundedness of $\beta_m(x,u)$ on finite intervals w.r.t. u:

$$v_m^2 \leqslant |u_m|\int\limits_0^{|u_m|}(\beta_m'(x,\tau)-\varepsilon_m)d\tau \leqslant |u_m|\int\limits_0^{|u_m|}c_\beta\gamma'(\tau)d\tau.$$

The aforementioned uniform convergence $f_m(x,u)\to f(x,u)$ w.r.t. u as $x\in\Omega$ follows easily from the identity

$$f_m(x,u)-f(x,u)=\int\limits_0^{\varepsilon_m}((\beta_m'(x,\tau)-\varepsilon_m)^{\frac{1}{2}}-(\beta'(x,\tau))^{\frac{1}{2}})d\tau$$

and Cauchy-Schwarz inequality

$$\left(\int_0^{\varepsilon_m} (\beta'(x,\tau))^{\frac{1}{2}} d\tau\right)^2 \leqslant \varepsilon_m \int_0^{\varepsilon_m} \beta'(x,\tau) d\tau \to 0.$$

Employing Lemma 3 and the arbitrariness of $r > 0$, $T = 1, 2, \dots$, by the diagonal process we choose a subsequence v_{m_k} converging in D almost everywhere. Since $f(x,u)$ is an increasing in u continuous function having the inverse, by the identity $v_m = (f_m(x,u_m) - f(x,u_m)) + f(x,u_m) = \nu_m + f(x,u_m)$, $\nu_m \to 0$, we find that $u_m = f^{-1}(x, v_m - \nu_m)$. Then the convergence of sequence v_{m_k} follows the convergence of sequence u_{m_k} almost everywhere in D to u. The fact that the limiting function is exactly u is followed by

Lemma 4. *Suppose that sequence z_m converges to z almost everywhere in Q and is bounded in $L_B(Q)$. Then $z_m \to z$ weakly in $L_B(Q)$.*

The proof of this lemma provided in [2, Ch. 1, Sect. 1.4, Lm. 1.3]) for $L_q(Q)$, $q > 1$, is obviously transferred to the general case.

Thanks to the weak convergence $u_m \to u$ in $V(D^T)$ and the continuity of the embedding $V(D^T) \subset L_1([0,T] \times \Omega)$ the weak convergence $u_m \to u$ holds true in $L_1([0,T] \times \Omega)$. It also follows from Lemma 4 that $v_{m_k} \to v = f(x,u)$ weakly in $L_2(D^T)$ for each $T > 0$.

By Lemma 3 we know that $v_{m_k}(T) \to v(T)$ in $L_1(\Omega)$. Then we can choose a subsequence convergent almost everywhere in Ω: $v_{m_k}(T,x) \to v(T,x) \quad \Rightarrow \quad u_{m_k}(T,x) \to u(T,x)$ almost everywhere in Ω. In the same way we establish that $u_{m_k}(0,x) \to u(0,x)$ almost everywhere in Ω, i.e.,

$$u(0,x) = u_0(x).$$

The boundedness of sequence $u_m(T)$ in space $\mathring{W}^1_{G,B}(\Omega)$ by a constant $C(T)$ allows us to choose a subsequence such that $u_{m_k}(T) \to u(T)$ weakly in $\mathring{W}^1_{G,B}(\Omega)$ for a fixed T. Since constant $C(T)$ increases in T, it yields that $u \in L_{\infty,\mathrm{loc}}([0,\infty); \mathring{W}^1_{G,B}(\Omega))$.

We have $(v'_m, \varphi)_{D^T} = -(v_m, \varphi')_{D^T}$, $\varphi \in C_0^\infty(D^T)$. Passing to the limit, we obtain

$$(\tilde{u}, \varphi)_{D^T} = -(v, \varphi')_{D^T}.$$

It follows that $\tilde{u} = v' = (\beta'(x,u))^{\frac{1}{2}} u'$.

Let us show that sequence $\beta(x, u_m(t))$ is bounded in $L_{\overline{G}}(\Omega)$ by a constant independent of $t \in [0,T]$. We first establish an inequality by employing (5):

$$\overline{G}\left(\frac{\beta(x,u_m)}{c_\beta c_\gamma}\right) \leqslant \overline{G}\left(\frac{\gamma(u_m)}{c_\gamma}\right) \leqslant \overline{G}(u\gamma'(u_m)) = \overline{G}(G'(u_m)) \leqslant cG(u_m).$$

The boundedness of sequence $\beta(x, u_m(t))$ is proven. In particular, it is bounded in space $L_{\overline{G}}(D^T)$. Then we can choose a subsequence weakly converging to function $\beta(x,u)$ in $L_{\overline{G}}(D^T)$. Indeed, sequence u_m converges to u almost everywhere in D^T and Lemma 4 ensures the desired fact.

