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DIFFERENT TYPES OF LOCALIZATION FOR EIGENFUNCTIONS OF SCALAR MIXED BOUNDARY VALUE PROBLEMS IN THIN POLYHEDRA

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Abstract. We construct asymptotics for the eigenvalues and eigenfunctions of the Laplace operator in a thin polyhedron with parallel closely spaced bases and skewed narrow lateral faces. On the bases we impose the Dirichlet conditions, while on the lateral faces the Dirichlet or Neumann conditions are imposed. Their distribution over the faces, as well as the slope of the latter, significantly affect the behavior of eigenfunctions when the domain becomes thinner. We find situations, in which the eigenfunctions are distributed along the entire polyhedron and localized near its lateral faces or vertices. The results are based on the analysis of the spectrum (cut–off point, isolated eigenvalues, threshold resonances, etc.) of auxiliary problems in a half–strip and a quarter of a layer with skewed end and lateral sides, respectively. We formulate open questions concerning both spectral and asymptotic analysis.

Keywords: Laplace operator, mixed problem in thin polyhedron, asymptotics for eigenvalues, localization of eigenfunctions, essential and discrete spectrum of problems in infinite domains.

Mathematics Subject Classification: 35J05

1. Introduction

1.1. Prelude. The eigenfunctions of the Dirichlet problem for the Laplace operator in a “flattened pyramidal” polyhedron are localized near the vertex furthest from the flat base, see Fig. 1. This result was obtained in paper [1] and it joins the results of the works [2]–[6] on thin domains of varying width. In the present paper we study the eigenfunctions of mixed boundary value problems in thin polyhedra with parallel bases and skewed narrow lateral faces, see Figs. 2a and 3a. On the bases we impose the Dirichlet condition, while on the lateral faces we impose either the Dirichlet or Neumann condition. Depending on the particular choice, one or another localization of eigenfunctions is realized or the absence of localization. Namely, we describe the situations, in which several first eigenfunctions are localized respectively near the edges or angles of thin plate or they are distributed along the entire plate. The absence of localization of eigenfunctions or its characteristics are determined by the properties of spectra (presence of localized eigenvalues and threshold resonances) of model problems on pointed semi–infinite strip (hereafter, semi–strip) or a quarter of layer with a skewed lateral surface, which are considered in Section 2. While the flat problem has already been completely studied

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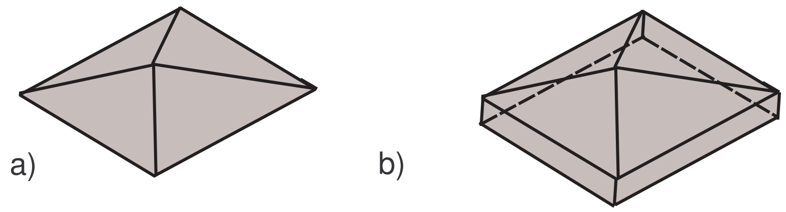

FIGURE 1. Pyramidal polyhedra

(see [7]–[9] and other works), the known results for the spatial problem are fragmentary (see [10]–[12]), and in what follows for the completeness we choose the geometric shape, see Fig. 3, which requires an independent study, in particular, it allows us to point out new approaches for analysis of the discrete spectrum.

The choice of particular thin polyhedra is motivated by the possibility of constructing from them thin–walled boxes and cubes[1], see Fig. 2b and 4b. It should be stressed that the thin–walled constructions appears everywhere, however, no full studies were made. The scalar Neumann problems are rather simple, while the vector problems of elasticity theory are very complicated. The scalar Dirichlet problem considered in the paper [12] and in the present work has an intermediate position. We note that the localization effect in constructions from [12] was first of all achieved by varying the widths of various elements (partition walls), but for the box the localization appears for the constant width of all walls due to the slope of walls with the Neumann conditions in an auxiliary polyhedron, Fig. 2a. We succeed to find, and, what is important, to justify rigorously the phenomenon of edge localization of eigenfunctions, which just was discussed in [12]. Finally, the results of [10], [11] allows one, by the scheme presented in Section 5, to verify that the first eight eigenvalues of the Dirichlet problem in the thin–walled cube are concentrated near its vertices, see Fig. 3.

1.2. Formulation of first group of problems. The mixed spectral problem

$$-\Delta_x u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), \quad x \in \Omega^\varepsilon, \tag{1.1}$$

$$u^\varepsilon(x) = 0, \quad x \in \Gamma^\varepsilon_D, \tag{1.2}$$

$$\partial_{\nu(x)} u^\varepsilon(x) = 0, \quad x \in \Gamma^\varepsilon_N := \partial\Omega^\varepsilon \setminus \overline{\Gamma^\varepsilon_D}, \tag{1.3}$$

is posed in a thin polyhedra

$$\Omega^\varepsilon = \{x = (y,z) : y_1 = x_1 \in (-1,1), |y_2| = |x_2| < 1 - z, z = x_3 \in (0,\varepsilon)\}, \tag{1.4}$$

see Fig. 1a. Here ε is a small positive parameter, $\nabla_x = \operatorname{grad}$, $\Delta_x = \nabla_x \cdot \nabla_x$ is the Laplace operator, $\partial_\nu = \partial_{\nu(x)}$ is the derivative along the outward normal and

$$\Gamma^\varepsilon_N = \{x \in \partial\Omega^\varepsilon : z \in (0,\varepsilon)\} \tag{1.5}$$

or

$$\Gamma^\varepsilon_D = \{x \in \partial\Omega^\varepsilon : |y_1| < 1\}. \tag{1.6}$$

In the first case the Neumann conditions are imposed on the entire lateral surface of the polyhedra Ω^ε, while in the second case it is imposed only on two thin faces

$$\Gamma^\varepsilon_{\sharp\pm} = \{x \in \Omega^\varepsilon : y_1 = \pm 1, |y_2| < 1, z \in (0,\varepsilon)\}, \tag{1.7}$$

perpendicular to the abscise axis. As the lower base of polyhedron (1.4) the square $\square_1 = (-1,1)^2$ serves.

[1] In some sense they can interpreted as fragments of spatial quantum waveguides, cf. the monograph [13].

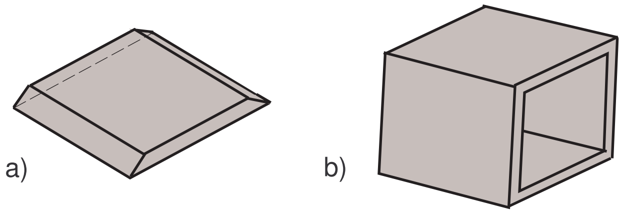

FIGURE 2. Thin polyhedron (a) and a box formed by four polyhedra (b)

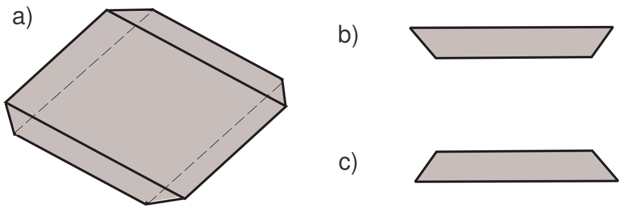

FIGURE 3. Thin polyhedron with differently skewed lateral faces (a) and its sections parallel to the ordinate (b) and abscise axis (c)

1.3. First group of asymptotic results. One of the aims of work is to construct the asymptotics for the eigenvalues

$$0 < \lambda_1^\varepsilon < \lambda_2^\varepsilon \leqslant \lambda_3^\varepsilon \leqslant \ldots \leqslant \lambda_m^\varepsilon \leqslant \cdots \to +\infty \tag{1.8}$$

and associated eigenfunctions $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon, \ldots, u_m^\varepsilon, \ldots \in H_0^1(\Omega^\varepsilon; \Gamma_D^\varepsilon)$ of the problem (1.1)–(1.3) as $\varepsilon \to +0$. The variational formulation of this problem is given by the integral identity [14], [15]

$$\left(\nabla_x u^\varepsilon, \nabla_x \psi^\varepsilon\right)_{\Omega^\varepsilon} = \lambda^h (u^\varepsilon, \psi^\varepsilon)_{\Omega^\varepsilon} \qquad \forall \psi^\varepsilon \in H_0^1(\Omega^\varepsilon; \Gamma_D^\varepsilon). \tag{1.9}$$

Here $(\,\cdot\,,\,\cdot\,)_{\Omega^\varepsilon}$ is the natural scalar product in the Lebesgue space $L^2(\Omega^\varepsilon)$, scalar or vector, while $H_0^1(\Omega^h; \Gamma_D^\varepsilon)$ is the Sobolev space of functions obeying the Dirichlet condition (1.2).

The pairs $\{\lambda^\varepsilon; u_m^\varepsilon\}$ are called the eigenpairs of problem (1.1)–(1.3). The first eigenvalue is simple, while the associated eigenfunction can be chosen positive in $\Omega^\varepsilon \cup \Gamma_N^\varepsilon$.

In the situation (1.6) the eigenvalues admit a simple asymptotic representation

$$\lambda_{(p,q)}^\varepsilon = \frac{\pi^2}{\varepsilon^2} + \frac{\pi^2}{4}\left(p^2 + q^2\right) + \widetilde{\lambda}_{(p,q)}^\varepsilon, \tag{1.10}$$

where $\widetilde{\lambda}_{(p,q)}^\varepsilon$ is a small remainder, see Section 5.3. Of course, the eigenvalues (1.10) indexed by the subscripts $q \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ are to be regrouped into the monotone sequence (1.8). The eigenfunctions becomes

$$u_{(p,q)}^\varepsilon(x) = \sin\left(\pi \frac{z}{\varepsilon}\right) \cos\left(\frac{\pi}{2} p(y_1 - 1)\right) \sin\left(\frac{\pi}{2} q(y_2 - 1)\right) + \widetilde{u}_{(p,q)}^\varepsilon(x), \tag{1.11}$$

with a small remainder $\widetilde{u}_{(p,q)}^\varepsilon$, see Section 3.5. It is easy to see that the leading terms in the formulas (1.10) and (1.11) form an eigenpair of the problem (1.1)–(1.3) in the parallelepiped $\Omega_\square^\varepsilon = \square_1 \times (0, \varepsilon) \subset \mathbb{R}^3$, where the separation of variables is possible.

The eigenpairs of the problem (1.1)–(1.3) acquires a completely different asymptotics structure in the case of the Neumann condition on the entire lateral surface (1.5), see Sections 4.1,

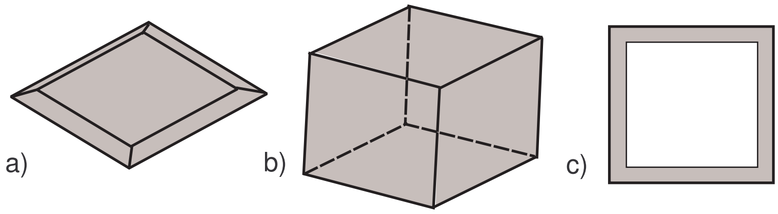

FIGURE 4. A thin truncacted pyramide (a), a thin–walled cube formed by six pyramides (b), and its central section (c)

4.4, namely,

$$\lambda^{\varepsilon}_{(p,j)} = \frac{\Lambda_1}{\varepsilon^2} + \frac{\pi^2}{4}p^2 + \widetilde{\lambda}^{\varepsilon}_{(k,j)}, \tag{1.12}$$

$$u^{\varepsilon}_{(p,j)}(x) = \frac{1}{\varepsilon}\sum_{\pm} K^{\pm}_{p,j} W_1\Big(\frac{1 \mp y_2}{\varepsilon}, \frac{z}{\varepsilon}\Big) \cos\Big(\frac{\pi}{2}p(y_1 - 1)\Big) + \widetilde{u}^{\varepsilon}_{(k,j)}(x), \tag{1.13}$$

where $K^{\pm}_{k,j}$ are some coefficients, $k \in \mathbb{N}$, $j = 1, 2$ and $p \in \mathbb{N}_0$, while $\Lambda_1 \in (\pi^2/2, \pi^2)$ is the eigenvalue and the associated eigenfunction $W_1 \in H^1(\Pi)$ exponentially decaying at infinity (see Sect. 2.1) of the auxiliary problem (2.1)–(2.3) on the pointed semi–strip

$$\Pi = \{\eta = (\eta_1, \eta_2) \in \mathbb{R}^2 : \eta_2 \in (0, 1), \eta_1 > \eta_2\}. \tag{1.14}$$

The set $(-1, 1) \times \Pi$ is obtained by the formal passage to the limit $\varepsilon = 0$ after the following rescaling of the ordinate and applicate:

$$x \quad \mapsto \quad (y_1, \eta_1^{\pm}, \eta_2^{\pm}) = \Big(y_1, \frac{1 \mp y_2}{\varepsilon}, \frac{z}{\varepsilon}\Big). \tag{1.15}$$

Owing to the definition (1.4) the result is independent of the subscript $\pm$ of the face

$$\Gamma^{\varepsilon}_{\pm} = \{x : |y_1| < 1, \pm y_2 = 1 - z, z \in (0, \varepsilon)\}. \tag{1.16}$$

The eigenfunctions (1.13) are localized in a small neighbourhood of narrow faces (1.16) and they exponentially fast decay while going from the faces, see Section 4.3.

We stress that both formulas (1.11) are (1.13) provide just some non–normalized eigenfunctions, but in the next sections we suppose that they obey the orthogonality and normalization conditions

$$\big(u^{\varepsilon}_j, u^{\varepsilon}_k\big)_{\Omega^{\varepsilon}} = \delta_{j,k}, \quad j, k \in \mathbb{N}, \tag{1.17}$$

where $\delta_{j,k}$ is the Kronecker delta.

1.4. Brief review of known forms. A wide literature is devoted to the localization of eigenfunctions of boundary value problems, see the works [1]–[6], [16]–[18], the review [19] and many other publications. As it has been already mentioned, for thin domains with the Dirichlet condition on one or both bases the concentration of eigenfunctions is observed near the height with the maximal length, see Fig. 2a and 2b, while there are known shapes of domains, for which the discussed phenomenon appears in a different way, see Fig. 2c–2f.

We note that in the paper [20] there was found a similar phenomenon of concentration of modes of eigenoscillations of cylindrical elastic (homogeneous and isotropic) thin plates with with rigidly fixed bases and a narrow side surface, which is free of external forces.

The choice of the polyhedron (1.4) is motivated by the following observation: the odd in the case (1.5) and the even in the case (1.6) continuation of the eigenfunction from the horizontal wall via the sides (1.16) and the repetition of this procedure for two formed vertical walls gives

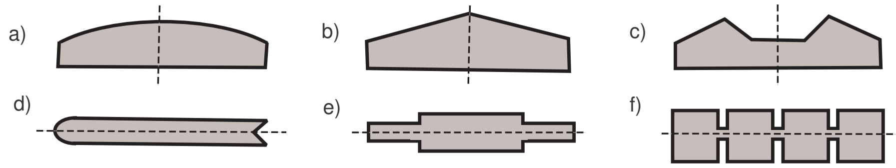

FIGURE 5. Localization near the height of maximal length (a and b). Localization near two points or circumference (after rotation of section) (c), left end (d), on the segment (e) and on a single cell (f). The dashed–dotted line indicates the rotation axes under the admitted passage from flat figures to spatial bodies.

the associated with the same eigenvalue eigenfunction of the mixed boundary value in a thin box, see Fig. 1b,

$$\mathcal{K}^\varepsilon = \left(\square_1 \setminus \overline{\square_{1-\varepsilon}}\right) \times (-1,1), \tag{1.18}$$

where $\square_a = \{\eta = (\eta_1,\eta_2) \in \mathbb{R}^2 : |\eta_j| < a, j = 1,2\}$ is the square with side $2a$. At the same time on the outer and inner lateral surfaces of the box $\{x \in \partial\mathcal{K}^\varepsilon : |y_1| < 1\}$ the Dirichlet conditions are imposed, while on the ends $\left(\square_1 \setminus \overline{\square_{1-\varepsilon}}\right) \times \{\pm 1\}$ the Neumann or Dirichlet condition is imposed. The asymptotic formulas obtained in Section 3 and Section 4 show that eigenfunctions of the aforementioned problem in the thin–walled construction (1.18) can have completely different behavior as $\varepsilon \to +0$, namely, they can concentrate near the edges or appear everywhere in the box.

The found options of distribution of eigenfunctions appear also in the Dirichlet problem for the Laplace operator in a thin–walled (hollow) cube

$$\mathcal{K}^\varepsilon \cup \left(\square_1 \times \left((-1,-1+\varepsilon) \cup (1-\varepsilon,1)\right)\right), \tag{1.19}$$

whereby it primarily manifests a different method of localization, already mentioned in Section 1.1: the first eight eigenfunctions are concentrated near the vertices of the cube and decay exponentially far from the vertices. This property of the eigenfunctions is derived using the approach described in Section 5 on the base of the results in [10], [11] on the spectrum of the Dirichlet problem in the «Fichera layer»

$$\bigcup_{j=1,2,3} \{\xi = (\xi_1,\xi_2,\xi_3) : \xi_j < 1, \quad \xi_k > 0, k = 1,2,3\},$$

called similarly to the well–known Fichera angle [21]. For a detailed description of the reasons for such near–vertex localization, a thin body will be formed in one of the sections, see Fig. 3a, and in Section 2 we carry out a spectral analysis, namely, we determine the essential spectrum and veirfy the non–emptiness of the discrete spectrum for a model boundary value problem on a quarter of layer with differently skewed lateral edges

$$\Xi = \{\xi = (\xi_1,\xi_2,\xi_3) : \xi_1 > -\xi_3, \xi_2 > \xi_3, \xi_3 \in (0,1)\}. \tag{1.20}$$

We stress that the discrete spectrum is absent for the mixed boundary value problem on the quarter of layer

$$\Xi_\sqcup = \{\xi \in \mathbb{R}^3 : \xi_1 > 0, \xi_2 > \xi_3, \xi_3 \in (0,1)\} \tag{1.21}$$

with one flat side, while the case of quarter of layer

$$\Xi_\wedge = \{\xi \in \mathbb{R}^3 : \xi_1 > \xi_3, \xi_2 > \xi_3, \xi_3 \in (0,1)\} \tag{1.22}$$

with same skewed lateral faces is directly related with the thin–walled cube (1.19). The presence of an eigenvalue in the discrete spectrum of the mixed boundary value problem in the domain (1.22) was established in the work [11].

1.5. Aggravation of localization effect. As it has been mentioned, we consider one more domain, in which the problem (1.1)–(1.3) is posed. This polyhedron is shown on Fig. 3 and is defined by the formula

$$\Omega^\varepsilon = \{x : |y_1| < 1 + z, |y_2| < 1 - z, z \in (0, \varepsilon)\}, \tag{1.23}$$

while the set, on which the Neumann condition is imposed, is given by (1.5). We stress that in contrast to (1.4), the pairs of sides (1.16) and (1.7) of the polyhedron (1.23) are located at angles $\pi/4$ and $3\pi/4$ to the plane $\{x : z = 0\}$. The quarter of layer (1.20) has the same angles of the lateral faces, but for the quarter of layer (1.22) both angles are $\pi/4$.