For further limiting passages we define the operator $T_k u = u^{(a_k, b_k)}$, where

$$u^{(a,b)} = \begin{cases} 0, & \text{as } |u| \leqslant a, \\ u - a\,\mathrm{sign} u, & \text{as } a < |u| < b, \\ (b-a)\,\mathrm{sign} u, & \text{as } |u| \geqslant b. \end{cases}$$

We also let

$$T_k^\beta \beta(x,u_m) = \beta(x, a_k \mathrm{sign} u_m + T_k u_m) - \beta(x, a_k \mathrm{sign} u_m), \quad T_k^\beta \beta(x,u_m) \in L_{\overline{G}}(D^T).$$

We observe obvious formulae which will be employed in what follows:

$$(T_k^\beta \beta(x,u_m))' = \beta'(x,u_m)(T_k u_m)' = \beta'(x,u_m)u_m'\chi(|u_m| \in I_k)), \quad u_m = \sum_k T_k u_m.$$

Let us show that the sequence

$$\beta'(x,u_m)u_m'\chi(|u_m| \in I_k) = \beta'(x,u_m)(T_k u_m)', \quad k \in N^-,$$

is bounded in $L_{\overline{H_k}}(D^T)$. Indeed by (36), (5), and Lemma 2

$$\begin{aligned} |[\beta'(x,u_m)u_m'\varphi\chi(|u_m| \in I_k)]| \leqslant & c[(\beta')^{\frac{1}{2}}|u_m'\varphi|(\gamma'(u_m))^{\frac{1}{2}}\chi(|u_m| \in I_k)] \\ \leqslant & c_1\|\varphi(\gamma'(u_m))^{\frac{1}{2}}\chi(|u_m| \in I_k)\|_{L_2(D^T)} \\ \leqslant & c_2[H_k(\varphi) + G(u_m)]^{\frac{1}{2}} \leqslant c_3 \text{ as } [H_k(\varphi)] \leqslant 1. \end{aligned}$$

Therefore, we can choose a weakly convergent subsequence

$$\beta'(x,u_m)u_m'\chi(|u_m| \in I_k) \to \overline{u} \in L_{\overline{H_k}}(D^T).$$

Applying the diagonal process, we can get the weak convergence of a chosen sequence for each $k \in N^-$. Passing to the limit in the identity

$$[T_k^\beta \beta(x,u_m)\varphi'] = -[\beta'(x,u_m)(T_k u_m)'\varphi], \ \varphi \in C_0^\infty(D^T),$$

we obtain that $[T_k^\beta \beta(x,u)\varphi'] = -[\overline{u}\varphi]$, i.e.,

$$(T_k^\beta \beta(x,u))_t' = \overline{u} \in L_{\overline{H_k}}(D^T), k \in N^-. \tag{38}$$

Let us prove the boundedness of sequence $(T_k u_m)' = u_m'\chi(|u_m| \in I_k)$ in space $L_{\overline{J_k}}(D^T)$ as $k \in N^+$. Indeed, by (37), (36), and Lemma 2,

$$[|u_m'\varphi|\chi(|u_m| \in I_k)] \leqslant I_\beta/2 + [J_k(\varphi) + G(u_m)] \leqslant c_1, \text{ as } [J_k(\varphi)] \leqslant 1.$$

By analogy with (38) we find that

$$(T_k u)' = u'\chi(|u| \in I_k) \in L_{\overline{J_k}}(D^T), \ k \in N^+. \tag{39}$$

Step 3. We proceed to the proof of identity $\chi = \tilde{a}(u)$. We multiply equation (32) by a smooth function $d_j(t)$ and integrate w.r.t. t denoting $d_j(t)\omega_j(x)$ by φ in the final expression

$$[\beta_m'(x,u_m)u_m'\varphi] + (\tilde{a}(u_m),\varphi)_{D^T} = 0. \tag{40}$$

We rewrite the first term

$$[\beta_m'(x,u_m)u_m'\varphi] = [\beta'(x,u_m)u_m'\varphi] + A_m,$$

where

$$\begin{aligned} &A_m = [(\varepsilon_m + c_+(\beta'(x,\varepsilon_m) - \beta'(x,u_m))\chi(0 < |u_m| < \varepsilon_m)))u_m'\varphi] = -[(\mu_m)'\varphi], \\ &\mu_m = (c_+\beta(x,u_m) - (\varepsilon_m + c_+\beta'(x,\varepsilon_m))u_m) \text{ as } |u_m| < \varepsilon_m. \end{aligned}$$