In Section 5 we shall demonstrate that the eigenfunctions $u_1^\varepsilon, \ldots, u_4^\varepsilon$ associated with the first four eigenvalues in the sequence (1.8) feature the concentration in $c\varepsilon$–neighbourhoods of short edges incident to the points

$$P^{\pm+} = (\pm 1, +1, 0), \qquad P^{\pm-} = (\pm 1, -1, 0), \tag{1.24}$$

and the exponential decay far from them, while the eigenvalues have the asymptotics

$$\lambda_k^\varepsilon = \varepsilon^{-2} M_1 + \widetilde{\lambda}_k^\varepsilon, \qquad k = 1, \ldots, 4, \tag{1.25}$$

where $M_1 \in (0, \Lambda_1)$ is the eigenvalue of the problem (2.13) in the infinite domain (1.20), see Section 2.3, and $\widetilde{\lambda}_k^\varepsilon$ is a small remainder, see Section 5.2. Due to the reasons mentioned in Section 5.3, the author has no information about the eigenvalues $\{\lambda_k^\varepsilon; u_k^\varepsilon\}$ for $k > 4$.

1.6. Preliminary description of results. In the next section we study mixed spectral boundary value problem in the half–strip (1.14) and quarter of layer (1.20). While for the planar problem all results presented in Section 2.1 are known, for the spatial problem, in Sections 2.2–2.4 we have to prove the formula for the essential spectrum (Theorem 2.1), the non–emptiness of the discrete spectrum (Theorem 2.2), as well as the exponential decay at infinity of the eigenfunction (Theorem 2.3). We stress that the mentioned results is the key point of the work and, as in the paper [12], they serve as the base for finding out the near–vertex localization of eigenfunctions. However, in Section 2.5 we count all disadvantages of the analysis of spatial problem being the obstacle for a complete study of the problem in the polyhedron (1.23), in particular, we discuss the phenomenon of threshold resonance and its influence on the asymptotic structures.

In Section 3 we provide asymptotic formulas for the spectral pairs of problem (1.1)–(1.3) in the situation (1.6) including the spectral pairs of problem (3.3) in the square $\square_1$. The construction and justification of asymptotics are traditional, see, for instance, [22]–[25], while the passage from the Neumann conditions to the Dirichlet condition requires a modification of the procedure. The calculations and arguing is presented in detail in Sectoin 3 for the reader's convenience and also as a preliminary material for clarifying the differences in constructing and justifying the asymptotic formulas in further sections under the appearance of the localization effect. We first construct a formal asymptotics and then provide the classical lemma 3.1 on almost eigenvalues and eigenvectors, which is used for finding the eigenvalues of original problems with the constructed asymptotics, and finally, Lemma 3.2 allows us to establish the final statements (Theorems 2.1 and 2.2) on asymptotic expansions of the eigenpairs $\{\lambda_m^\varepsilon; u_m^\varepsilon\}$.

In Section 4 we study the spectrum of problem (1.1)–(1.3) in the situation (1.5), which features the concentration of eigenfunctions near narrow faces (1.16) and this is reflected in change of asymptotic ansätze, which now involve the eigenpair $\{\Lambda_1; W_1\}$ of the problem (2.1)–(2.3), as well as the eigenpairs $\{\mu_m; v_m\}$ of the Neumann problem (4.1) for an ordinary differential equation on the segment $(-1, 1) \ni y_1$. On one hand, the procedure of justification of asymptotics becomes simpler, since by imposing artificial boundary conditions on the central plane $\{x : y_2 = 0\}$ of the body Ω^ε the eigenvalues become simple. On the other hand, the proof of

Lemma 4.1 on convergence required a significant revision of the material from Section 3.4. Finally, in Section 4.5 we point out other series of eigenvalues with stable asymptotics, which are constructed along the lines of Section 3, but with changes in arguing on imposing the boundary conditions on the sides of square $\square_1$.

In Section 5 we provide asymptotic results on the problem (1.1)–(1.3) in the domain (1.23). The near–vertex localization originates from the found in Section 2.3 point M_1 in the discrete spectrum of problem (2.13) in the quarter of layer (1.20). The presence of eigenpair $\{M_1; V_1\}$ simplifies essentially the asymptotic ansätze and the justification also becomes trivial thanks to imposing the artificial boundary conditions on two symmetry planes of the body (1.23), see Theorem 5.2 on the eigenvalues $\lambda_1^\varepsilon, \dots, \lambda_4^\varepsilon$. At the same time, because of the incompleteness of spectral analysis of problem (2.13), cf. the comments in Section 2.5, we failed to get the information about the eigenpairs with the indices $m > 4$. Other open questions are discussed in Section 5.3.

The methods and results of the asymptotic analysis carried out below admit various generalizations (of course, under understandable restrictions), namely, a variation in the number of faces, solutions of dihedral angles and the distribution of boundary conditions (1.2) and (1.3), as well as for second–order scalar equations in divergence form with smooth coefficients, but such generalizations are left without attention for clarity and simplification of asymptotic constructions and, of course, to facilitate the formulation of results.

2. Spectral problems in infinite domains

2.1. Auxiliary planar problem. In the half–strip (1.14) with a skewed end $\gamma = \{\eta : \eta_2 \in (0,1), \eta_1 = \eta_2\}$ and lateral sides $\sigma^j = \{\eta : \eta_2 = j, \eta_1 > j\}$, $j = 0,1$, we consider the problem

$$-\Delta_\eta W(\eta) = \Lambda W(\eta), \quad \eta \in \Pi, \tag{2.1}$$

$$W(\eta) = 0, \quad \eta \in \sigma := \sigma^0 \cup \sigma^1, \tag{2.2}$$

$$\partial_{\nu(\eta)} W(\eta) = 0 \quad \text{or} \quad W(\eta) = 0, \quad \eta \in \gamma. \tag{2.3}$$

The latter boundary conditions are denoted $(2.3)_N$ or $(2.3)_D$, respectively.

The continuous spectrum of both problems is the ray $[\pi^2, +\infty)$. The classical trick [26] shows that the point spectrum of the Dirichlet problem (2.1)–$(2.3)_D$ is empty.

It is known that the discrete spectrum of the mixed boundary value problem (2.1)–$(2.3)_N$ consists of the single point $\Lambda_1 \in (0, \pi^2)$; the approximate value $0.93\pi^2$ was calculated in work [27], while the existence and uniqueness was rigorously established in the works [7], [8], [16]. The corresponding eigenfunction $W_1 \in H_0^1(\Pi; \sigma)$ decays at infinity with the rate $O\big(e^{-\eta_1\sqrt{\pi^2-\Lambda_1}}\big)$ and can be represented as (see, for instance, [28, Ch. 2]),

$$W_1(\eta) = \chi(r_1) C_1 r_1^{\frac{2}{3}} \sin\frac{2\varphi_1}{3} + \widehat{W}_1(\eta), \tag{2.4}$$

where C_1 is the so–called intensity factor, $(r_j, \varphi_j) \in \mathbb{R}_+ \times (0, (2j+1)\pi/4)$ is the system of polar coordinates centered at the point $\mathcal{P}_j = (j,j)$ (Fig. 6a, 6b), $j = 0,1$, $\widehat{W}_1 \in H^2(\Pi)$, $W_1(\mathcal{P}_0) = 0$ and $W_1(\mathcal{P}_1) = 0$, while $\chi \in C^\infty(\mathbb{R})$ is an etalon cut–off function,

$$\chi(r) = 1 \quad \text{for} \quad r < \frac{1}{3} \quad \text{and} \quad \chi(r) = 0 \quad \text{for} \quad r > \frac{2}{3}. \tag{2.5}$$

We note that near the point $\mathcal{P}_0$ the function W_1 behaves as $C_0 r_0^2 \sin(2\varphi) + O(r_0^4)$, i.e., it turns out to be smooth. We normalize the first eigenfunction in the space $L^2(\Pi)$

$$\|W_1; L^2(\Pi)\| = 1. \tag{2.6}$$

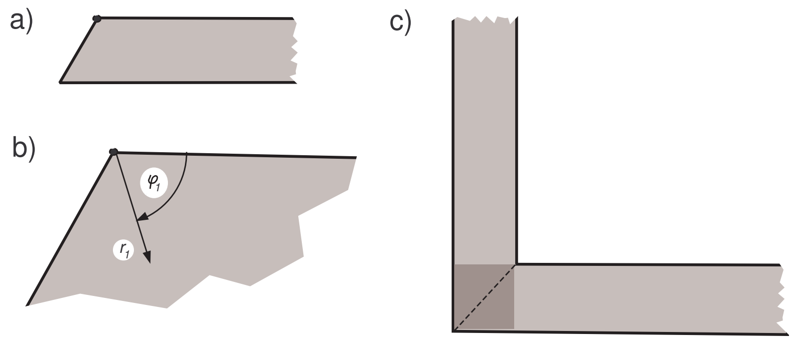

FIGURE 6. Pointed semi–strip (a), angle $3\pi/4$ and polar coordinates (r_1, φ_1) (b). L–shaped domain and deeply toned unit square in it (c)

By the Krein — Rutman theorem, see, for instance, [29, Thm. 1.2.5], it can be fixed positive in $\Pi \cup \gamma$. In this case the coefficient C_1 is positive since, except for one detached in the expansion (2.4), all harmonic functions in the angle $3\pi/4$ with the Dirichlet and Neumann conditions on its sides are sign–changing; this fact is to be applied iteratively. Finally, the expansion at infinity

$$W_1(\eta) = K_1 e^{-\beta_1 \eta_1} \sin(\pi \eta_2) + \widetilde{W}_1(\xi) \tag{2.7}$$

is valid, in which $\widetilde{W}_1(\xi) = O\big(e^{-\beta_2 \eta_1}\big)$ as $\eta_1 \to +\infty$ and

$$\beta_k = \sqrt{\pi^2 k^2 - \Lambda_1}, \qquad k \in \mathbb{N}. \tag{2.8}$$

The coefficient K_1 is positive since again among the terms $K_k e^{-\beta_k \eta_1} \sin(\pi k \eta_2)$ of the Fourier series for the function W_1, which converges as $\eta_1 > 1$, only the term detached in the relation (2.7) is sign–definite.

In view of the importance of result on the discrete spectrum of mixed boundary value problem in Π and for the reader's convenience, we present simple and shortened proofs.

Lemma 2.1. *On the interval* $(0, \pi^2)$ *the problem* (2.1)–(2.3)$_N$ *has a unique eigenvalue* $\Lambda_1 \in (\pi^2/2, \pi^2)$.

Proof. The existence of eigenvalue in the discrete spectrum was verified in [7], [8], [16] and others. Let us show how to establish its uniqueness and get a simplest lower bound. By the even continuation through the diagonal of first quadrant (dash–dotted line on Fig. 6c), we reduce the problem (2.1)–(2.3)$_N$ to the Dirichlet problem in L–shaped domain

$$\mathbf{L} = \bigcup_{j=1,2} \Big\{ \eta : \eta_j > 0, 0 < \eta_{3-j} < 1 \Big\},$$

which we partition in two ($j = 1, 2$) semi–strips $\varpi_j = \{\eta \in \mathbf{L} : \eta_j > 1\}$ with right ends and unit square $\blacksquare = (0,1)^2$ (deeply toned on Fig. 6c). By the Dirichlet condition on the boundary $\partial\mathbf{L}$ the one–dimensional Friedrichs inequality on the segment $(0, 1)$ shows that

$$\|\nabla_\eta W; L^2(\varpi_j)\|^2 \geqslant \pi^2 \|W; L^2(\varpi_j)\|^2 \quad \forall W \in H^1_0(\mathbf{L}). \tag{2.9}$$

The first two eigenvalues of the Laplace operator in the square $\blacksquare$ with the Dirichlet condition on two adjacent sides and the Neumann condition on two others are equal to $\frac{\pi^2}{2}$ and $\frac{5\pi^2}{2}$. The first eigenvalue is simple with the positive eigenfunction $\sin\left(\frac{\pi}{2}\eta_1\right) \sin\left(\frac{\pi}{2}\eta_2\right)$, which obeys the Neumann condition on the diagonal of the square, that is, the condition (2.3)$_N$ on γ, while the second eigenvalue is double. Thus, the relations hold:

$$\|\nabla_\eta W; L^2(\blacksquare)\|^2 \geqslant \frac{\pi^2}{2}\|W; L^2(\blacksquare)\|^2 \quad \forall \quad W \in H^1_0(\mathbf{L}) \tag{2.10}$$

$$\|\nabla_\eta W; L^2(\blacksquare)\|^2 \geqslant \frac{5\pi^2}{2}\|W; L^2(\blacksquare)\|^2 \quad \text{for all} \quad W \in H^1_0(\mathbf{L}), \tag{2.11}$$

$$\text{obeying the restriction} \quad \int\limits_{\mathbf{Q}} \sin\Big(\frac{\pi}{2}\eta_1\Big)\sin\Big(\frac{\pi}{2}\eta_1\Big)W(\eta)d\eta = 0.$$

Finally, the needed facts are ensured by the minimax principle, see [30, Thms. 10.2.1, 10.2.2] and respectively by the inequalities (2.9), (2.10) and inequalities (2.9), (2.11). The proof is complete. □

Remark 2.1. *Similar results are known also for the pointed strips*

$$\Pi^\alpha = \{\eta : \eta_2 \in (0,1), \eta_1 > \eta_2 \cot\alpha\}, \quad \alpha \in \Big(0, \frac{\pi}{2}\Big),$$

see papers [8], [9] *and others, however, the multiplicity of discrete spectrum increases unboundedly as* $\alpha \to +0$. *The paper* [9], *in which this fact was observed, there is a flaw: the constructed asymptotic approximation for the eigenfunction is not in the domain of the self–adjoint operator because of the singularity* $O\big(r^{\frac{\pi}{2(\pi-\alpha)}}\big)$ *at the point* $(\cot\alpha, 1)$. *The way of correcting this flaw was provided in the paper* [31].

2.2. Essential spectrum of problem in quarter of layer. The rescaling of all three coordinates

$$x \quad \mapsto \quad \xi = \varepsilon^{-1}\big(y_1+1, y_2+1, z\big) \tag{2.12}$$

with respect to the point $P^{--} \in \mathbb{R}^3$ and the formal passage to $\varepsilon = 0$ transforms the thin domain (1.23) into the set (1.20), on which we consider the mixed spectral boundary value problem

$$\begin{aligned} &-\Delta_\xi V(\xi) = MV(\xi), \quad \xi \in \Xi, \\ &V(\xi) = 0, \quad \xi \in \Theta := \Theta^1 \cup \Theta^2, \\ &\partial_{\nu(\xi)} V(\xi) = 0, \quad \xi \in \Upsilon := \Upsilon^0 \cup \Upsilon^1. \end{aligned} \tag{2.13}$$

Here $\Theta^j = \{\xi \in \partial\Xi : \xi_3 = j\}$, $j = 0, 1$, are quadrants, that is, the bases of the infinite polyhedron (1.20), and

$$\Upsilon^k = \{\xi \in \partial\Xi : \xi_k > (-1)^k\xi_3, \xi_3 \in (0,1)\}, \qquad k = 1, 2,$$

are its lateral sides. By the definition (1.23) of the thin finite polyhedron Ω^ε, the rescaling of the coordinates with respect to other points in the list (1.24) and rotations of the Cartesian coordinates, similar to (2.12), give the same quarter of layer (1.20).

To the variational formulation of the problem (2.13)

$$\big(\nabla_\xi V, \nabla_\xi \Psi\big)_\Xi = M\big(V, \Psi\big)_\Xi \quad \forall \quad \Psi \in H^1_0(\Xi; \Upsilon) \tag{2.14}$$

we assign [30, Ch. 10, Sect. 1] a self–adjoint positive definite unbounded operator B in the Hilbert space $\mathcal{L} = L^2(\Xi)$. The nearest aim is to confirm that the essential spectrum of the operator reads

$$\wp_e = [M_\dagger, +\infty) = [\Lambda_1, +\infty), \tag{2.15}$$

where $\Lambda_1 \in (\pi^2/2, \pi^2)$ is the eigenvalue of problem (2.1)–(2.3), see Lemma 2.1. We essentially reproduce the arguing from [10], [12].

We begin with confirming the inclusion $[\Lambda_1, +\infty) \subset \wp_e$. In order to do this, we define the singular Weyl sequence for the operator B at the point $M \geqslant M_\dagger$ as

$$\mathcal{Z}_k(\xi) = \|X_k Z_M; \mathcal{L}\|^{-1} X_k(\xi_1) Z_M(\xi), \quad k \in \mathbb{N}, \tag{2.16}$$

where

$$Z_M(\xi) = e^{i\xi_1\sqrt{M-M_\dagger}}W_1(\xi_2,\xi_3), \quad i=\sqrt{-1},$$
$$X_k(\xi_1) = \chi(\xi_1 - 2^{k+1}+1)\big(1-\chi(\xi_1-2^k)\big).$$

Here χ is the cut–off function (2.5). Thus, the support of the function (2.16) is located in the set Ξ_k^1, where

$$\Xi_k^j = \Big\{\xi \in \overline{\Xi} : \xi_1 \in \Big[2^k + \frac{j}{3}, 2^{k+1} - \frac{j}{3}\Big]\Big\}, \quad j=1,2.$$

As a result, the formula $\operatorname{supp}\mathcal{Z}_k \cap \operatorname{supp}\mathcal{Z}_j = \varnothing$ holds for $k\neq j$, as well as first two properties of the singular Weyl sequence

1° $\|\mathcal{Z}_k;\mathcal{L}\| = 1$,

2° $\mathcal{Z}_k \to 0$ weakly in $\mathcal{L}$

become clear. Since $X_k = 1$ on Ξ_k^2, by the identity (2.6) we have

$$\|X_k Z_M;\mathcal{L}\|^2 \geqslant \int\limits_{2^k+\frac{2}{3}}^{2^{k+1}-\frac{2}{3}} \int\limits_{\Pi} \big|W_1(\xi_2,\xi_3)\big|^2 d\xi_2 d\xi_3 d\xi_1 = 2^{k+1} - 2^k - \frac{4}{3}.$$

Moreover, $(\Delta_\xi + M)Z_M = 0$, and hence the function $(\Delta_\xi + M)\mathcal{Z}_k$ vanishes on the set Ξ_k^2 and

$$\|(B-M)(X_kZ_M);\mathcal{L}\|^2 \leqslant \int\limits_{\Xi_k^1\setminus\Xi_k^2} \big|\big[\Delta_\xi, X_k\big]Z_M\big|^2 d\xi \leqslant 2c_{\chi M}.$$

Thus, the third property is also obvious:

3° $\|B\mathcal{Z}_k - M\mathcal{Z}_k;\mathcal{L}\| \to 0$.

As a result $M \in \wp_e$ and $[\Lambda_1,+\infty) \subset \wp_e$ by the Weyl criterion, see [30, Thm. 9.1.2].

Now let $M \in (0, M_\dagger)$. In what follows we shall establish the unique solvability of the problem

$$\big(\nabla_\xi V, \nabla_\xi \Psi\big)_\Xi - M\big(V,\Psi\big)_\Xi + t\big(V,\Psi\big)_{\Xi(R)} = f(\Psi), \quad \Psi \in H_0^1(\Xi;\Upsilon), \tag{2.17}$$

in which $f \in \big(H_0^1(\Xi;\Upsilon)\big)^*$ is a linear continuous functional on the space $H_0^1(\Xi;\Upsilon)$, and a number $t>0$ and a bounded set $\Xi(R)\subset\Xi$ will be appropriately fixed. The fact that mapping

$$H_0^1(\Xi;\Upsilon) \ni V \quad\mapsto\quad \mathcal{B}_t(R)V = f \in \big(H_0^1(\Xi;\Upsilon)\big)^*$$

is an isomorphism yield the Fredholm property of the operatro $\mathcal{B}_0$ of the original problem (2.14), since the difference $\mathcal{B}_t(R) - \mathcal{B}_0$ is a compact operator due to the compactness of the embedding $H^1(\Xi) \subset L^2(\Xi(R))$, namely,

$$(\mathcal{B}_t(R)V - \mathcal{B}_0V, \Psi)_\Xi = t(V,\Psi)_{\Xi(R)}.$$

We begin with verifying simple and mostly known facts, see [10], [32] and others.

Proposition 2.1. *Let $T>1$ and Λ_1^T is the first eigenvalue of mixed boundary value problem on the trapezoid $\Pi^T = \{\eta : \eta_1 \in (\eta_2, T), \eta_2 \in (0,1)\}$*

$$\begin{aligned}&-\Delta_\eta W^T(\eta) = \Lambda^T W^T(\eta) \quad \textit{for} \quad \eta\in\Pi^T,\\ &W^T(\eta) = 0 \quad \textit{for} \quad \eta_2<\eta_1<T, \quad \eta_2 = 0 \quad \textit{or} \quad \eta_2 = 1,\\ &\partial_{\nu(\eta)}W^T(\eta) = 0 \quad \textit{for} \quad \eta\in\partial\Pi^T, \quad \eta_2\in(0,1).\end{aligned} \tag{2.18}$$

The function $(1,+\infty)\ni T\mapsto \Lambda_1^T$ is smooth and strictly monotonically increasing. It satisfies the estimate

$$\big|\Lambda_1^T - \Lambda_1 + \beta_1 K_1^2 e^{-2\beta_1 T}\big| \leqslant Ce^{-\beta_2 T}, \tag{2.19}$$

and the numbers K_1 and β_k are taken from the formulas (2.7) and (2.8).

Proof. The simplest way to confirm the properties of the eigenvalue Λ_1^T as of the function of parameter T is to construct the asymptotics. Here, it is sufficient to realize the formal procedure, the justification of obtained representations follows standard schemes repeatedly published, see [22, Ch. 5, Sect. 6, Ch. 9], [32] and others. We stress that by the dilatation of coordinates the "long" domain Π^T becomes a "thin" domain, while for such bodies, even elastic, the published literature is vast, see, for instance, surely incomplete lists of references in the monographs [22], [25], [33]–[36], and it answers almost all meaningful questions.

We seek the asymptotic for eigenpairs of the problem (2.18) in the form

$$\Lambda_1^T = \Lambda_1 + e^{-2\beta_1 T}\Lambda' + \dots, \tag{2.20}$$

$$W_1^T(\xi) = W_1(\xi) + K_1 e^{-2\beta_1 T} e^{\beta_1\eta_1}\sin(\pi\eta_2) + e^{-2\beta_1 T} W'(\xi) + \dots, \tag{2.21}$$

where the dots replaces higher–order asymptotic terms inessential for the made analysis, and the pair $\{\Lambda'; W'\}$ is to be determined. According to the expansion (2.7), the sum of first two terms in the right hand side of ansätze (2.21) gives the error $O\big(e^{-\beta_2 T}\big)$ in the boundary condition on the segment $\gamma^T = \{\eta : \eta_1 = T, \eta_2 \in 0,1)\}$. For the pair $\{\Lambda'; W'\}$ we obtain the equation

$$-\Delta_\eta W'(\eta) - \Lambda_1 W'(\eta) = \Lambda' W_1(\eta), \quad \eta \in \Pi,$$

with the boundary condition (2.2) on the lateral sides and the Neumann condition on the end

$$\partial_{\nu(\eta)} W'(\eta) = -K_1 \partial_{\nu(\eta)}\left(e^{\beta_1\eta_2}\sin(\pi\eta_2)\right), \quad \eta \in \gamma,$$

already found according to the mentioned expansion (2.7). Since Λ_1 is a simple eigenvalue, there is one solvability condition of the obtained problem in the class of functions decaying at infinity, see, for instance, [28, Chs. 2, 5], which in view of the normalization (2.6), we satisfy as follows:

$$\begin{aligned}
\Lambda' &= \Lambda'\|W'; L^2(\Pi)\|^2 = -\int\limits_\Pi W_1(\eta)\left(\Delta_\eta + \Lambda_1\right) W'(\eta)d\eta \\
&= K_1 \int\limits_\gamma W_1(\eta)\partial_{\nu(\eta)}\left(e^{\beta_1\eta_1}\sin(\pi\eta_2)\right) ds_\eta \\
&= -K_1 \lim_{t\to+\infty} \int\limits_{\gamma^t} \left(W_1(\eta)\frac{\partial}{\partial\eta_1}\left(e^{\beta_1\eta_1}\sin(\pi\eta_2)\right) - e^{\beta_1\eta_1}\sin(\pi\eta_2)\frac{\partial W_1}{\partial\eta_1}(\eta)\right) d\eta_2 \\
&= -K_1^2 2\beta_1 \int\limits_0^1 \left(\sin(\pi\eta_2)\right)^2 d\eta_2 = -\beta_1 K_1^2.
\end{aligned}$$

The corrector in the representation (2.20) is calculated.

To verify the monotonicity property, we take a small parameter $h > 0$ and compare the eigenvalues Λ_1^{T-h} and Λ_1^T for the trapezoids $\Pi^{T-h} \subset \Pi^T$. We again admit the simplest asymptotic ansätze

$$\begin{aligned}
\Lambda_1^{T-h} &= \Lambda_1^T + h\Lambda_\bullet^T + \dots, \\
W_1^{T-h}(\eta) &= W_1^T(\eta) + hW_\bullet^T(\eta) + \dots.
\end{aligned}$$

In view of the Taylor series

$$\frac{\partial W_1^T}{\partial\eta_1}(\eta_1 - h, \eta_2) = \frac{\partial W_1^T}{\partial\eta_1}(\eta_1,\eta_2) - h\frac{\partial^2 W_1^T}{\partial\eta_1^2}(\eta_1,\eta_2) + \dots$$

we find that the correctors in ansätze are found from the equation

$$-\Delta_\eta W_\bullet^T(\eta) - \Lambda_1^T W_\bullet^T(\eta) = \Lambda_\bullet^T W_1^T(\eta) \quad \text{in} \quad \Pi^T,$$

with the homogeneous Dirichlet condition on the bases σ_0^T and σ_1^T of the trapezoid Π^T and the Neumann condition on the lateral sides

$$\frac{\partial W_\bullet^T}{\partial\nu(\eta)} = 0 \quad \text{on} \quad \gamma, \quad \frac{\partial W_\bullet^T}{\partial\eta_1}(T,\eta_2) = \frac{\partial^2 W_1^T}{\partial\eta_1^2}(T,\eta_2) \quad \text{for} \quad \eta_2 \in (0,1).$$

In view of one (Λ_1^T is a simple eigenvalue) solvability condition of the formed problem, we find

$$\begin{aligned}
\Lambda_\bullet^T \|W_1^T; L^2(\Pi^T)\|^2 &= -\int_0^1 W_1^T(T,\eta_2)\frac{\partial^2 W_1^T}{\partial\eta_1^2}(T,\eta_2)d\eta_2 \\
&= \int_0^1 W_1^T(T,\eta_2)\Big(\frac{\partial^2 W_1^T}{\partial\eta_2^2}(T,\eta_2) + \Lambda_1^T W_1^T(T,\eta_2)\Big)d\eta_2 \qquad (2.22)\\
&= \int_0^1 \Big(\Lambda_1^T\big|W_1^T(T,\eta_2)\big|^2 - \Big|\frac{\partial W_1^T}{\partial\eta_2}(T,\eta_2)\Big|^2\Big)d\eta_2.
\end{aligned}$$

The function W_1^T is at least twice continuously differentiable at the angular points, which are the ends of the segment γ^T. Thus, by means of the Friedrichs inequality, the formula (2.22) implies the estimate

$$\Lambda_\bullet^T \leqslant \big(\Lambda_1^T - \pi^2\big)\|W_1^T; L^2(\Pi^T)\|^{-2}\|W_1^T; L^2(\gamma^T)\|^2,$$

and both norms of the positive in $\Pi^T \cup \gamma \cup \gamma^T$ function W_1^T do not vanish. Thus, the derivative of the function $T \mapsto \Lambda_1^T$ at some point $T > 1$ is strictly positive under the condition $\Lambda_1^T < \pi^2$. The inequality $\Lambda_1^T \geqslant \pi^2$ is impossible for all $T > 1$ since by the formula (2.19) for large T the needed condition is satisfied.

We note that the almost identical change of variables

$$\eta \mapsto (\eta_1\chi(\eta_1 - 1) + (\eta_1 - h)(1 - \chi(\eta_1 - 1)), \eta_2)$$

transforms the trapezoid Π^T into the trapezoid Π^{T-h}, that is, the translation of boundary is a regular perturbation of the problem and the justification of asymptotics in this case is very simple, see the monograph [37, Ch. 7, Sect. 6].

We return to the problem (2.17) for $M < M_\dagger$. We partition the domain Ξ into three sets

$$\begin{aligned}
\Xi(R) &= \{\xi \in \Xi : \xi_1 + 1 < R, \xi_2 < R\},\\
\Xi^+(R) &= \{\xi \in \Xi : \xi_1 + 1 > R, \xi_2 < \xi_1 + 1\}, \qquad (2.23)\\
\Xi^-(R) &= \{\xi \in \Xi : \xi_2 > R, \xi_2 > \xi_1 + 1\},
\end{aligned}$$

and choose the size $R > 1$ so that the relation

$$\Lambda_1^T > \frac{1}{2}\big(M_\dagger + M\big) > M \quad \text{for} \quad T > R \qquad (2.24)$$

holds, where Λ_1^T is the first eigenvalue of the problem (2.18). Proposition 2.1 shows that the condition (2.24) can be satisfied for each $M \in \big(0, M_\dagger\big)$.

Owing to the Dirichlet condition on the bases of infinite truncated pyramide $\Xi^+(R)$, the relation

$$\|\nabla_\xi V; L^2(\Xi^+(R))\|^2 \geqslant \int\limits_{R-1}^{+\infty} \int\limits_{\Pi^{\xi_1+1}} |\nabla_\eta V(\xi_1,\eta)|^2 d\eta d\xi_1$$
$$\geqslant \int\limits_{R-1}^{+\infty} \Lambda_1^{\xi_1+1} \int\limits_{\Pi^{\xi_1+1}} |V(\xi_1,\eta)|^2 d\eta d\xi_1 \geqslant \frac{M+M_\dagger}{2} \|V; L^2(\Xi^+(R))\|^2$$

holds. Exactly the same inequality holds on the set $\Xi^-(R)$, which is congruent to the set $\Xi^+(R)$. Therefore, for a symmetric bilinear form $b(V,\Psi;\Xi)$ in the left hand side of integral identity (2.17) restricted to the subdomains (2.23), the formulas

$$b(V,V;\Xi^\pm(R)) = \|\nabla_\xi V; L^2(\Xi^\pm(R))\|^2 - M\|V; L^2(\Xi^\pm(R))\|^2$$
$$\geqslant \delta\|\nabla_\xi V; L^2(\Xi^\pm(R))\|^2 + \frac{1}{2}\left((M_\dagger - M) - \delta(M_\dagger + M)\right)\|V; L^2(\Xi^\pm(R))\|^2,$$
$$b(V,V;\Xi(R)) = \|\nabla_\xi V; L^2(\Xi^\pm(R))\|^2 + (t-M)\|V; L^2(\Xi(R))\|^2.$$

are valid. Fixing the numbers $t > M$ and $\delta \in \big(0, (M_\dagger + M)^{-1}(M_\dagger - M)\big)$, we find that the form $b(V,\Psi;\Xi)$ is positive definite on the space $H_0^1(\Xi;\Upsilon)$, that is, by the Riesz theorem on representation of continuous linear functional in a Hilbert space the problem (2.17) is uniquely solvable. □

Thus, we have proved the next theorem.

Theorem 2.1. *The essential spectrum of problem* (2.13) *in the domain* (1.20) *with the Dirichlet conditions on the bases* $\Upsilon^\pm$ *is the ray* (2.15)*, the bottom* $M_\dagger$ *of which is the eigenvalue* Λ_1 *in the discrete spectrum of problem* (2.1)–(2.3) *in the pointed strip* (1.14).

2.3. Discrete spectrum of problem in quarter of layer. The approaches of this section slightly differ from ones used in works [11] and [12] for checking the non–emptiness of discrete spectrum in layer–type domains of similar shapes. According to the minimax principle [30, Thm. 10.2.1], the bottom $\underline{\wp}$ of spectrum $\wp$ of problem (2.14) (or (2.13) in the differential form) obeys the relation

$$\underline{\wp} = \min_{\Psi \in H_0^1(\Xi;\Upsilon)\setminus\{0\}} \frac{\|\nabla_\xi \Psi; L^2(\Xi)\|^2}{\|\Psi; L^2(\Xi)\|^2}.$$

Thus, to check the non–emptiness of the spectrum $\wp_d$, it is sufficient to find a test function $\Psi \in H_0^1(\Xi;\Upsilon)$, which satisfies the inequality

$$\|\nabla_\xi \Psi; L^2(\Xi)\|^2 - M_\dagger \|\Psi; L^2(\Xi)\|^2 < 0. \tag{2.25}$$

At the same time it turns out that $\underline{\wp}$ is the first eigenvalue in the discrete spectrum $\wp_d$.

We let

$$\Psi_\delta(\xi) = \begin{cases} W_1(\xi_2,\xi_3) & \text{for} \quad \xi_1 \leqslant 0, \\ W_1(\xi_2,\xi_3)e^{-\delta\xi_1} & \text{for} \quad \xi_1 \geqslant 0. \end{cases}$$

It is clear that $\Psi_\delta \in H_0^1(\Xi;\Upsilon)$ for $\delta > 0$. By the normalization (2.6) we have

$$\|\Psi_\delta; L^2(\Xi)\|^2 = \|W_1; L^2(\mathbb{T})\|^2 + \int\limits_0^\infty e^{-2\delta\xi_1} d\xi_1 \int\limits_\Pi |W_1(\xi_2,\xi_3)|^2 d\xi_2 d\xi_3 = \|W_1; L^2(\mathbb{T})\|^2 + \frac{1}{2\delta}. \tag{2.26}$$

Here $\mathbb{T} = \{\xi : \xi_1 \in (-\xi_3, 0), \xi_2 > \xi_3, \xi_3 \in (0,1)\}$ is a truncated prism with a triangular section. Supposing the parameter δ is small, in the same way we obtain

$$\|\nabla_\xi \Psi_\delta; L^2(\Xi)\|^2 = \|\nabla_\xi W_1; L^2(\mathbb{T})\|^2 + \int_0^\infty e^{-2\delta\xi_1} d\xi_1 \int_\Pi \big|\nabla_\eta W_1(\eta)\big|^2 d\eta + O(\delta)$$
$$= \|\nabla_\xi W_1; L^2(\mathbb{T})\|^2 + \frac{1}{2\delta}\Lambda_\dagger + O(\delta).$$

We consider the difference

$$\|\nabla_\xi W_1; L^2(\mathbb{T})\|^2 - \Lambda_1 \|W_1; L^2(\mathbb{T})\|^2$$
$$= \mathbf{J} := -\int_{\mathbb{P}} W_1(\xi_2,\xi_3)\big(\Delta_\xi + \Lambda_1\big) W_1(\xi_2,\xi_3) d\xi + \int_{\Theta^2} W_1(\xi_2,\xi_3)\partial_{\nu(\xi)} W_1(\xi_2,\xi_3) ds_\xi.$$

The first integral in the right hand side vanishes due to Equation (2.1) for the pair $\{\Lambda_1; W_1\}$. The second integral is equal to

$$\mathbf{J} = -\frac{1}{\sqrt{2}} \int_{\mathbb{P}} W_1\Big(\eta_1, \frac{\eta_2}{\sqrt{2}}\Big) \frac{\partial W_1}{\partial \eta_2}\Big(\eta_1, \frac{\eta_2}{\sqrt{2}}\Big) d\eta, \tag{2.27}$$

where $\eta = (\eta_1, \eta_2)$ is the system of Cartesian coordinates in the plane of face Υ^2, and $\eta_1 = \xi_2$ and $\eta_2 = 2^{-\frac{1}{2}}(\xi_3 - \xi_1)$, while $\mathbb{P} \subset \mathbb{R}_+ \times \big(0, \sqrt{2}\big)$ is the pointed semi–strip with the vertices $\eta = (0,0)$ and $\eta = \big(1, \sqrt{2}\big)$. We denote the segment connecting these points by $\mathbb{I}$, and we integrate by parts to obtain

$$\mathbf{J} = -\frac{1}{2} \int_{\mathbb{I}} \Big| W_1\Big(\eta_1, \frac{\eta_2}{\sqrt{2}}\Big)\Big|^2 ds < 0. \tag{2.28}$$

The strict inequality holds since the first eigenfunction W_1 of the problem (2.1)–(2.3) is positive on the end of semi–strip (1.14); in any case it can not vanish everywhere on the end by the uniqueness continuation theorem, see, for instance, the book [38].

Gathering the formulas (2.26)–(2.28), we see that the left hand side of inequality (2.25) does not exceed the sum $\mathbf{J} + C\delta$ and this is why it indeed becomes negative for sufficiently small $\delta > 0$.

We formulate the obtained result.