Let us show that $A_m \to 0$ as $m \to \infty$. Indeed,

$$A_m = [\mu_m\varphi'] - \langle \mu_m(T)\varphi(T)\rangle + \langle \mu_m(0)\varphi(0)\rangle \to 0.$$

The latter follows from (5) since φ is compactly supported and

$$|\mu_m + \varepsilon_m u_m| \leqslant \gamma(\varepsilon_m) + \varepsilon_m\gamma'(\varepsilon_m) \to 0.$$

After the integration by parts in formula (40) we have

$$A_m - [\beta(x,u_m)\varphi'] + \langle \beta(x,u_m(T))\varphi(T)\rangle - \langle \beta(x,u_m(0))\varphi(0)\rangle + (\tilde{a}(u_m),\varphi)_{D^T} = 0.$$

We have mentioned above the weak convergence of sequences $\beta(x,u_m)$, $\beta(x,u_m(T))$ in spaces $L_{\overline{G}}(D^T)$, $L_{\overline{G}}(\Omega)$, respectively. Then by passing to the limit, we obtain

$$-[\beta(x,u)\varphi'] + \langle \beta(x,u(T))\varphi(T)\rangle - \langle \beta(x,u(0))\varphi(0)\rangle + (\chi,\varphi)_{D^T} = 0. \tag{41}$$

In particular, it yields that $\beta(x,u) = \chi$ in the distribution sense and thus $\beta(x,u) \in C_w([0,\infty); L_{\overline{G}}(\Omega))$.

Hence, u is a generalized solution to problem (1)–(3) if $\chi = \tilde{a}(u)$. Let us justify the possibility of substitution $\varphi = u$ into formula (41). In order to od it, we first substitute function $\varphi \in C_0^\infty(D_{-1}^{T+1})$ vanishing as $|u| \leqslant \varepsilon$ and we integrate by parts:

$$-[T_k^\beta \beta(x,u)\varphi'] + \langle T_k^\beta \beta(x,u(T))\varphi(T)\rangle - \langle T_k^\beta \beta(x,u(0))\varphi(0)\rangle = [(T_k^\beta \beta(x,u))'\varphi] = [\beta'(x,u)(T_k u)'\varphi].$$

The convergence of the integral in the right hand side follows from (38), (39) and inequality (see (5))

$$\beta'(x,u) \leqslant c_\beta \gamma'(|u|) \leqslant c_1\gamma(|u|)/|u| \leqslant c_2, \ |u| \in [\varepsilon, b_k].$$

Thus, (41) casts into the form

$$[\beta'(x,u)u'\varphi] + (\chi,\varphi)_{D^T} = 0. \tag{42}$$

By passing to the limit we justify the substitution in (42) of a bounded function $\varphi = u^{(\varepsilon,k)}\xi$, where $\xi(x)$ is a Lipschitz function with a bounded support. We have

$$[\beta'(x,u)u'u^{(\varepsilon,k)}\xi] + (\chi, u^{(\varepsilon,k)}\xi)_{D^T} = 0. \tag{43}$$

Let $w_m = (g(x,u_m))^{\frac{1}{2}}$, $w_m \to w = (g(x,u))^{\frac{1}{2}}$ almost everywhere in D^T. If we prove that $w \in L_2(D^T)$ has a generalized derivative $w_t \in L_2(D^T)$, then the identity

$$[\beta'(x,u)u'u] = \int\limits_0^T \frac{\partial}{\partial t}\|w\|_2^2 dt = \|w(T)\|_{L_2(\Omega)} - \|w(0)\|_{L_2(\Omega)} \tag{44}$$

holds true. We employ (5) to obtain

$$\int\limits_\Omega g(x,u_m(T,x))dx \leqslant c_1 \int\limits_\Omega G(u_m(T,x))dx < c_2. \tag{45}$$

Hence, sequence $w_m(T)$ is bounded in $L_2(\Omega)$ and by Lemma 4 we can choose a subsequence converging to $w(T)$ weakly in $L_2(\Omega)$. We observe that then $\|w\|_2^2 = \lim(w,w_m) \leqslant \liminf \|w\|_2\|w_m\|_2$. It implies the inequality

$$\liminf \|g(x,u_m(T))\|_{L_1(\Omega)} \geqslant \|g(x,u(T))\|_{L_1(\Omega)}. \tag{46}$$

Integrating inequality (45) w.r.t. T, we obtain the sequence w_m is bounded in $L_2(D^T)$ and by Lemma 4 we can choose a subsequence converging to w weakly in $L_2(D^T)$.