Theorem 2.2. *The discrete spectrum of problem* (2.13) *(or* (2.14) *in the variational form) contains at least one eigenvalue.*

2.4. Exponential decay of eigenfunction. Let M_1 be the first (smallest) eigenvalue of problem (2.13) given by Theorem 2.2. We normalize the associated eigenfunction $V_1 \in H^1_0(\Xi; \Upsilon)$ in $L^2(\Xi)$ and fix it positive in $\Xi \cup \Theta$. Into the integral identity (2.14) we substitute the test function $\Psi^\kappa_T = \mathcal{R}^\kappa_T \mathcal{V}^\kappa_T$, where $\mathcal{V}^\kappa_T = \mathcal{R}^\kappa_T V_1$. A continuous piecewise–smooth weight factor reads

$$\mathcal{R}^\kappa_T(\xi) = \begin{cases} e^\kappa & \text{for} \quad \rho \leqslant 1, \\ e^{\kappa\rho} & \text{for} \quad \rho \in (1, T), \\ e^{\kappa T} & \text{for} \quad \rho \geqslant T, \end{cases} \tag{2.29}$$

with $\rho^2 = \xi_1^2 + \xi_2^2$, while κ and R are positive parameters chosen small and large, respectively. We stress that the functions $\mathcal{V}^\kappa_T$ and Ψ^κ_T belong to the space $H^1_0(\Xi; \Theta)$ since the weight factor (2.29) is constant for large radius ρ. By simple transformations (several commutations of the operator–gradient ∇_ξ with the function $\mathcal{R}^\kappa_T$) we obtain the identity

$$\|\nabla_\xi \mathcal{V}^\kappa_T; L^2(\Xi)\|^2 - \|\mathcal{V}^\kappa_T (\mathcal{R}^\kappa_T)^{-1} \nabla_\xi \mathcal{R}^\kappa_T; L^2(\Xi)\|^2 = M_1 \|\mathcal{V}^\kappa_T; L^2(\Xi)\|^2. \tag{2.30}$$

We note that

$$\nabla_\xi \mathcal{R}_T^\kappa(\xi) = 0 \quad \text{for} \quad \rho \notin (1,T), \quad \mathcal{R}_T^\kappa(\xi)^{-1}|\nabla_\xi \mathcal{R}_T^\kappa(\xi)| \leqslant \kappa \quad \text{for} \quad \rho \in (1,T). \tag{2.31}$$

We partition the set Ξ into four parts: the set $\Xi(R)$ from the formula (2.23) and also the sets

$$\Sigma_R^1 = \Big\{\xi \in \Xi : \xi_2 < R, \xi_1 > R - \frac{1}{2}\Big\}, \qquad \Sigma_R^2 = \Big\{\xi \in \Xi : \xi_1 < R - \frac{1}{2}, \xi_2 > R\Big\}$$

and $K_R = \Xi \setminus \big(\Xi(R) \cup \Sigma_R^1 \cup \Sigma_R^1\big)$. Recalling Proposition 2.1, we choose the size $R > 1$ to satisfy the relation $\Lambda_1^R > \frac{1}{2}(M_1 + \Lambda_1)$. Then on the subdomains Σ_R^1 and Σ_R^2, which are congruent to the set $(R, +\infty) \times \Pi^R \ni (\tau, \eta)$, the estimates hold:

$$\frac{1}{2}(M_1 + \Lambda_1)\,\|\mathcal{V}_T^\kappa : L^2(\Sigma_R^j)\|^2 \leqslant \Lambda_1^R \|\mathcal{V}_T^\kappa : L^2(\Sigma_R^j)\|^2 \leqslant \|\nabla_\xi \mathcal{V}_T^\kappa : L^2(\Sigma_R^j)\|^2, \tag{2.32}$$

which are obtained by integrating in τ the Friedrichs inequality on the trapezoid Π^R. After an additional integration, the one–dimensional Freidrichs inequality on the segment $(0,1) \ni \xi_3$ gives the relation

$$\pi^2 \|\mathcal{V}_T^\kappa : L^2(K_R)\|^2 \leqslant \|\nabla_\xi \mathcal{V}_T^\kappa : L^2(K_R)\|^2. \tag{2.33}$$

Now by means of the formulas (2.31)–(2.33) we transform the identity (2.30) into the estimate

$$\begin{aligned} M_1 e^{R\sqrt{2}\kappa} \geqslant & M_1 \|\mathcal{V}_T^\kappa; L^2(\Xi(R))\|^2 \geqslant \delta \|\nabla_\xi \mathcal{V}_T^\kappa; L^2(\Xi)\|^2 \\ & + ((1-\delta)\pi^2 - M_1 - \kappa)\|\mathcal{V}_T^\kappa; L^2(K_R)\|^2 \\ & + \sum_{j=1,2} \Big(\frac{1}{2}(M_1 + \Lambda_1) - M_1 - \kappa\Big)\|\mathcal{V}_T^\kappa; L^2(\Sigma_R^j)\|^2. \end{aligned}$$

Taking sufficiently small $\delta > 0$ and $\kappa > 0$, we find that the factors in the norms $\|\mathcal{V}_T^\kappa; L^2(K_R)\|$ and $\|\mathcal{V}_T^\kappa; L^2(\Sigma_R^j)\|$ are positive and hence, the uniform estimate

$$\|\nabla_\xi \mathcal{V}_T^\kappa; L^2(\Xi)\|^2 + \|\mathcal{V}_T^\kappa; L^2(\Xi)\|^2 \leqslant \mathcal{M} \tag{2.34}$$

holds.

Since the weight factor (2.29) grows monotonically as the parameter T grows, the passage to limit as $T \to +\infty$ in the inequality (2.34) ensures the following statement, which confirms the aforementioned decay at infinity of the eigenfunction V_1.

Theorem 2.3. *The found first eigenfunction $V_1 \in H_0^1(\Xi; \Upsilon)$ of problem (2.13) normalized in the space $L^2(\Xi)$ satisfies the weight estimate*

$$\|e^{\kappa\rho}\nabla_\xi V_1; L^2(\Xi)\|^2 + \|e^{\kappa\rho} V_1; L^2(\Xi)\|^2 \leqslant \mathcal{K}, \tag{2.35}$$

where κ and $\mathcal{K}$ are some positive numbers and $\rho = \sqrt{\xi_1^2 + \xi_2^2}$.

2.5. Remarks on threshold resonances. The planar problem (2.1)–(2.3) has already been studied for a long time in detail sufficient for the asymptotic analysis in the present work. At the same time for the spatial problem (2.13) a series of important question remained open, for instance, the multiplicity of discrete spectrum and the emergence of threshold resonances.

In the planar domain (1.14) the threshold resonance (see papers [39], [40] and others) is due to the appearance of a non–trivial bounded solution for the problem with the threshold spectral parameter $\Lambda = \pi^2$; this solution is either trapped (decaying at infinity) or almost standing (stabilizing at infinity) wave. It is easy to establish the absence of such solutions in the problem (2.1)–(2.3): in the case of the Dirichlet condition on the end the method from [26] works, while in the case of the Neumann condition we need to apply the inequality (2.11), which means that the second eigenvalue of the problem in the triangle $\{\eta \in \Pi : \eta_1 < 1\}$ is strictly greater than the threshold π^2. We also need to apply the sufficient condition [41], [42] or the first of two criterions [32] of the absence of threshold resonance.

In the spatial problem (2.13) on layer–type domains (1.20)—(1.22) the notion of threshold resonance requires a specification since the asymptotics of its solution at infinity for $M = \Lambda_1$ is unknown; Theorem 2.3 concerns only the case $M < \Lambda_1$. However, in this problem on the domain (1.21) the even continuation to the set $\mathbb{R}\times\Pi$ is admitted and it is followed by the Fourier transform in the variable ξ_1, while the needed bounded solution is of the form $\xi \mapsto W_1(\xi_2, \xi_3)$. By the way, exactly due to the mentioned threshold resonance in the domain $\Xi_\sqcup$ defined by the formula (1.21) the limiting problem (4.1) on the segment $(-1, 1) \ni y_1$ acquires the Neumann condition. The influence of threshold resonances on the boundary conditions in the limiting problems is also discussed in Section 5.3.

The next statement on solvability of the Helmholtz equation in the skewed semi–strip (1.14)

$$-\Delta_\eta w(\eta) - \pi^2 w(\eta) = F(\eta), \quad \eta \in \Pi, \tag{2.36}$$

with the boundary conditions (2.2) and (2.3) is obtained by specification of general results from the book [28, Ch. 5] and paper [40]. However, for the reader's convenience we reproduce its short proof. In order to do this, we define the exponential weight Sobolev space $\mathcal{W}^1_\beta(\Pi)$ (the Kondratiev space; see the original work [43] and, for instance, the books [28], [44]) as the completion of the linear set $C^\infty_c(\overline{\Pi})$ by the norm

$$\|w; \mathcal{W}^1_\beta(\Pi)\| = \|e^{\beta\eta_1} w; H^1(\Pi)\|, \tag{2.37}$$

where $\beta \in \mathbb{R}$ is the weight index. The space $\mathcal{W}^1_\beta(\Pi)$ consists of the functions $w \in H^1_{loc}(\overline{\Pi})$, for which the norm (2.37) is finite and in the case $\beta = 0$ it coincides with $H^1(\Pi)$, but for $\beta > 0$ the functions in $\mathcal{W}^1_\beta(\Pi)$ decay at infinity, while for $\beta < 0$ a certain growth is allowed for them and the decay/growth rate is controlled by the weight index. By $\mathcal{W}^{1,0}_\beta(\Pi)$ we denote the subspace of functions obeying the Dirichlet condition from the list (2.2), (2.3).

As usually, by a weak solution to the problem (2.36), (2.2), (2.3) in the weight classes we mean a function $w \in \mathcal{W}^{1,0}_\beta(\Pi)$ obeying the integral identity

$$(\nabla_\eta w, \nabla_\eta \psi)_\Pi - \pi^2 (w, \psi)_\Pi = f(\psi) \quad \forall \psi \in \mathcal{W}^{1,0}_{-\beta}(\Pi), \tag{2.38}$$

where $f \in \big(\mathcal{W}^{1,0}_{-\beta}(\Pi)\big)^*$ is a linear continuous functional on the space $\mathcal{W}^{1,0}_{-\beta}(\Pi)$, for instance,

$$f(\psi) = (F, \psi)_\Pi \quad \text{with} \quad e^{\beta\eta_1} F \in L^2(\Pi).$$

The problem (2.38) is associated with the continuous mapping

$$\mathcal{W}^{1,0}_\beta(\Pi) \ni w \quad \mapsto \quad \mathcal{A}_\beta w := f \in \big(\mathcal{W}^{1,0}_{-\beta}(\Pi)\big)^*.$$

Proposition 2.2. *The following assertions hold.*

1) *The operators $\mathcal{A}_\beta$ and $\mathcal{A}_{-\beta}$ are mutually adjoint. They turn out to be Fredholm in the case $\beta \in (0, \pi\sqrt{3})$, but they lose this property for $\beta = 0$ and $\beta = \pi\sqrt{3}$.*
2) *If $\beta \in (0, \pi\sqrt{3})$ and $f \in \big(\mathcal{W}^{1,0}_{-\beta}(\Pi)\big)^* \subset \big(\mathcal{W}^{1,0}_{\beta}(\Pi)\big)^*$, then the problem (2.38) with the replacement $\beta \mapsto -\beta$ has a unique (bounded) solution $w \in \mathcal{W}^{1,0}_{-\beta}(\Pi)$, which can be represented as*

$$w(\eta) = (1 - \chi(\eta_1 - 1)) a \sin(\pi\eta_2) + \widetilde{w}(\eta), \tag{2.39}$$

where $\widetilde{w} \in \mathcal{W}^{1,0}_\beta(\Pi)$, $a \in \mathbb{R}$, χ is the cut–off function (2.5), and the estimate

$$\big(\|\widetilde{w}; \mathcal{W}^{1,0}_\beta(\Pi)\|^2 + |a|^2\big)^{\frac{1}{2}} \leqslant c_\beta \|f; \big(\mathcal{W}^{1,0}_{-\beta}(\Pi)\big)^*\| \tag{2.40}$$

holds, and the factor c_β is independent of the functional f, but it grows unboundedly as $\beta \to +0$ or $\beta \to \pi\sqrt{3} - 0$.

Proof. We first of all observe that the solutions of the Dirichlet problem for the homogeneous ($F=0$) equation (2.36) in the entire space $\mathbb{R}\times(0,1)$ are the functions

$$\eta_1\sin(\pi\eta_2) \quad \text{and} \quad e^{\pm\eta_1\pi\sqrt{k^2-1}}\sin(\pi k\eta_2), \quad k=1,2,3,\ldots. \tag{2.41}$$

Thus, the first statement is a corollary of classical Kondratiev theorem [43] (a simple exposition of this theory is presented in the introductory chapter 2 in the monograph [28]), and the restriction for the quantity β puts the weight indices $\pm\beta$ between "prohibited" indices 0 and $\pm\pi\sqrt{3}$ taken from the latter formula (2.41) with $k=1,2$.

The absence of threshold resonance in the problem (1.1)–(1.3) in particular means that the operator $\mathcal{A}_\beta$ is a monomorphism for $\beta>0$. Therefore, in the case $\beta\in\big(0,\pi\sqrt{3}\big)$ the operator $\mathcal{A}_{-\beta}$ is an epimorphism. Theorem 4.3.3 in [28] on the index increment applied for cylindrical domains show that $\operatorname{Ind}\mathcal{A}_\beta-\operatorname{Ind}\mathcal{A}_{-\beta}=-2$, where two is the number of solutions in the list (2.41) with the polynomial growth at the infinity. Since

$$\operatorname{Ind}\mathcal{A}=\operatorname{dimker}\mathcal{A}-\operatorname{dimcoker}\mathcal{A} \quad \text{and} \quad \operatorname{dimker}\mathcal{A}_\beta=0, \quad \operatorname{dimcoker}\mathcal{A}_{-\beta}=0,$$

we find out that $\operatorname{Ind}\mathcal{A}_{-\beta}=-1$, and hence the restriction of the operator $\mathcal{A}_{-\beta}$ to the subspace $\mathcal{W}^{1,0}_{\beta\oplus}(\Pi)$ of the functions in $\mathcal{W}^{1,0}_{-\beta}(\Pi)$ admitting the representation (2.39), which we equip with the norm in the left hand side (2.40), takes the zero index and becomes the isomorphism due to the absence of the trapped waves on the threshold frequency. Thus, we have proved the second statement and have completed the proof. We note that the introduced subspace is called a weight class with the detached asymptotics. ☐

Remark 2.2. *In the proof of Proposition 2.2 the identity*

$$\operatorname{dimker}\mathcal{A}_{-\beta}=1$$

was established. It is easy to see that for $K=N,D$ the subspace $\ker\mathcal{A}_{-\beta}$ is spanned over the solution $\mathbf{W}_K$ of homogeneous problem (2.1)–(2.3)$_K$ with the parameter $\Lambda=\pi^2$, which has a linear growth at infinity and admits the representation

$$\mathbf{W}_K(\eta)=\sin(\pi\eta_2)(\eta_1-\mathbf{C}_K)+\widetilde{\mathbf{W}}_K(\eta), \tag{2.42}$$

where $\mathbf{C}_K$ is some constant, while the remainder $\widetilde{\mathbf{W}}_K\in\mathcal{W}^1_\beta(\Pi)$ decays exponentially at infinity with the rate $O\big(e^{-\eta_1\pi\sqrt{3}}\big)$ and it turns out to be infinitely differentiable everywhere on the set $\overline{\Pi}$ except for the angular points $\mathcal{P}_0$ and $\mathcal{P}_1$. We stress that the function (2.42) is not in the space $\mathcal{W}^{1,0}_{\beta\oplus}(\Pi)$ and we introduce the difference

$$\widehat{\mathbf{W}}_K(\eta)=\mathbf{W}_K(\eta)-\sin(\pi\eta_2)\eta_1=-\mathbf{C}_K\sin(\pi\eta_2)+\widetilde{\mathbf{W}}_K(\eta), \tag{2.43}$$

which possesses the needed behavior at infinity, belongs to some weigth space with a separated asymptotics, but it does not satisfy the boundary condition (2.3).

3. Absence of localization effect

3.1. Usual asymptotic constructions. The approaches used in this section are widely known in the case of Neumann boundary conditions on the bases of thin domains, see the monograph [25] and the references therein, and its adaption to the Dirichlet condition requires minimal efforts in the case of the passage to the mixed boundary conditions exclusively owing to the Neumann condition only on the perpendicular bases on the lateral sides of polyhedron Ω^ε. The passage to the Dirichlet condition on the entire boundary $\partial\Omega$ requires only literal reproducing the arguing and calculations given below.

In the situation (1.6) we admit the simplest ansatz for the eigenpairs of problem (1.1)–(1.3)

$$\lambda^\varepsilon(x) = \varepsilon^{-2}\pi^2 + \mu + \dots, \tag{3.1}$$

$$u^\varepsilon(x) = \sin\big(\pi\varepsilon^{-1}z\big)v(y) + \dots, \tag{3.2}$$

where, as usually, the dots replace the higher–order terms. After substituting the ansätze into the differential equation and boundary conditions we find that the leading asymptotic terms mutually cancel out, while for the pair $\{\mu; v\}$ we obtain the mixed boundary value problem in the square

$$\begin{aligned} -\Delta_y v(y) &= \mu v(y), \quad y \in \square_1, \\ v(y_1, \pm 1) = 0, \quad |y_1| < 1, &\quad \pm\frac{\partial v}{\partial y_1}(\pm 1, y_2) = 0, \quad |y_2| < 1. \end{aligned} \tag{3.3}$$

The eigenpairs of this problem

$$\{\mu_{(p,q)}; v_{(p,q)}(y)\} = \Big\{\frac{\pi^2}{4}(p^2+q^2), \cos\Big(\frac{\pi}{2}p(y_1-1)\Big)\sin\Big(\frac{\pi}{2}q(y_2-1)\Big)\Big\} \tag{3.4}$$

appear in the formulas (1.10) and (1.11). We renumerate the eigenvalues forming the monotone unbounded sequence

$$0 < \mu_1 < \mu_2 \leqslant \mu_3 \leqslant \dots \leqslant \mu_m \leqslant \dots \to +\infty. \tag{3.5}$$

The associated eigenfunctions of problem (3.3) obey the orthogonality and normalization conditions

$$(v_m, v_n)_\omega = \delta_{m,n}, \quad m, n \in \mathbb{N}. \tag{3.6}$$

Let us clarify the choice of boundary conditions: while the Neumann conditions are obtained by the substitution of ansätze (3.2) into the boundary condition (1.3) on sides (1.7), the Dirichlet condition in order to eliminate the errors in the boundary conditions (1.2) on other sides (1.6). The main term is generated by the Taylor series

$$\begin{aligned} v_{(p,q)}(y) &= C_p(y_1)\sin\Big(\frac{\pi}{2}q(y_2-1)\Big) = C_p(y_1)\big(A_q^\pm(y_2 \mp 1) + O(|y_2 \mp 1|^3)\big) \\ &= \varepsilon C_p(y_1)\big(\mp A_q^\pm \eta_1^\pm + O(\varepsilon^2|\eta_1^\pm|^3)\big). \end{aligned} \tag{3.7}$$

Here we use the stretched coordinates (1.15), as well as the function and numbers

$$C_p(y_1) = \cos\Big(\frac{\pi}{2}p(y_1-1)\Big) \quad \text{and} \quad A_q^\pm = \frac{\pi}{2}(\pm 1)^q q. \tag{3.8}$$

The terms of order ε in (3.7) multiplied by $\sin(\pi\eta_2^\pm)$ in according to the ansätze (3.2) is compensated by boundary layer

$$\varepsilon\widetilde{w}^\pm(y_1, \eta^\pm) = \mp\varepsilon C_p(y_1)A_q^\pm \widetilde{\mathbf{W}}_D(\eta^\pm), \tag{3.9}$$

where $\widetilde{\mathbf{W}}_D$ is an exponentially decaying as $\eta_1^\pm \to +\infty$ remainder in the solution (2.42) of problem (2.1)–(2.3)$_D$ for $\Lambda = \pi^2$.

In view of the representation (2.42), the functions (3.9) generate additional remainders in the boundary Dirichlet conditions on the skewed sides (1.16), which we eliminate by means of specifying asymptotic ansätze (3.1) and (3.2) by higher–order terms $\varepsilon\mu'_{(p,q)}$ and $\varepsilon\sin(\pi\varepsilon^{-1}z)v'_{(p,q)}$ respectively, which are determined by the problem

$$\begin{aligned} -\Delta_y v'_{(p,q)}(y) - \mu_{(p,q)}v'_{(p,q)}(y) &= \mu'_{(p,q)}v_{(p,q)}(y), \quad y \in \square_1, \\ v'_{(p,q)}(y_1, \pm 1) = \pm\mathbf{C}_D A_q^\pm C_p(y_1), \quad |y_1| < 1, &\qquad \pm\frac{\partial v'_{(p,q)}}{\partial y_1}(\pm 1, y_2) = 0, \quad |y_2| < 1. \end{aligned} \tag{3.10}$$

The coefficient $\mathbf{C}_D$ is taken from the representation (2.43) of the function $\mathbf{W}_D$, and from the boundary condition on the end γ of the semi–strip Π for the exponentially decaying remainder in this representation

$$\widetilde{\mathbf{W}}_D(\eta) = \sin(\pi\eta_2)\big(\mathbf{C}_D - \eta_1\big) \quad \text{on} \quad \gamma.$$

In the case of the simple eigenvalue $\mu_{(p,q)}$ the solvability condition of problem (3.10) is the relation

$$\mu'_{(p,q)} = \mu'_{(p,q)}\|v_{(p,q)}; L^2(\square_1)\|^2 = \int\limits_{-1}^{1} \sum_{\pm} \pm \frac{\partial v_{(p,q)}}{\partial y_2}(y) v'_{(p,q)}(y)\Big|_{y_2=\pm 1} dy_1 = \mathbf{C}_D \frac{\pi^2}{2} q^2. \qquad (3.11)$$

As a result the number (3.11) and solution of problem (3.10), as well as the boundary layers (3.9) determine the correctors in the asymptotic ansätze. In the case of a multiple eigenvalue $\mu_{(p,q)}$ the procedure of constructing the remainders becomes a little bit more complicated, see the monographs [22, Ch. 16], [25, Ch. 7] and many separated publications, but we shall not reproduce the corresponding arguing since in the derivation of estimate for the remainder $\widetilde{\lambda}^{\varepsilon}_{(p,q)}$ in the expansion (1.10) the remainders are not needed.

Remark 3.1. *In this and the next section the boundary layers do not appear near the faces* (1.7) *of polyhedron* (1.4). *This fact can be easily explained. We continue the eigenfunctions* u^{ε}_m *of problem* (1.1)–(1.3) *evenly via the plane* $\{x : y_1 = 1\}$ *and impose the periodicity conditions*

$$u^{\varepsilon}(+1, y_2, z) = u^{\varepsilon}(-1, y_2, z),$$
$$\frac{\partial u^{\varepsilon}}{\partial y_1}(+1, y_2, z) = \frac{\partial u^{\varepsilon}}{\partial y_1}(-1, y_2, z), \quad z \in (0, \varepsilon), \quad |y_2| < 1 - z,$$

on the faces of polyhedron $\{x : y_1 \in (-1, 3), |y_2| < 1 - z, z \in (0, \varepsilon)\}$ *perpendicular to the abscise axis. Then the eigenfunctions of the new problem become smooth and periodic in the variable* $y_1 \in [-1, 3]$, *and this dependence is inherited by the eigenfunctions of original problem in* Ω^{ε} *and hence, the boundary layers can not appear in the direction of the axis* y_1.

3.2. Abstract formulation of original problem. In the Hilbert space $\mathcal{H}^{\varepsilon} := H^1_0(\Omega^{\varepsilon}; \Gamma^{\varepsilon}_D)$ we introduce the scalar product

$$\langle u^{\varepsilon}, \psi^{\varepsilon}\rangle_{\varepsilon} = \big(\nabla_x u^{\varepsilon}, \nabla_x \psi^{\varepsilon}\big)_{\Omega^{\varepsilon}}, \qquad (3.12)$$

as well as a positive symmetric continuous and hence self–adjoint operator $\mathcal{T}^{\varepsilon}$,

$$\langle \mathcal{T}^{\varepsilon} u^{\varepsilon}, \psi^{\varepsilon}\rangle_{\varepsilon} = \big(u^{\varepsilon}, \psi^{\varepsilon}\big)_{\Omega^{\varepsilon}} \qquad \forall u^{\varepsilon}, \psi^{\varepsilon} \in \mathcal{H}^{\varepsilon}. \qquad (3.13)$$

The operator $\mathcal{T}^{\varepsilon}$ is compact and hence, according to Theorems 10.1.5 and 10.2.2 in [30] its essential spectrum is the single point $\tau = 0$, while the discrete spectrum forms a monotone positive infinitesimal sequence of eigenvalues

$$\tau^{\varepsilon}_1 \geqslant \tau^{\varepsilon}_2 \geqslant \tau^{\varepsilon}_3 \geqslant \ldots \geqslant \tau^{\varepsilon}_m \geqslant \ldots \to +0. \qquad (3.14)$$

Comparing the formulas (3.12), (3.13) and (1.9), we see that the variational formulation of problem (1.1)–(1.3) is equivalent to the abstract equation

$$\mathcal{T}^{\varepsilon} u^{\varepsilon} = \tau^{\varepsilon} u^{\varepsilon} \quad \text{in the space} \quad \mathcal{H}^{\varepsilon}$$

with the spectral parameter

$$\tau^{\varepsilon} = \big(\lambda^{\varepsilon}\big)^{-1}. \qquad (3.15)$$

Then next statement known as the lemma of almost eigenvalues and eigenvectors, see the source [45] is ensured by the spectral expansion of resolvent, see, for instance [30, Ch. 6].

Lemma 3.1. *Let $U^\varepsilon \in \mathcal{H}^\varepsilon$ and $t^\varepsilon \in \mathbb{R}_+$ be such that*

$$\|U^\varepsilon; \mathcal{H}^\varepsilon\| = 1, \quad \|\mathcal{T}^\varepsilon U^\varepsilon - t^\varepsilon U^\varepsilon; \mathcal{H}^\epsilon\| =: \delta^\varepsilon \in [0, t^\varepsilon). \tag{3.16}$$

Then the operator $\mathcal{T}^\varepsilon$ possesses an eigenvalue $\tau^\varepsilon_{n(\varepsilon)}$ obeying the inequality

$$\left|t^\varepsilon - \tau^\varepsilon_{n(\varepsilon)}\right| \leqslant \delta^\varepsilon.$$

Moreover, for each $\delta^\varepsilon_* \in (\delta^\varepsilon, t^\varepsilon)$ there exists the column of coefficients

$$\mathcal{C}^\varepsilon = \left(\mathcal{C}^\varepsilon_{\mathcal{N}^\varepsilon}, \dots, \mathcal{C}^\varepsilon_{\mathcal{N}^\varepsilon + \mathcal{X}^\varepsilon - 1}\right),$$

which satisfies the relations

$$\left\| U^\varepsilon - \sum_{\ell=\mathcal{N}^\varepsilon}^{\mathcal{N}^\varepsilon + \mathcal{X}^\varepsilon - 1} \mathcal{C}^\varepsilon_\ell \mathcal{U}^\varepsilon_{(\ell)}; \mathcal{H}^\varepsilon \right\| \leqslant 2\frac{\delta^\varepsilon}{\delta^\varepsilon_*}, \qquad \sum_{\ell=\mathcal{N}^\varepsilon}^{\mathcal{N}^\varepsilon + \mathcal{X}^\varepsilon - 1} \left|\mathcal{C}^\varepsilon_\ell\right|^2 = 1, \tag{3.17}$$

where $\tau^\varepsilon_{\mathcal{N}^\varepsilon}, \dots, \tau^\varepsilon_{\mathcal{N}^\varepsilon + \mathcal{X}^\varepsilon - 1}$ is the set of all eigenvalues (3.14) *of operator $\mathcal{T}^\varepsilon$ in the segment $\left[t^\varepsilon - \delta^\varepsilon_*, t^\varepsilon + \delta^\varepsilon_*\right]$, and the associated eigenvectors $\mathcal{U}^\varepsilon_{\mathcal{N}^\varepsilon}, \dots, \mathcal{U}^\varepsilon_{\mathcal{N}^\varepsilon + \mathcal{X}^\varepsilon - 1}$ obey the orthogonality and normalization conditions*

$$\left\langle \mathcal{U}^\varepsilon_p, \mathcal{U}^\varepsilon_q \right\rangle_\varepsilon = \delta_{p,q}. \tag{3.18}$$

3.3. Asymptotics of eigenvalues. As the component of almost eigenpair $\{t^\varepsilon_{(p,q)}; U^\varepsilon_{(p,q)}\}$ we take the expressions

$$t^\varepsilon_{(p,q)} = \varepsilon^2 \left(\pi^2 + \varepsilon^2 \mu_{(p,q)}\right)^{-1}, \qquad U^\varepsilon_{(p,q)} = \|v^\varepsilon_{(p,q)}; \mathcal{H}^\varepsilon\|^{-1} v^\varepsilon_{(p,q)}, \tag{3.19}$$

and

$$\begin{aligned} v^\varepsilon_{(p,q)}(x) = & \sin\left(\pi \frac{z}{\varepsilon}\right) X^\varepsilon(y_2) \Big(v_{(p,q)}(y) - \sum_\pm \chi_\pm(y_2) C_p(y_1) A^\pm_q (y_2 \mp 1) \Big) \\ & + \varepsilon \sum_\pm \mp \chi_\pm(y_2) A^\pm_q C_p(y_1) \mathbf{W}_q(\eta^\pm). \end{aligned} \tag{3.20}$$

Here the eigenpair (3.4) with the subscripts $p \in \mathbb{N}_0$, $q \in \mathbb{N}$ of problem (3.3) is involved, as well as the quantities (3.8) and (2.42) and the cut–off functions

$$\begin{aligned} & \chi_\pm(y_2) = \chi(1 \mp y_2), \\ & X^\varepsilon(y_2) = 1 \quad \text{for} \quad |y_2| \leqslant 1 - 2\varepsilon \quad \text{and} \quad X^\varepsilon(y_2) = 0 \quad \text{for} \quad |y_2| \geqslant 1 - \varepsilon, \\ & X^\varepsilon \in C^\infty_c(\mathbb{R}), \quad 0 \leqslant X^\varepsilon(y_2) \leqslant 1, \quad \left| \frac{d^j X^\varepsilon}{dy_2^j}(y_2) \right| \leqslant c_j \varepsilon^{-j}, \quad j \in \mathbb{N}_0. \end{aligned} \tag{3.21}$$

We note that first, owing to the choice of ingredients $v^\varepsilon_{(p,q)}$ and C_p the function (3.20) fulfils the boundary conditions (1.2), (1.3), and second

$$\frac{\partial v^\varepsilon_{(p,q)}}{\partial z}(x) - v_{(p,q)}(y) \frac{\pi}{\varepsilon} \cos\left(\frac{\pi}{\varepsilon} z\right) = O\left(\varepsilon + e^{-\frac{1-|y_2|}{\varepsilon}} \max\left\{1, \left(\varrho^\varepsilon_\pm\right)^{-\frac{1}{3}}\right\}\right),$$

where $\varrho^\varepsilon_\pm = \left((|y_2| - 1 + \varepsilon)^2 + z^2\right)^{\frac{1}{2}} / \varepsilon$, cf. the expansion (2.4). Thus,

$$\left| \langle v^\varepsilon_{(p,q)}, v^\varepsilon_{(m,n)} \rangle_\varepsilon - \frac{\pi^2}{2\varepsilon} \delta_{p,m} \delta_{q,n} \right| \leqslant c_{pq,mn} \varepsilon,$$

and, in particular,

$$\begin{aligned} & \left| \langle U^\varepsilon_{(p,q)}, U^\varepsilon_{(m,n)} \rangle_\varepsilon - \delta_{p,m} \delta_{q,n} \right| \leqslant C_{pq,mn} \varepsilon, \\ & \|v^\varepsilon_{(p,q)}; \mathcal{H}^\varepsilon\| \geqslant \mathbf{c}_{(p,q)} \varepsilon^{-\frac{1}{2}}, \quad \mathbf{c}_{(p,q)} > 0. \end{aligned} \tag{3.22}$$

We treat the quantity $\delta^\varepsilon_{(p,q)}$ from the formula (3.16) calculated by the pair (3.19). We have

$$\begin{aligned}\delta^\varepsilon_{(p,q)} &= \sup_{\dots} \big|\langle \mathcal{T}^\varepsilon U^\varepsilon_{(p,q)} - t^\varepsilon_{(p,q)} U^\varepsilon_{(p,q)}, \psi^\varepsilon\rangle_\varepsilon\big| \\ &= t^\varepsilon_{(p,q)} \|v^\varepsilon_{(p,q)}; \mathcal{H}^\varepsilon\|^{-1} \sup_{\dots} \big|(\nabla_x v^\varepsilon_{(p,q)}, \nabla_x \psi^\varepsilon)_{\Omega^\varepsilon} - (\pi^2\varepsilon^{-2} + \mu_{(p,q)})(v^\varepsilon_{(p,q)}, \psi^\varepsilon)_{\Omega^\varepsilon}\big| \\ &= t^\varepsilon_{(p,q)} \|v^\varepsilon_{(p,q)}; \mathcal{H}^\varepsilon\|^{-1} \sup_{\dots} \big|((\Delta_x + \pi^2\varepsilon^{-2} + \mu_{(p,q)}) v^\varepsilon_{(p,q)}, \psi^\varepsilon)_{\Omega^\varepsilon}\big|.\end{aligned} \tag{3.23}$$

Here the supremum is calculated over the unit ball in the space $\mathcal{H}^\varepsilon$, that is, $\|\psi^\varepsilon; \mathcal{H}^\varepsilon\| \leqslant 1$, and by the one–dimensional Friedrichs inequality the relation

$$\frac{1}{\varepsilon^2} \int_{\Omega^\varepsilon} \big|\psi^\varepsilon(x)\big|^2 dydz \leqslant \frac{1}{\pi^2} \int_0^\varepsilon \int_{\square_1} \mathbf{h}_\varepsilon(y_2)^{-2} \big|\psi^\varepsilon(x)\big|^2 dydz \leqslant \frac{1}{\pi^2} \Big\| \frac{\partial \psi^\varepsilon}{\partial z}; L^2(\Omega^\varepsilon)\Big\|^2 \tag{3.24}$$

holds. Here $\mathbf{h}_\varepsilon(y_2) = \min\{\varepsilon, 1 - |y_2|\}$ is the width of the domain (1.4) and $\square_1$ is its square base. Bearing in mind the differential equations satisfied by the functions $v_{(p,q)}$ and $\mathbf{W_D}$ and the formulas (2.42) and (2.43) for the latter function, we find that the first factor

$$I^\varepsilon_{(p,q)} = (\Delta_x + \pi^2\varepsilon^{-2} + \mu_{(p,q)}) v^\varepsilon_{(p,q)}$$

in the last scalar product in (3.24) becomes

$$\begin{aligned} I^\varepsilon_{(p,q)} = &\sin\Big(\pi \frac{z}{\varepsilon}\Big) \Big[\frac{d^2}{dy_2^2}, X^\varepsilon\Big] \Big(v_{(p,q)} - \sum_\pm C_p A^\pm_q (y_2 \mp 1)\Big) \\ &+ \varepsilon \sum_\pm A^\pm_q C_p \Big[\frac{d^2}{dy_2^2}, \chi_\pm\Big] \widehat{\mathbf{W}}_D + \varepsilon \mu_{(p,q)} \sum_\pm A^\pm_q C_p \chi_\pm \widehat{\mathbf{W}}_D =: I^{1\varepsilon}_{(p,q)} + I^{2\varepsilon}_{(p,q)} + I^{3\varepsilon}_{(p,q)}. \end{aligned}$$

Owing to the Taylor formula (3.7) and the estimate (3.24) we obtain that

$$\big|\big(I^{1\varepsilon}_{(p,q)}, \psi^\varepsilon\big)_{\Omega^\varepsilon}\big| \leqslant c\Big(\operatorname{mes}_3 Y^\varepsilon \max_{y \in Y^\varepsilon} \sum_{j=0,1} \varepsilon^{-2j} (1 - |y_2|)^{4+2j}\Big)^{\frac{1}{2}} \sup_{\dots} \|\psi^\varepsilon; L^2(\Omega^\varepsilon)\| \leqslant c_{(p,q)} \varepsilon^4.$$

In the above relations $[\mathfrak{D}, \mathfrak{x}]$ is the commutator of the differential operator $\mathfrak{D}$ with the cut–off function $\mathfrak{x}$, Y^ε is the set

$$\operatorname{supp} |\nabla_x X^\varepsilon| = \{x : |y_1| \leqslant 1, |y_2| \in [1 - 2\varepsilon, 1 - \varepsilon], z \in (0, \varepsilon)\}$$

and its volume $\operatorname{mes}_3 Y^\varepsilon$ is equal to $4\varepsilon^2$. Since $\operatorname{mes}_3 \operatorname{supp} |\nabla_x \chi_\pm| = O(\varepsilon)$ and the function $\widehat{\mathbf{W}}_D$ is bounded in the semi–strip $\overline{\Pi}$, while its derivative in η_1 decays exponentially at infinity, we find out that

$$\begin{aligned} &\big|\big(I^{2\varepsilon}_{(p,q)}, \psi^\varepsilon\big)_{\Omega^\varepsilon}\big| + \big|\big(I^{3\varepsilon}_{(p,q)}, \psi^\varepsilon\big)_{\Omega^\varepsilon}\big| \\ &\qquad \leqslant c\varepsilon\Big(\big(\varepsilon^{-2} e^{-\frac{2\kappa}{3\varepsilon}} + 1 + \mu_{(p,q)}\big) \operatorname{mes}_3 \operatorname{supp} |\nabla_x \chi_\pm|\Big)^{\frac{1}{2}} \sup_{\dots} \|\psi^\varepsilon; L^2(\Omega^\varepsilon)\| \leqslant c_{(p,q)} \varepsilon^{\frac{5}{2}}. \end{aligned}$$

Finally, in view of the formulas (3.19) and (3.22), for the quantity (3.23) we get the estimate

$$\delta_{(p,q)} \leqslant c_{(p,q)} \varepsilon^2 \varepsilon^{\frac{1}{2}} \varepsilon^{\frac{5}{2}} = c_{(p,q)} \varepsilon^5, \tag{3.25}$$

and hence, by Lemma 3.1 there exists an eigenvalue $\tau_{n_{(p,q)}(\varepsilon)}$ of the operator $\mathcal{T}^\varepsilon$, which obeys the inequality

$$\big|\tau^\varepsilon_{n_{(p,q)}(\varepsilon)} - \varepsilon^2 (\pi^2 + \varepsilon^2 \mu_{(p,q)})^{-1}\big| \leqslant c_{(p,q)} \varepsilon^5. \tag{3.26}$$

By the relation (3.15) of spectral parameters this implies

$$\big|\lambda^\varepsilon_{n_{(p,q)}(\varepsilon)} - \varepsilon^{-2}\pi^2 - \mu_{(p,q)}\big| \leqslant c_{(p,q)} \varepsilon^3 \lambda^\varepsilon_{n_{(p,q)}(\varepsilon)} \big(\pi^2 + \varepsilon^2 \mu_{(p,q)}\big). \tag{3.27}$$

Moreover,

$$\begin{aligned}&\lambda^\varepsilon_{n_{(p,q)}(\varepsilon)} \leqslant \varepsilon^{-2}\pi^2 + \mu_{(p,q)} + c_{(p,q)}\varepsilon^3\lambda^\varepsilon_{n_{(p,q)}(\varepsilon)}\big(\pi^2+\varepsilon^2\mu_{(p,q)}\big)\\ &\Rightarrow \quad \lambda^\varepsilon_{n_{(p,q)}(\varepsilon)} \leqslant \frac{2}{\varepsilon^2}\big(\pi^2+\varepsilon^2\mu_{(p,q)}\big) \quad \text{for} \quad c_{(p,q)}\varepsilon^3\big(\pi^2+\varepsilon^2\mu_{(p,q)}\big) \leqslant \frac{1}{2}.\end{aligned} \tag{3.28}$$

We hence find

$$\big|\lambda^\varepsilon_{n_{(p,q)}(\varepsilon)} - \varepsilon^{-2}\pi^2 - \mu_{(p,q)}\big| \leqslant 2c_{(p,q)}\varepsilon\big(\pi^2+\varepsilon^2\mu_{(p,q)}\big)^2 \tag{3.29}$$

or finally

$$\big|\lambda^\varepsilon_{n_{(p,q)}(\varepsilon)} - \varepsilon^{-2}\pi^2 - \mu_{(p,q)}\big| \leqslant C_{(p,q)}\varepsilon \quad \text{as} \quad \varepsilon \in \big(0, \varepsilon_{(p,q)}\big] \tag{3.30}$$

with positive quantities $C_{(p,q)}$ and $\varepsilon_{(p,q)}$ chosen in accordance with the formula (3.28).