In order to prove that $w' \in L_2(D^T)$, we apply condition (18):

$$\left[\left((g^{\frac{1}{2}}(x,u_m))'\right)^2\right] = \left[\left(\frac{g'(x,u_m)u_m'}{2g^{\frac{1}{2}}(x,u_m)}\right)^2\right] \leqslant \left[\frac{\alpha(g'(x,u_m))^2(u_m')^2}{2ug'(x,u_m)}\right]$$
$$= \frac{\alpha}{2}\left[\beta'(x,u_m)(u_m')^2\right] = \frac{\alpha}{2}I_\beta < c_3.$$

Hence, w_m' converges weakly to $\overline{w}$ in $L_2(D^T)$. Then, $[w_m\varphi'] = -[w_m'\varphi]$, $\varphi \in C_0^\infty(D^T)$. Passing to the limit, we obtain $[w\varphi'] = -[\overline{w}\varphi]$. Thus, $\overline{w} = w' = (g^{\frac{1}{2}}(x,u))_t'$, (44) is proven and $\beta'(x,u)u'u \in L_1(D^T)$.

We let $u^{(h)} = u^{(h,\infty)}$. By Lebesgue dominated convergence theorem we can pass to the limit in (43) first as $k \to \infty$

$$[\beta'(x,u)u'u^{(\varepsilon)}\xi] + (\chi, u^{(\varepsilon)}\xi)_{D^T} = 0,$$

and the as $\varepsilon \to 0$. We obtain ($u^{(0)} = u$)

$$[\beta'(x,u)u'u\xi] + (\chi, u\xi)_{D^T} = 0.$$

Since $\beta'(x,u)u'u \in L_1(D^T)$, passing to the limit by an appropriate non-decreasing sequence $\xi_m \to 1$, such that $u\xi_m \to u$ in $V(D^T)$, we obtain

$$[\beta'(x,u)u'u] + (\chi, u)_{D^T} = 0.$$

Applying (44), we get

$$(-\chi, u)_{D^T} = [\beta'(x,u)u'u] = \langle g(x,u(T)) - g(x,u(0))\rangle. \tag{47}$$

We then employ the monotonicity of operator $\tilde{a}$:

$$X_m = (\tilde{a}(u_m) - \tilde{a}(h), u_m - h)_{D^T} \geqslant 0, \qquad \forall h \in V(D^T).$$

Equations (40) as $\varphi = u_m$ imply easily the relations

$$(\tilde{a}(u_m), u_m)_{D^T} = \|g_m(x,u_m(0))\|_{L_1(\Omega)} - \|g_m(x,u_m(T))\|_{L_1(\Omega)}.$$

Hence,

$$X_m = \|g_m(x,u_m(0))\|_{L_1(\Omega)} - \|g_m(x,u_m(T))\|_{L_1(\Omega)} - (\tilde{a}(u_m), h)_{D^T} - (\tilde{a}(h), u_m - h)_{D^T}.$$

It follows from (46) that

$$0 \leqslant \limsup X_m \leqslant \|g(x,u(0))\|_{L_1(\Omega)} - \|g(x,u(T))\|_{L_1(\Omega)} - (\chi, h)_{D^T} - (\tilde{a}(h), u - h)_{D^T}.$$

Applying (47), we get

$$(\chi - \tilde{a}(h), u - h)_{D^T} \geqslant 0.$$

We let $h = u - \lambda\omega$, $\lambda > 0$, $\omega \in L_\infty((0,T); \mathring{W}^1_{G,B}(\Omega))$, then

$$\lambda(\chi - \tilde{a}(u - \lambda\omega), \omega)_{D^T} \geqslant 0.$$

Letting $\lambda \to 0$, we obtain $(\chi - \tilde{a}(u), \omega)_{D^T} \geqslant 0$, $\forall \omega \in L_\infty((0,T); \mathring{W}^1_{G,B}(\Omega))$. Thus, $\chi = \tilde{a}(u)$.

5. Examples

We provide examples of equations satisfying assumptions in the Introduction.