In order to confirm the coincidence of eigenvalues of problem (1.1)–(1.3) from the formulas (1.10) and (3.30), we need additional calculations and arguing.

Let us verify that for the eigenvalue $\mu_{(p,q)}$ of multiplicity $\varkappa_{(p,q)} > 1$ there exist at least $\varkappa_{(p,q)}$ different eigenvalues $\lambda^\varepsilon_{n_{(p,q)}(\varepsilon)}, \dots, \lambda^\varepsilon_{n_{(p,q)}(\varepsilon)+\varkappa_{(p,q)}-1}$ in the sequence (1.8). We employ the second part of Lemma 3.1 and denote by δ^ε the maximal of above treated quantities $\delta^\varepsilon_{(p,q)}$, while by δ^ε_* we denote the product $t^{-1}\delta^\varepsilon$ with a factor $t \in (0,1)$. Let also $\mathcal{S}^\varepsilon_{n_{(p,q)}(\varepsilon)+k}$, $k = 0, \dots, \varkappa_{(p,q)} - 1$, be the sums over $\ell = \mathcal{N}^\varepsilon, \dots, \mathcal{N}^\varepsilon + \mathcal{X}^\varepsilon - 1$ from the first formula in the list (3.17), and $\mathcal{C}^\varepsilon_{n_{(p,q)}(\varepsilon)+k} \in \mathbb{R}^{\mathcal{X}^\varepsilon}$ the columns of coefficients of these linear combinations; if it is needed, we add zero terms to align the sizes of columns. Then by the relations (3.18) and (3.22) we find

$$\begin{aligned}\big|(\mathcal{C}^\varepsilon_{n_{(p,q)}(\varepsilon)+j}, \mathcal{C}^\varepsilon_{n_{(p,q)}(\varepsilon)+k})_{\mathbb{R}^{\mathcal{X}^\varepsilon}} - \delta_{j,k}\big| =&\big|\langle \mathcal{S}^\varepsilon_{n_{(p,q)}(\varepsilon)+j}, \mathcal{S}^\varepsilon_{n_{(p,q)}(\varepsilon)+k}\rangle_\varepsilon - \delta_{j,k}\big|\\ \leqslant&\big|\langle \mathcal{S}^\varepsilon_{n_{(p,q)}(\varepsilon)+j}, \mathcal{S}^\varepsilon_{n_{(p,q)}(\varepsilon)+j} - U^\varepsilon_{n_{(p,q)}(\varepsilon)+k}\rangle_\varepsilon\big|\\ &+ \big|\langle \mathcal{S}^\varepsilon_{n_{(p,q)}(\varepsilon)+j} - U^\varepsilon_{n_{(p,q)}(\varepsilon)+j}, U^\varepsilon_{n_{(p,q)}(\varepsilon)+k}\rangle_\varepsilon\big|\\ &+ \big|\langle U^\varepsilon_{n_{(p,q)}(\varepsilon)+j}, U^\varepsilon_{n_{(p,q)}(\varepsilon)+k}\rangle_\varepsilon - \delta_{j,k}\big| \leqslant 2t + 2t + C_{(p,q)}\varepsilon.\end{aligned}$$

Therefore, for small t and ε the columns $\mathcal{C}^\varepsilon_{n_{(p,q)}(\varepsilon)+1}, \dots, \mathcal{C}^\varepsilon_{n_{(p,q)}(\varepsilon)+\varkappa_{(p,q)}-1}$ are almost orthonormalized in the Euclidean space $\mathbb{R}^{\mathcal{X}^\varepsilon}$, and this is possible only in the case

$$\varkappa_{(p,q)} \leqslant \mathcal{X}^\varepsilon.$$

In other words, having fixed appropriate $t \in (0,1)$ and $\varepsilon \in (0, \varepsilon_{(p,q)}]$, we find at least $\varkappa_{(p,q)}$ different eigenvalues of the operator $\mathcal{T}^\varepsilon$ obeying the estimate (3.26) with the enlarged in t^{-1} times majorant. By means of the previous calculations (3.27), (3.28) we get that at least $\varkappa_{(p,q)}$ different terms of the sequence (1.8) satisfy the inequality (3.30) with new positive numbers $C_{(p,q)}$ and $\varepsilon_{(p,q)}$. Using this observation and sorting out the eigenvalues of limiting problem (3.3) not exceeding $\mu_{(p,q)}$, we obtain the apriori estimate for the eigenvalues of original problem (1.1)–(1.3)

$$\lambda^\varepsilon_m \leqslant \varepsilon^{-2}\pi^2 + c_m. \tag{3.31}$$

3.4. Convergences. We continue the eigenfunction u^ε_m, normalized by the identity (1.17) for $j, k = m$, by zero from the polyhedron Ω^ε to the parallelepiped

$$\Omega^\varepsilon_\square = (-1,1)^2 \times (0, \varepsilon)$$

and define the functions

$$u^{\varepsilon 0}_m(y) = \int_0^\varepsilon S^\varepsilon(z) u^\varepsilon_m(y,z)dz, \qquad u^{\varepsilon\perp}_m(y,z) = u^\varepsilon_m(y,z) - S^\varepsilon(z)u^{\varepsilon 0}_m(y).$$

Here

$$S^\varepsilon(z) = \sqrt{\frac{2}{\varepsilon}} \sin\Big(\pi \frac{z}{\varepsilon}\Big). \tag{3.32}$$

It is clear that the function $u_m^{\varepsilon 0}$ vanishes on the sides $\upsilon_1^\pm$ of the square $\square_1 = (-1,1)^2$ and belongs to the space $H_0^1(\square_1; \upsilon_1^+ \cup \upsilon_1^-)$; here

$$\upsilon_k^\pm = \{y : |y_k| < 1, y_{3-k} = \pm 1\}, \quad k = 1, 2. \tag{3.33}$$

Moreover, the orthogonality condition holds

$$\int_0^\varepsilon S^\varepsilon(z) u_m^{\varepsilon\perp}(y,z)dz = 0 \quad \text{for} \quad y \in \square_1, \tag{3.34}$$

and hence, the Poincaré inequality yields the estimate

$$\|u_m^{\varepsilon\perp}; L^2(\Omega_\square^\varepsilon)\|^2 \leqslant \frac{\varepsilon^2}{4\pi^2} \|\partial_z u_m^{\varepsilon\perp}; L^2(\Omega_\square^\varepsilon)\|^2. \tag{3.35}$$

The relations

$$\|u_m^\varepsilon; L^2(\Omega_\square^\varepsilon)\|^2 = \|u_m^{\varepsilon\perp}; L^2(\Omega_\square^\varepsilon)\|^2 + \|u_m^{\varepsilon 0}; L^2(\square_1)\|^2,$$
$$\|\nabla_y u_m^\varepsilon; L^2(\Omega_\square^\varepsilon)\|^2 = \|\nabla_y u_m^{\varepsilon\perp}; L^2(\Omega_\square^\varepsilon)\|^2 + \|\nabla_y u_m^{\varepsilon 0}; L^2(\square_1)\|^2,$$
$$\|\partial_z u_m^\varepsilon; L^2(\Omega_\square^\varepsilon)\|^2 = \|\partial_z u_m^{\varepsilon\perp}; L^2(\Omega_\square^\varepsilon)\|^2 + \int_0^\varepsilon \big(\partial_z S^\varepsilon(z)\big)^2 dz \|u_m^{\varepsilon 0}; L^2(\square_1)\|^2$$
$$+ 2\int_{\Omega_\square^\varepsilon} \partial_z u_m^{\varepsilon\perp}(x) u_m^{\varepsilon 0}(y) \partial_z S^\varepsilon(z)dx = \|\partial_z u_m^{\varepsilon\perp}; L^2(\Omega_\square^\varepsilon)\|^2 + \frac{\pi^2}{\varepsilon^2}\|u_m^{\varepsilon 0}; L^2(\square_1)\|^2$$

hold. The latter integral over the parallelepiped $\Omega_\square^\varepsilon$ vanishes by means of integration by parts. Thus, we transform the integral identity (1.9) with the test function $\psi^\varepsilon = u_m^\varepsilon$ into the form

$$\|\partial_z u_m^{\varepsilon\perp}; L^2(\Omega_\square^\varepsilon)\|^2 + \|\nabla_y u_m^{\varepsilon\perp}; L^2(\Omega_\square^\varepsilon)\|^2 + \|\nabla_y u_m^{\varepsilon 0}; L^2(\square_1)\|^2$$
$$= \Big(\lambda_m^\varepsilon - \varepsilon^{-2}\pi^2\Big)\|u_m^{\varepsilon 0}; L^2(\square_1)\|^2 + \lambda_m^\varepsilon \|u_m^{\varepsilon\perp}; L^2(\Omega_\square^\varepsilon)\|^2.$$

In view of the formulas (3.31) and (3.35) this gives the estimates

$$\begin{aligned}
&\|\partial_z u_m^{\varepsilon\perp}; L^2(\Omega_\square^\varepsilon)\|^2 - \lambda_m^\varepsilon \|u_m^{\varepsilon\perp}; L^2(\Omega_\square^\varepsilon)\|^2 + \|\nabla_y u_m^{\varepsilon 0}; L^2(\square_1)\|^2 \\
&\qquad \leqslant \big(\lambda_m^\varepsilon - \varepsilon^{-2}\pi^2\big)\|u_m^{\varepsilon 0}; L^2(\square_1)\|^2 \leqslant c_m \|u_m^{\varepsilon 0}; L^2(\square_1\|^2 \leqslant c_m \\
&\qquad \Rightarrow \quad \|\nabla_y u_m^{\varepsilon 0}; L^2(\square_1)\|^2 \leqslant c_m \quad \text{and} \\
&3\frac{\pi^2}{\varepsilon^2}\|u_m^{\varepsilon\perp}; L^2(\Omega_\square^\varepsilon)\|^2 \leqslant \Big(4\frac{\pi^2}{\varepsilon^2} - \lambda_m^\varepsilon\Big)\|u_m^{\varepsilon\perp}; L^2(\Omega_\square^\varepsilon)\|^2 \leqslant c_m.
\end{aligned} \tag{3.36}$$

Thus, along some infinitesimal positive sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ we have the convergences

$$\begin{aligned}
&u_m^{\varepsilon 0} \quad \to \quad u_m^{00} \quad \text{weakly in} \quad H_0^1(\square_1; \upsilon_1^+ \cup \upsilon_1^-) \quad \text{and strongly in} \quad L^2(\square_1), \\
&\lambda_m^\varepsilon - \varepsilon^{-2}\pi^2 \quad \to \quad \mu_m^0, \quad \|u_m^{\varepsilon 0}; L^2(\square_1)\| \quad \to \quad 1.
\end{aligned} \tag{3.37}$$

Now we substitute the test function $\psi^\varepsilon = S^\varepsilon\varphi$ into the integral identity (1.9), where $\varphi \in C_c^\infty\big(\overline{\square_1} \setminus (\upsilon_1^+ \cup \upsilon_1^-)\big)$. As above, the orthogonality condition (3.34) shows that the right hand side of the obtained relation

$$\big(\nabla_y u_m^{\varepsilon_j 0}, \nabla_y \varphi\big)_{\square_1} - \big(\lambda_m^{\varepsilon_j} - \varepsilon^{-2}\pi^2\big)\big(u_m^{\varepsilon_j 0}, \varphi\big)_{\square_1}$$
$$= \lambda_m^{\varepsilon_j}\big(u_m^{\varepsilon_j\perp}, S^\varepsilon\varphi\big)_{\Omega_\square^\varepsilon} - \big(\partial_z u_m^{\varepsilon_j\perp}, \varphi\partial_z S^\varepsilon\big)_{\Omega_\square^\varepsilon} - \big(\nabla_y u_m^{\varepsilon_j\perp}, S^\varepsilon\nabla_y\varphi\big)_{\Omega_\square^\varepsilon}$$

is zero and hence, the convergences (3.37) ensure the integral identity

$$\big(\nabla_y u_m^{00}, \nabla_y \varphi\big)_{\square_1} = \mu_m^0 \big(u_m^{00}, \varphi\big)_{\square_1},$$

which serves the limiting problem (3.3) since taking the closure we can pass to the test functions $\varphi \in H_0^1(\square_1; \upsilon_1^+ \cup \upsilon_1^-)$.

Lemma 3.2. *The limits* (3.37) *provide an eigenpair of the limiting problem* (3.3)*, and the eigenfunction u_m^{00} is normalized in the space $L^2(\square_1)$.*

3.5. Final theorems on asymptotics. We complete the justification of asymptotic formulas. In particular, we need to verify that the index $n_{(p,q)}(\varepsilon)$ of the eigenvalue of original problem (1.1)–(1.3) in the formula (3.30) coincides with the index m of the eigenvalue $\mu_{(p,q)}$ of limiting problem (3.3) in the monotone sequence (3.5). The arguing, which led us to the estimate (3.31), gives the inequality $n_{(p,q)}(\varepsilon) \geqslant m$. Suppose that $n_{(p,q)}(\varepsilon) > m$ for an infinitesimal positive sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$. Then for the indices $j \in \mathbb{N}$ there exist the eigenvalues $\lambda_{\mathbf{n}_j}^{\varepsilon_j} \leqslant \varepsilon^{-2}\pi^2 + \mu_{(p,q)} + c_{(p,q)}\varepsilon_j^{\frac{1}{2}}$, the associated eigenfunctions of which satisfy the orthogonality conditions

$$(u_{\mathbf{n}_j}^{\varepsilon_j}, u_q^{\varepsilon})_{\Omega^\varepsilon} = 0, \qquad q = 1, \dots, m + \varkappa_m - 1,$$

where $\varkappa_m$ is the multiplicity of eigenvalue μ_m. Hence, the limits (3.37) and estimates (3.36) provide the eigenvalue $\mu^{00} \leqslant \mu_m$ of the problem (3.3), the associated eigenfunction $u^{00} \in H_0^1(\square_1; \upsilon_1^+ \cup \upsilon_1^-)$ of which is orthogonal in the space $L^2(\square_1)$ to the eigenfunctions $v_1, \dots, v_{m-1}, v_m, \dots, v_{m+\varkappa_m-1}$. This observation contradicts the way of constructing the sequence (3.5), that is, we indeed have $n_{(p,q)}(\varepsilon) = m$. Thus, the following statement hold.

Theorem 3.1. *The terms of sequences* (1.8) *and* (3.5) *of eigenvalues of problem* (1.1)–(1.3) *and* (3.3) *respectively satisfy the relation*

$$\Big|\lambda_m^\varepsilon - \varepsilon^{-2}\pi^2 - \mu_m\Big| \leqslant c_m\varepsilon \quad \text{for} \quad \varepsilon \in (0, \varepsilon_m],$$

where c_m and ε_m are some positive numbers.

We shall formulate the theorem on asymptotics of the eigenfunctions of problem (1.1)–(1.3) for a simple ($p = 0$, $q \in \mathbb{N}$ or $p = q \in \mathbb{N}$) eigenvalue of problem (3.3); the case of a multiple eigenvalue can be treated in the same way but the final formula becomes not so explicit, while its derivation in similar situation was published many times. Moreover, in view of the symmetry of domain (1.4) some multiple eigenvalues (for instance, the pair (p, q) includes an odd and an even number) can be split by imposing[1] artificial Dirichlet or Neumann conditions on the sections

$$\Upsilon_{0k}^\varepsilon = \{x \in \Omega^\varepsilon : y_k = 0\}, \quad k = 1, 2, \tag{3.38}$$

Theorem 3.2. *Let μ_m be a simple eigenvalue of problem* (3.3)*, and v_m be the associated eigenfunction (see* (3.4)*). Then the sign of normalized in the space $L^2(\Omega^\varepsilon)$ eigenfunction u_m^ε of problem* (1.1)–(1.3) *can be chosen so that the asymptotic formula*

$$\varepsilon\Big\|\nabla_x\Big(u_m^\varepsilon - \Big(\frac{2}{\varepsilon}\Big)^{\frac{1}{2}} S^\varepsilon v_m\Big); L^2(\Omega^\varepsilon)\Big\| + \Big\|u_m^\varepsilon - \Big(\frac{2}{\varepsilon}\Big)^{\frac{1}{2}} S^\varepsilon v_m; L^2(\Omega^\varepsilon)\Big\| \leqslant C_m\varepsilon, \tag{3.39}$$

holds, where S^ε is the function (3.32)*, C_m and ε_m are some positive numbers and $\varepsilon \in (0, \varepsilon_m]$.*

Proof. By Theorem 3.1 for some $h > 0$ the interval $\big(\varepsilon^{-2}\pi^2 + \mu_m - h, \varepsilon^{-2}\pi^2 + \mu_m + h\big)$ contains a unique eigenvalue λ_m^ε. The relation (3.15) of spectral parameters again shows that for some $h > 0$ the closed segment

$$\big[\varepsilon^2(\pi^2 + \varepsilon^2\mu_m)^{-1} - \varepsilon^4 h, \varepsilon^2(\pi^2 + \varepsilon^2\mu_m)^{-1} + \varepsilon^4 h\big] \tag{3.40}$$

[1] This approach is employed in next two sections.

contains the unique eigenvalue τ_m^ε of the operator $\mathcal{T}^\varepsilon$. In Lemma 3.1 we take the numbers $\delta^\varepsilon \leqslant c_m \varepsilon^5$ and $\delta_*^\varepsilon = \varepsilon^4 h$ from the formulas (3.25) and (3.40). Then the sums in the relations (3.17) involve a single term, and hence, the inequality

$$\left\|U_m^\varepsilon - \mathcal{C}_m \mathcal{U}_m^\varepsilon; \mathcal{H}^\varepsilon\right\| \leqslant 2\left(\delta_*^\varepsilon\right)^{-1} \delta^\varepsilon \leqslant 2c_m h^{-1}\varepsilon \tag{3.41}$$

holds and $\mathcal{C}_m = \pm 1$ depending on the choice of sign of eigenfunction $\mathcal{U}_m^\varepsilon$.

It remains to compare the normalization conditions (1.17) and (3.6) of eigenfunctions of problems (1.1)–(1.3) and (3.3) with the relation (3.18) for the eigenvectors of operator $\mathcal{T}^\varepsilon$: the inequality (3.41) implies the estimate (3.39). We just note that $L^2(\Omega^\varepsilon)$–norm of the subtrahend $\sqrt{2/\varepsilon} S^\varepsilon v_m$ in the left hand side of (3.39) is equal to $1 + O(\varepsilon)$. □

4. Localization near thin faces of polyhedron

4.1. Formal asymptotic constructions. In the situation (1.5) we make the rescaling of coordinates (1.15) and formally let $\varepsilon = 0$. In the both cases $\pm$ the domain (1.4) is transformed in the set $(-1,1) \times \Pi \ni (y_1, \eta_1, \eta_2)$, where Π is the pointed semi–strip (1.14). On the base of results in Section 2.1 we admit the asymptotic ansätze

$$\lambda^\varepsilon(x) = \frac{\Lambda_1}{\varepsilon^2} + \mu + \dots,$$
$$u^\varepsilon(x) = W_1\Big(\frac{1 \mp y_2}{\varepsilon}, \frac{z}{\varepsilon}\Big) w_\pm(y_1) + \dots.$$

Substituting them into original problem (1.1)–(1.3) and equating the coefficients at the like powers of small parameter, we find that the factors at ε^{-2} in the relation (1.1) cancel out, while the factors at $1 = \varepsilon^0$ form an ordinary differential equation on the segment $(-1,1) \ni y_1$. Bearing in mind the boundary conditions (1.3) on the faces (1.7) we derive two ($\vartheta = \pm$) limiting Neumann problems

$$-\frac{\partial^2 w_\vartheta}{\partial y_1}(y_1) = \mu w_\vartheta(y_1), \quad y_1 \in (-1,1), \quad \pm\frac{\partial w_\vartheta}{\partial y_1}(\pm 1) = 0 \tag{4.1}$$

(cf. Remark 3.1). In what follows we omit the subscript ϑ. The eigenpairs

$$\{\mu_p; w_p\} = \Big\{\frac{\pi^2}{4}p^2; \cos\Big(\frac{\pi}{2}p(y_1 - 1)\Big)\Big\}$$

of problem (4.1) were involved in the expansions (1.12) and (1.13).

4.2. Asymptotics of eigenvalues. In order to simplify the justification of asymptotics, we use the symmetry of domain (1.4) with respect to the central section $\Upsilon_{02}^\varepsilon$ (the rectangle from the formula (3.38)) and impose artificial boundary conditions

$$\frac{\partial u^\varepsilon}{\partial x_2}(x) = 0 \quad \text{or} \quad u^\varepsilon(x) = 0 \quad \text{for} \quad x \in \Upsilon_{20}^\varepsilon. \tag{4.2}$$

We recall that the Neumann conditions are imposed on the entire lateral surface (1.5), that is, in what follows we suppose that respectively

$$\Gamma_{D+}^\varepsilon = \{\partial\Omega_+^\varepsilon : z \notin (0,\varepsilon)\} \quad \text{or} \quad \Gamma_{D+}^\varepsilon = \{\partial\Omega_+^\varepsilon : z \notin (0,\varepsilon)\} \cup \Upsilon_{02}^\varepsilon. \tag{4.3}$$

At the same time the original problem in the polyhedron Ω^ε is restricted to its half

$$\Omega_+^\varepsilon = \{x \in \Omega^\varepsilon : y_2 > 0\}.$$

We shall employ the notation (1.1)–(1.3), (4.2) independently on the choice of artificial boundary condition; it does not influence the asymptotic formulas. At the same time, the even in the case $(4.2)_N$ and odd in the case $(4.2)_D$ continuation of eigenfunction of this problem from the domain Ω_+^ε through the abscise axis to the domain Ω^ε provides a smooth eigenfunction of original problem (1.1)–(1.3).

We make necessary changes in the definitions in Section 3.2, but we keep the notation for the Hilbert space $\mathcal{H}^\varepsilon$ with the scalar product $\langle , \rangle_\varepsilon$ and the operator $\mathcal{T}^\varepsilon$ in $\mathcal{H}^\varepsilon$.

As almost eigenpairs of the problem in Ω_+^ε we take

$$\{t_p^\varepsilon; U_p^\varepsilon\} = \Big\{\varepsilon^2\left(\Lambda_1 + \varepsilon^2\mu_p\right)^{-1}; \|w_p^\varepsilon; \mathcal{H}^\varepsilon\|^{-1} w_p^\varepsilon\Big\}, \tag{4.4}$$

where $p \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$,

$$w_p^\varepsilon(x) = \chi_+(y_2) W_1\Big(\frac{1-y_2}{\varepsilon}, \frac{z}{\varepsilon}\Big) w_p(y_1), \tag{4.5}$$

and $\{\Lambda_1; W_1\}$ and $\{\mu_p; w_p\}$ are the eigenpairs of problems (2.1)–(2.3) and (4.1), respectively. Finally, χ_+ is the cut–off function in the list (3.21).

We note that the constructions (4.4) and (4.5) are same for both cases (4.3) since the boundary conditions on remote from the faces (1.16) part of the boundary $\partial\Omega_+^\varepsilon$ do not influence the leading terms of asymptotics. At the same time, owing to the presence of cut–off function χ_+ the only remainder of function (4.5) in the boundary value problem Ω_+^ε with the parameter $\lambda^\varepsilon = \varepsilon^{-2}\Lambda_1 + \mu_p$ appears in the differential equation

$$(\Delta_x + \varepsilon^{-2}\Lambda_1 + \mu_p) v_p^\varepsilon(x) = \Big[\frac{\partial^2}{\partial y_2^2}, \chi_+(y_2)\Big] W_1\Big(\frac{1-y_2}{\varepsilon}, \frac{z}{\varepsilon}\Big) w_p(y_1).$$

The support of this remainder is located in the set $\big\{x \in \overline{\Omega^\varepsilon} : 1/3 \leqslant y_2 \leqslant 2/3\big\} \supset \operatorname{supp} |\nabla_y \chi_+|$, where the factor W_1 turns out to be exponentially small in accordance with the expansion (2.7).

We note that

$$\|v_p^\varepsilon; \mathcal{H}^\varepsilon\|^2 = \int_{-1}^{1} |w_p(y_1)|^2 dy_1 \int_\Pi \big|\nabla_\eta(\chi(\varepsilon^{-1}\eta_1) W_1(\eta))\big|^2 d\eta = \Lambda_1 + O(e^{-\kappa/\varepsilon}) \tag{4.6}$$

for some $\kappa > 0$, see the representation (2.7), and treat the quantity δ_p^ε in the formula (3.16) found by the pair (4.4). We have

$$\begin{aligned} \delta_p^\varepsilon =& \|\mathcal{T}^\varepsilon U_p^\varepsilon - t_p^\varepsilon U_p^\varepsilon; \mathcal{H}^\varepsilon\| = \sup_{\dots} \big|\langle \mathcal{T}^\varepsilon U_p^\varepsilon - t_p^\varepsilon U_p^\varepsilon, \psi^\varepsilon\rangle_\varepsilon\big| \\ =& t_p^\varepsilon \|v_p^\varepsilon; \mathcal{H}^\varepsilon\|^{-1} \sup_{\dots} \Big|(\nabla_x v_p^\varepsilon, \nabla_x \psi^\varepsilon)_{\Omega_+^\varepsilon} - (\varepsilon^{-2}\Lambda_1 + \mu_p)(v_p^\varepsilon, \psi^\varepsilon)_{\Omega_+^\varepsilon}\Big| \\ =& t_p^\varepsilon \|v_p^\varepsilon; \mathcal{H}^\varepsilon\|^{-1} \sup_{\dots} \big|\big(\chi_+(\Delta_x + \varepsilon^{-2}\Lambda_1 + \mu_p)(W_1 w_p), \psi^\varepsilon\big)_{\Omega_+^\varepsilon} \\ &+ ([\Delta_x, \chi_+](W_1 w_p), \psi^\varepsilon)_{\Omega_+^\varepsilon}\big|. \end{aligned} \tag{4.7}$$

Here the supremum is taken over the unit ball in the space $\mathcal{H}^\varepsilon$, that is, $\|\psi^\varepsilon; \mathcal{H}^\varepsilon\| \leqslant 1$, and hence,

$$\|\psi^\varepsilon; L^2(\Omega_+^\varepsilon)\|^2 \leqslant c_+ \varepsilon^2 \|\nabla_x \psi^\varepsilon; L^2(\Omega_+^\varepsilon)\|^2 = c_+ \varepsilon^2 \|\psi^\varepsilon; \mathcal{H}^\varepsilon\|^2 \leqslant c_+ \varepsilon^2, \quad c_+ > 0. \tag{4.8}$$

We stress that the inequality (4.8) differs from the inequality (3.24) since in the situation (1.5) the Dirichlet conditions are imposed not on faces (1.16), but the estimate (4.8) is ensured by Proposition 2.1, where we can take, for instance, $c_+ = \Lambda_1/2$.

By the definition of the functions W_1 and w_p the first term in the sum under between the last modulus in (4.7) is equal to zero. Therefore, according the formulas (4.4)–(4.6), (4.8) and (2.7), the inequalities

$$\delta_p^\varepsilon \leqslant c_p' \varepsilon^2 \big(\varepsilon(1 + \varepsilon^{-2}) e^{-\frac{2\kappa}{\varepsilon}}\big)^{\frac{1}{2}} \sup_{\dots} \|\psi^\varepsilon; L^2(\Omega_+^\varepsilon)\| \leqslant c_p \varepsilon^{\frac{5}{2}} e^{-\frac{\kappa}{\varepsilon}}$$

hold. Thus, by Lemma 3.1 there exists an eigenvalue $\tau^\varepsilon_{n_p(\varepsilon)}$ of the operator $\mathcal{T}^\varepsilon$, for which the estimate

$$\left|\tau^\varepsilon_{n_p(\varepsilon)} - t^\varepsilon_p\right| \leqslant c_p \varepsilon^{\frac{5}{2}} e^{-\frac{\kappa}{\varepsilon}} \tag{4.9}$$

is true. As a result, the relation (3.16) of spectral parameters and transformations similar to (3.26)–(3.29) establish the existence of an eigenvalue $\lambda^\varepsilon_{n_p(\varepsilon)}$ of problem (1.1)–(1.3), (4.2) in the domain Ω^ε_+ obeying the relation

$$\left|\lambda^\varepsilon_{Kn_p(\varepsilon)} - \varepsilon^{-2}\Lambda_1 - \mu_p\right| \leqslant C_p \varepsilon^{-\frac{3}{2}} e^{-\frac{\kappa}{\varepsilon}} \quad \text{for} \quad \varepsilon \in (0, \varepsilon_p]. \tag{4.10}$$

Here C_p and ε_p are some positive numbers. We note that the relations (4.9) and (4.10) concern both artificial consitions (4.2), that is, the formula (4.10) provides two eigenvalues of the original problem in the entire domain Ω^ε.

4.3. Convergences. The change of coordinates (1.15) with the plus sign, which we omit in what follows, transforms the domain Ω^ε_+ into the set $(-1,1)\times\Pi^{\frac{1}{\varepsilon}}$, and Proposition 2.1 for each $h>0$ and small $\varepsilon\in(0,\varepsilon_h]$, $\varepsilon_h>0$, gives the inequality

$$\varepsilon^{-2}(\Lambda_1 - h)\|\psi^\varepsilon; L^2(\Omega^\varepsilon_+)\|^2 \leqslant \|\nabla_x \psi^\varepsilon; L^2(\Omega^\varepsilon_+)\|^2 \quad \forall \psi^\varepsilon \in H^1_0(\Omega^\varepsilon_+; \Gamma^\varepsilon_{D+}). \tag{4.11}$$

In the integral identity (1.9) corresponding to the problem (1.1)–(1.3), (4.2), we substitute the test function $\psi^\varepsilon = E^\varepsilon_\kappa \mathbf{u}^\varepsilon_m$, where $\mathbf{u}^\varepsilon_m(x) = E^\varepsilon_\kappa(y) u^\varepsilon_m(x)$ and

$$E^\varepsilon_\kappa(y) = \begin{cases} e^{\frac{\kappa}{\varepsilon}(1-y_2)} & \text{for} \quad y_2 \leqslant 1-\varepsilon, \\ e^\kappa & \text{for} \quad y_2 \geqslant 1-\varepsilon, \end{cases} \tag{4.12}$$

and u^ε_m is a normalized in $L^2(\Omega^\varepsilon_+)$ eigenfunction associated with some eigenvalue

$$\lambda^\varepsilon_m \leqslant \varepsilon^{-2}\Lambda_1 + \mathbf{c}_m. \tag{4.13}$$

At the same time $\mathbf{c}_m \geqslant 0$, and $\varepsilon > 0$ is a temporarily fixed small value of geometric parameter. In particular, the eigenvalues appearing in the estimate (4.10) satisfy the condition (4.13).

Several times commuting the operator–gradient ∇_x with the weight function E^ε_κ, we arrive at the identity

$$\|\nabla_x \mathbf{u}^\varepsilon_m; L^2(\Omega^\varepsilon_+)\|^2 - \|\mathbf{u}^\varepsilon_m E^\varepsilon_{-\kappa} \nabla_y E^\varepsilon_\kappa; L^2(\Omega^\varepsilon_+)\|^2 = \lambda^\varepsilon_m \|\mathbf{u}^\varepsilon_m; L^2(\Omega^\varepsilon_+)\|^2. \tag{4.14}$$

We note that

$$E^\varepsilon_{-\kappa}(y)\left|\nabla_x E^\varepsilon_\kappa(y)\right| = \begin{cases} \dfrac{\kappa}{\varepsilon} & \text{for} \quad y_2 \leqslant 1-\varepsilon, \\ 0 & \text{for} \quad y_2 \geqslant 1-\varepsilon, \end{cases} \tag{4.15}$$

and reproduce with some changes the calculations presented in Section 2.4. As a result, the formulas (4.11)–(4.14) imply the following weight estimate indicating the concentration of the eigenfunctions of both problems in the domain Ω^ε_+ near its face Γ^ε_+.

Theorem 4.1. *If an eigenvalue λ^ε_m of problem (1.1)–(1.3), (4.2) obeys the relation (4.13), then the associated normalized in $L^2(\Omega^\varepsilon_+)$ eigenfunction u^ε_m satisfies the estimate*

$$\|E^\varepsilon_\kappa \nabla_x u^\varepsilon_m; L^2(\Omega^\varepsilon_+)\|^2 + \varepsilon^{-2}\|E^\varepsilon_\kappa u^\varepsilon_m; L^2(\Omega^\varepsilon_+)\|^2 \leqslant \varepsilon^{-2}\mathbf{C}_m \quad \textit{for} \quad \varepsilon \in (0, \varepsilon_m], \tag{4.16}$$

where $E^\varepsilon_\kappa(y)$ is the weight factor (4.12), and κ and $\mathbf{C}_m$, ε_m are some positive numbers.

Proof. We introduce a thin triangular prism $\triangle^\varepsilon_+ = \{x \in \Omega^\varepsilon_+ : y_2 > 1-\varepsilon\}$. The difference $\Omega^\varepsilon_+ \setminus \overline{\triangle^\varepsilon_+}$ is a parallelepiped of height ε, and the Dirichlet conditions on its bases ensure the Friedrichs inequality

$$\|\mathbf{u}^\varepsilon_m; L^2(\Omega^\varepsilon_+ \setminus \triangle^\varepsilon_+)\|^2 \leqslant \frac{\varepsilon^2}{\pi^2}\|\nabla_x \mathbf{u}^\varepsilon_m; L^2(\Omega^\varepsilon_+ \setminus \triangle^\varepsilon_+)\|^2. \tag{4.17}$$

We have

$$\begin{aligned}\varepsilon^{-2}\Lambda_1 e^{2\kappa} \geqslant&\varepsilon^{-2}\Lambda_1 e^{2\kappa}\|u_m^\varepsilon; L^2(\triangle_+^\varepsilon\|^2 \geqslant \varepsilon^{-2}\Lambda_1\|\mathbf{u}_m^\varepsilon; L^2(\triangle_+^\varepsilon\|^2\\ =&\|\nabla_x\mathbf{u}_m^\varepsilon; L^2(\Omega_+^\varepsilon)\|^2 - \|\mathbf{u}_m^\varepsilon E_{-\kappa}^\varepsilon\nabla_y E_\kappa^\varepsilon; L^2(\Omega_+^\varepsilon\setminus\triangle_+^\varepsilon)\|^2 - \lambda_m^\varepsilon\|\mathbf{u}_m^\varepsilon; L^2(\Omega_+^\varepsilon\setminus\triangle_+^\varepsilon)\|^2\\ &-\big(\lambda_m^\varepsilon - \varepsilon^{-2}\Lambda_1\big)\|\mathbf{u}_m^\varepsilon; L^2(\triangle_+^\varepsilon)\|^2 \geqslant \delta\|\nabla_x\mathbf{u}_m^\varepsilon; L^2(\Omega_+^\varepsilon)\|^2\\ &+\Big((1-\delta)\frac{\pi^2}{\varepsilon^2} - \frac{\kappa^2}{\varepsilon^2} - \lambda_m^\varepsilon\Big)\|\mathbf{u}_m^\varepsilon; L^2(\Omega_+^\varepsilon\setminus\triangle_+^\varepsilon)\|^2 + \mathbf{c}_m\|\mathbf{u}_m^\varepsilon; L^2(\triangle_+^\varepsilon)\|^2.\end{aligned}$$

Here we have used the formulas (4.15) and (4.17), (4.11). It remains to choose small positive quantities δ and κ so that the last factor at the norm $\|\mathbf{u}_m^\varepsilon; L^2(\Omega_+^\varepsilon\setminus\triangle_+^\varepsilon)\|$ exceeds $(2\varepsilon)^{-2}(\pi^2-\Lambda_1)$. To estimate the first term in the left hand side of (4.16) we once again make the commutation, take into consideration the formula (4.15) and impose the restriction $\varepsilon\in(0,\varepsilon_m]$. The proof is complete. □