5.1. Example 1.

Let us consider the equation

$$(\gamma(u))_t = \sum_{i=1}^{n} (B_i'(u_{x_i}) + \Psi_i(x))_{x_i} + \Phi(x), \tag{48}$$

where B_i are N-functions, $\Psi_i(x) \in L_{\overline{B}_i}(\Omega)$, $\Phi(x) \in L_{\overline{G}}(\Omega)$, $\gamma'(u)$ is an even unbounded function decaying on $(0,\infty)$ such that the condition (5) holds true. Then N^- is empty and $N^+ = \{1\}$.

It is easy to make sure that conditions (7), (12)–(16) are satisfied and Theorem 1 hold true.

Despite of a long list of conditions in the Introduction, there is a wide class of equations satisfying these conditions. However, we restrict ourselves by a particular example.

5.2. Example 2.

We introduce the notation

$$t^{[a,b]} = \begin{cases} |t|^a, & \text{as } |t| < 1, \\ |t|^b, & \text{as } |t| \geqslant 1. \end{cases}$$

Let $n = 2$ and domain Ω be bounded. We choose N-functions $B_1(s)$, $B_2(s)$, $G(s)$, as well as functions $\beta_1(x,u)$, $\gamma(x,u)$, $a(x,u,p)$, $b(x,u,p)$ as follows:

$$B_1(s) = s^{5/2},\ B_2(s) = s^{3/2},\ G(s) = |s|^{3/2} + |s|^{5/2},\ \gamma'(s) = \frac{3}{2}|s|^{-1/2} + \frac{5}{2}s^{1/2},$$

$$a(x,u,p) = 2/5B_1(p_1) + 2/3B_2(p_2)\frac{2+|x|}{1+|x|} + \frac{p_1^2 + |p_2|^{5/4}}{|u|+1} + u^2,$$

$$b(x,u,p) = a_u(x,u,p) + u|p_1|^{[2,1/2]}, \quad \beta(x,u) = (3|u|^{1/2} + \frac{5}{3}|u|^{3/2})\mathrm{sgn}u\frac{2+|x|}{1+|x|}.$$

Let us check that operator $\widetilde{a}$ is monotonous. Calculating the Hessian of the function $\frac{p_1^2}{|u|+1}$, we see that it is nonnegative and the function is thus convex. Function $\frac{|p_2|^{5/4}}{|u|+1}$ is also convex. Let us prove the inequality

$$F = (|s_1|^{3/2}\mathrm{sign}s_1 - s_2^{3/2}\mathrm{sign}s_2)(s_1 - s_2) + (u_1 - u_2)^2 + (|s_1|^{[2,1/2]} - |s_2|^{[2,1/2]})(u_1 - u_2) \geqslant 0$$

which together with the convexity of the functions mentioned above ensure monotonicity condition (6).

1) Suppose first that $s_1 s_2 \leqslant 0$. Then it is sufficient to establish the inequality

$$(|s_1|^{5/2} + |s_2|^{5/2}) + (u_1 - u_2)^2 + (|s_1|^{[2,1/2]} - |s_2|^{[2,1/2]})(u_1 - u_2) \geqslant 0,$$

which can be easily checked by considering the cases $|s_1| < 1$ and $|s_1| \geqslant 1$ and employing inequalities $B^2 < 4AC,\ A > 0 \Rightarrow A + B + C \geqslant 0$.

2) Suppose that s_1, s_2 have the same sign. After redenoting we can assume that these numbers are nonnegative and $s_1 \geqslant s_2$. Then

$$(|s_1|^{3/2}s_1 - |s_2|^{3/2}s_2)(s_1 - s_2) \geqslant (s_1 - s_2)^2(s_1^2 + s_2^2)/(|s_1|^{3/2} + |s_2|^{3/2}) \geqslant (s_1 - s_2)^2|s_1|^{1/2}/2;$$

$$|s_1|^{[2,1/2]} - |s_2|^{[2,1/2]} \leqslant 2(s_1 - s_2)|s_1|^{[1,-1/2]}.$$

Now it is easy to make sure that function F is nonnegative.

Employing the established relations, we complete the proof of the monotonicity of operator $\widetilde{a}$. By direct calculations we find that $\Theta(s) = s^{1/30}$, integral (10) diverges and $B^*(s) = (s/30)^{30}$. It is easy to check that these functions satisfy conditions (5), (7), (12)–(16) and Theorem 1 holds true.

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Elina Radikovna Andriyanova
Ufa State Aviation Technical University,
K. Marx str., 12,
450074, Ufa, Russia
E-mail: Elina.Andriyanov@mail.ru

Mukminov Farit Khamzaevich
Bashkir State University,
Z. Validi str., 32,
450074, Ufa, Russia
E-mail: mfkh@rambler.ru