By Remark 3.1 the eigenfunction u_m^ε depends smoothly on the variable y_1. We introduce the functions

$$\begin{aligned}w_m^{\varepsilon 0}(y_1) &= \frac{1}{\varepsilon}\int_\Pi W_1(\eta)\chi(\varepsilon\eta_1)u_m^\varepsilon\big(y_1, 1-\varepsilon\eta_1, \varepsilon\eta_2)d\eta,\\ w_m^{\varepsilon\perp}(y_1,\eta) &= \chi(\varepsilon\eta_1)\varepsilon^{-1}u_m^\varepsilon\big(y_1, 1-\varepsilon\eta_1,\varepsilon\eta_2) - W_1(\eta)w_m^{\varepsilon 0}(y_1).\end{aligned}\tag{4.18}$$

We recall that $\chi(\varepsilon\eta_1) = \chi_+(y_1)$. By the conditions (2.6) and definitions (4.18) the orthogonality condition

$$\int_\Pi W_1(\eta)w_m^{\varepsilon\perp}(y_1,\eta)d\eta = 0\tag{4.19}$$

holds, and hence, in view of Lemma 2.1, we get the inequalities

$$\begin{aligned}&\Lambda_\perp\|w_m^{\varepsilon\perp}(y_1,\cdot); L^2(\Pi)\|^2 \leqslant \|\nabla_\eta w_m^{\varepsilon\perp}(y_1,\cdot); L^2(\Pi)\|^2\\ &\text{for all}\quad y_1\in(-1,1)\quad\text{and some}\quad \Lambda_\perp\in(\Lambda_1,\pi^2].\end{aligned}\tag{4.20}$$

According to the normalization (1.17), Theorem 4.1 and relation (4.19) we have

$$\begin{aligned}1 + O(e^{-\frac{\kappa}{3\varepsilon}}) &= \|\chi_+u_m^\varepsilon; L^2(\Omega_+^\varepsilon)\|^2 = \|w_m^{\varepsilon 0} + w_m^{\varepsilon\perp}; L^2((-1,1)\times\Pi)\|^2\\ &= \|w_m^{\varepsilon 0}; L^2(-1,1)\|^2 + \|w_m^{\varepsilon\perp}; L^2((-1,1)\times\Pi)\|^2.\end{aligned}\tag{4.21}$$

We transform the integral identity with the test function $\varepsilon^{-2}\chi_+^2u_m^\varepsilon\in H_0^1(\Omega_+^\varepsilon;\Gamma_{D+}^\varepsilon)$ into the form

$$\begin{aligned}\varepsilon^{-2}&\big(u_m^\varepsilon\nabla_x\chi_+, \nabla_x(\chi_+u_m^\varepsilon)\big)_{\Omega_+^\varepsilon} - \varepsilon^{-2}\big(\chi_+\nabla_x u_m^\varepsilon, u_m^\varepsilon\nabla_x\chi_+\big)_{\Omega_+^\varepsilon}\\ =&\varepsilon^{-2}\|\nabla_x(\chi_+u_m^\varepsilon); L^2(\Omega_+^\varepsilon)\|^2 - \varepsilon^{-2}\lambda_m^\varepsilon\|\chi_+u_m^\varepsilon; L^2(\Omega_+^\varepsilon)\|^2\\ =&\varepsilon^{-2}\|\nabla_\eta(W_1w_m^{\varepsilon 0} + w_m^{\varepsilon\perp}); L^2(\Omega_+^\varepsilon)\|^2 - \lambda_m^\varepsilon\|W_1w_m^{\varepsilon 0} + w_m^{\varepsilon\perp}; L^2(\Omega_+^\varepsilon)\|^2\\ &+\varepsilon^{-2}\left\|\frac{\partial(\chi_+u_m^\varepsilon)}{\partial y_1}; L^2(\Omega_+^\varepsilon)\right\|^2\\ =&\|w_m^{\varepsilon 0}; L^2(-1,1)\|^2\|\nabla_\eta W_1; L^2(\Pi)\|^2 - \varepsilon^2\lambda_m^\varepsilon\|w_p^{\varepsilon 0}; L^2(-1,1)\|^2\\ &+\|\nabla_\eta w_m^{\varepsilon\perp}; L^2((-1,1)\times\Pi)\|^2 - \varepsilon^2\lambda_m^\varepsilon\|w_m^{\varepsilon\perp}; L^2((-1,1)\times\Pi)\|^2\\ &+2\int_{-1}^1 w_m^{\varepsilon 0}(y_1)\int_\Pi\nabla_\eta W_1(\eta)\nabla_\eta w_m^{\varepsilon\perp}(y_1,\eta)d\eta dy_1 + \frac{1}{\varepsilon^2}\Big\|\frac{\partial}{\partial y_1}\big(\chi_+u_m^\varepsilon); L^2(\Omega_+^\varepsilon)\Big\|^2.\end{aligned}\tag{4.22}$$

The estimate (4.16) shows that the absolute value of left hand side in the identity (4.22) does not exceed the quantity $c_p\varepsilon^{-3}e^{-\frac{2\kappa}{3\varepsilon}}$. The integral over the set $(-1,1)\times\Pi$ (the penultimate term in (4.22)) is eliminated by means of integration by parts, the Helmholtz equation for the factor W_1 and the orthogonality condition (4.19). We neglect the last term since it is not needed. Thus, by the formula (4.20) and restriction (4.13) we obtain the estimate

$$\begin{aligned} c_m^{\perp}\|w_m^{\varepsilon\perp}; L^2((-1,1)\times\Pi)\|^2 &\leqslant (\Lambda_\perp - \varepsilon^2\lambda_m^\varepsilon)\|w_m^{\varepsilon\perp}; L^2((-1,1)\times\Pi)\|^2 \\ &\leqslant (\varepsilon^2\lambda_m^\varepsilon - \Lambda_1)\|w_m^{\varepsilon 0}; L^2(-1,1)\|^2 + c_m\varepsilon^{-3}e^{-\frac{2\kappa}{3\varepsilon}} \leqslant C_m\varepsilon^2 \end{aligned} \tag{4.23}$$

with some independent of the small parameter $\varepsilon \in (0, \varepsilon_m]$ factor

$$\frac{c_m^{\perp}(\Lambda_\perp - \Lambda_1)}{2} > 0.$$

The made calculations show that on an infinitesimal positive sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ the convergences

$$\begin{array}{lll} \lambda_m^\varepsilon - \varepsilon^{-2}\Lambda_1 & \to & \mu_m^{00}, \\ w_m^{\varepsilon 0} & \to & w_m^{00} \quad \text{strongly in} \quad L^2(-1,1), \quad \text{and} \quad \|w_m^{00}; L^2(-1,1)\| = 1 \end{array} \tag{4.24}$$

hold.

We take some infinitely differentiable function Ψ of the variable $y_1 \in [-1, 1]$ obeying the conditions

$$\pm\frac{d\Psi}{dy_1}(\pm 1) = 0, \tag{4.25}$$

and multiply the Helmoholtz equation for the eigenpair $\{\lambda_m^\varepsilon; u_m^\varepsilon\}$ by $\varepsilon^{-1}\chi_+ W_1\Psi$. Integrating by parts in the domain Ω_+^ε and differentiating, we obtain the identity

$$\begin{aligned} \int_{-1}^{1}\Big(\Big(\lambda_m^\varepsilon - \frac{\Lambda_1}{\varepsilon^2}\Big)\Psi(y_1) + \frac{d^2\Psi}{dy_1^2}(y_1)\Big)\frac{1}{\varepsilon}\int_\Pi W_1(\eta)\chi_+(y_2)u_m^\varepsilon(y,z)dy_1dzdy_1 \\ = \frac{1}{\varepsilon}\int_{\Omega_+^\varepsilon}\Psi(y_1)u_m^\varepsilon(x)\Big[\frac{d^2}{dy_2^2}, \chi_+(y_2)\Big]W_1(\eta)dx. \end{aligned}$$

The right hand side is infinitesimal as $\varepsilon \to +0$ due to the exponential decay of the functions u_m^ε and W_1 (Theorem 4.1 and the formula (2.7)). The convergences (4.24) allow us to pass to the limit in the left hand side and in view of the first definition (4.18) we get the relation

$$\int_{-1}^{1} w_m^{00}(y_1)\Big(\mu_m^{00}\Psi(y_1) + \frac{d^2\Psi}{dy_1^2}(y_1)\Big)dy_1 = 0,$$

which in view of the arbitrariness of the test function $\Psi \in C^\infty[-1, 1]$ obeying only the boundary conditions (4.24), imply the inclusions $w_m^{00} \in H^2(-1, 1)$ and the differential equation and boundary conditions in the problem (4.1) (cf. the smoothness improving in [15, Ch. 2]).

Lemma 4.1. *The limits* (4.24) *provide an eigenpair* $\{\mu_m^{00}; w_m^{00}\}$ *of the limiting problem* (4.1) *and by the formulas* (4.21) *and* (4.23) *the eigenfunction* w_p^{00} *is normalized in the space* $L^2(-1, 1)$.

4.4. Theorem on asymptotics. The already standard arguing from Section 3.5 with simplifications caused by imposing the artificial boundary conditions (4.2) and the simplicity of eigenvalues of limiting problem (4.1) lead us to the following statements on the eigenpairs of the original problem (1.1)–(1.3) in the entire domain Ω^ε.

Theorem 4.2. *In the situation* (1.5) *for each* $p \in \mathbb{N}$ *there exist positive quantities* c_p, C_p *and* ε_p *such that for* $\varepsilon \in (0, \varepsilon_p]$ *the eigenvalues* $\lambda^\varepsilon_{(p,1)} := \lambda^\varepsilon_{2p}$ *and* $\lambda^\varepsilon_{(p,2)} := \lambda^\varepsilon_{2p-1}$ *in the sequence* (1.8) *obey the representation* (1.12), *where the absolute values of the remainders* $\widetilde{\lambda}^\varepsilon_{(p,j)}$ *do no exceed the expression* $c_p \varepsilon^{-\frac{3}{2}} e^{-\frac{\kappa}{\varepsilon}}$, *while the normalized in* $L^2(\Omega^\varepsilon)$ *eigenfunctions* $u^\varepsilon_{p,1} = u^\varepsilon_{2p}$ *and* $u^\varepsilon_{p,2} = u^\varepsilon_{2p-1}$, *even and odd in the variable* y_2, *obey the representations* (1.13) *with* $K^\pm_{p,1} = 2^{-\frac{1}{2}}$, $K^\pm_{p,2} = \pm 2^{-\frac{1}{2}}$ *and*

$$\varepsilon \big\| \nabla_x \widetilde{u}^\varepsilon_{p,j}; L^2(\Omega^\varepsilon) \| + \big\| \widetilde{u}^\varepsilon_{p,j}; L^2(\Omega^\varepsilon) \| \leqslant C_p \varepsilon^{-\frac{3}{2}} e^{-\frac{\kappa}{\varepsilon}}.$$

The formulas (1.12) *and* (1.13) *involve the eigenpair* $\{\Lambda_1; W_1\}$ *of problem* (2.1)–(1.13) *given by Lemma 2.1.*

4.5. Other asymptotic series of eigenvalues. The formal asymptotic constructions from Section 3.1 can be easily adapted for the problem (1.1)–(1.3) in the situation (1.5), and as a result, for the terms of asymptotic ansätze (3.1) and (3.2) we derive the limiting problem (3.3). A bit unexpected fact is that despite the Neumann condition (1.3) on the faces $\Gamma^\varepsilon_\pm$, the Dirichlet condition is preserved on the sides $\upsilon^\pm_1$ of square $\square_1$ (cf. the formulas (1.16) and (3.33)). The reason is that according to the general principles in [22, Ch. 16] and [41], the boundary conditions in the limiting problem for the thin domain are determined by the phenomenon of threshold resonance in the problem on boundary layer and not by the type of boundary conditions on the end. Such resonance is absent in both problems (2.1)–$(2.3)_{N,D}$ on the semi–strip Π (see Section 2.5), and this ensures the Dirichlet condition on the sides $\upsilon^\pm_1$ in both situations.

Remark 4.1. *For the same conclusion on the limiting boundary conditions on* $\upsilon^\pm_1$ *one can employ the method of composite asymptotic expansions and the method of matching asymptotic expansions, see, for instance, the monographs* [22], [24] *and* [46], [47], *respectively. Indeed, the leading term in the remainder of* $\sin(\pi z/\varepsilon) v(y)$ *in the boundary condition* (1.3) *on the face* $\Gamma^\varepsilon_\pm$ *is equal to* $2^{-\frac{1}{2}} \pi \varepsilon^{-1} \cos(\pi z/\varepsilon) v(y)$ *and in the first method, to lessen the remainder, exactly the Dirichlet condition is needed. In the framework of the second method one needs to match the expression* $\sin(\pi z/\varepsilon) v(y_1, \pm 1)$ *with some solution of the problem* (2.1)–$(2.3)_N$ *in the semi–strip* Π, *but because of the absence of threshold resonance, this problem has only the trivial bounded solution and this is why we have to let* $v(y_1, \pm 1) = 0$.

Reproducing the calculations and arguing from Section 3.3 and using the function $\mathbf{W}_N$ instead of the function $\mathbf{W}_D$ (see the formula (2.42) in Remark 2.2), we obtain the following statement.

Theorem 4.3. *For each* $m \in \mathbb{N}$ *there exist positive quantities* c_m *and* ε_m *and the index* $n_m(\varepsilon) \in \mathbb{N}$ *such the eigenvalue of problem* (1.1)–(1.3) *satisfies the relation*

$$\left| \lambda^\varepsilon_{n_m(\varepsilon)} - \varepsilon^{-2}\pi^2 - \mu_m \right| \leqslant c_m \varepsilon \quad \textit{for} \quad \varepsilon \in (0, \varepsilon_m], \tag{4.26}$$

where μ_m *is the term in the sequence* (3.5) *of limiting problem* (3.3) *on the square* $\square_1$.

In contrast to Theorem 3.1, in Theorem 4.3 the index $n_m(\varepsilon)$ of eigenvalue $\lambda^\varepsilon_{n_m(\varepsilon)}$ appearing in the formula (4.26) is not defined. This is explained by the fact that according to Theorem 4.2, the sequence (1.8) contains eigenvalues of order $\varepsilon^{-2}\Lambda_1$, which is smaller that $\varepsilon^{-2}\pi^2$, and the total amount of such numbers in the interval $(0, \varepsilon^{-2}\pi^2)$ increases unboundedly as $\varepsilon \to +0$. Thus, the index $n_m(\varepsilon)$ depends on the parameter ε and also tends to infinity as the parameters decreases. In other words, Theorems 4.2 and 4.3 describe different asymptotics of eigenvalues of problem (1.1)–(1.3) from formally low– and middle–frequences ranges of the spectrum, respectively.

5. Localization near short edges of polyhedron

5.1. Formal asymptotic constructions. Now the domain Ω^ε on Fig. 3a is defined by the formula (1.23), and to simplify the asymptotic procedures on both central sections (3.38) we impose the artificial Dirichlet or Neumann conditions; totally four options. We restrict the problem (1.1)–(1.3) to the subdomain $\Omega^\varepsilon_\# = \{x \in \Omega^\varepsilon : y_j < 0, j = 1,2\}$ and assign the subscript # to its index and the attributes; the type of artificial boundary conditions does not influence further arguing, calculations and results. The objects introduced in Section 3.2 for this problem are also equipped by the subscript # and as the almost eigenpair we take

$$\{t^\varepsilon_{1\#}; U^\varepsilon_{1\#}(x)\} = \{\varepsilon^2 M_1^{-1}; \|\chi_\# V_1; \mathcal{H}^\varepsilon\|^{-1}\chi_\#(y)V_1(\xi)\}. \tag{5.1}$$

Here $\{M_1; V_1\}$ is the eigenpair of problem (2.13) in the quarter of layer (1.20) presented by Theorem 2.2, the rescaled variables ξ are of form (2.12) and $\chi_\#(y) = \chi(r)$, while $r = |y - \mathcal{P}^{--}|$ is the polar radius, $\mathcal{P}^{--} = (-1,-1)$ is the vertex of square $\square_1$ and χ is the cut–off function (2.5). By Theorem 2.3 the normalized in $L^2(\Xi)$ eigenfunction V_1 decays exponentially as $|\xi| \to +\infty$ and hence, the relations

$$\begin{aligned}
&\|\chi_\# V_1; \mathcal{H}^\varepsilon\|^2 = \varepsilon\left(M_1 + O(e^{-\frac{\kappa}{\varepsilon}})\right),\\
&\|(\Delta_x + \varepsilon^{-2}M_1)(\chi_\# V_1); L^2(\Omega^\varepsilon_\#)\|^2 = \|[\Delta_x, \chi_\#]V_1; L^2(\Omega^\varepsilon_\#)\|^2\\
&\qquad\qquad\qquad \leqslant c\varepsilon\big(\varepsilon^{-2}+1\big)e^{-\frac{2\kappa}{3\varepsilon}} \leqslant C\varepsilon^{-1}e^{-\frac{2\kappa}{3\varepsilon}},
\end{aligned}$$

are true, where $\kappa > 0$ is the exponent from the formula (2.35). Thus, Lemma 3.1 provides an eigenvalue of the operator $\mathcal{T}^\varepsilon$ obeying the inequality

$$\left|\tau^\varepsilon_{n(\varepsilon)\#} - \varepsilon^2 M_1^{-1}\right| \leqslant c\varepsilon e^{-\frac{\kappa}{3\varepsilon}}.$$

Owing to the relation (3.16) of spectral parameters, similar to (3.26)–(3.29) calculations show that

$$\left|\lambda^\varepsilon_{n(\varepsilon)\#} - \varepsilon^{-2}M_1\right| \leqslant c_1\varepsilon^{-3}e^{-\frac{\kappa}{3\varepsilon}} \quad \text{quad} \quad \varepsilon \in (0, \varepsilon_1], \tag{5.2}$$

and c_1 and ε_1 are some positive numbers.

5.2. Justification of asymptotics. Since the multiplicity of discrete spectrum of problem (2.13) remains unknown, the usual way of confirming the identity $n(\varepsilon) = 1$ in the estimate (5.2), in particular, of proving the convergence theorem, is not appropriate. We follow another way.

First of all, by the minimax principle [30, Thm. 10.2.1] we get the relation

$$\begin{aligned}
\lambda^\varepsilon_1 &= \min_{\psi^\varepsilon \in \mathcal{H}^\varepsilon} \frac{\|\nabla_x \psi^\varepsilon; L^2(\Omega^\varepsilon_\#)\|^2}{\|\psi^\varepsilon; L^2(\Omega^\varepsilon_\#)\|^2} \leqslant \frac{\|\nabla_x(\chi_\# V_1); L^2(\Omega^\varepsilon_\#)\|^2}{\|\chi_\# V_1; L^2(\Omega^\varepsilon_\#)\|^2}\\
&\leqslant \frac{\varepsilon\|\nabla_\xi V_1; L^2(\Xi)\|^2 + c^1_V\varepsilon e^{-\frac{2\kappa}{3\varepsilon}}}{\varepsilon^3\|V_1; L^2(\Xi)\|^2 + c^0_V\varepsilon e^{-\frac{2\kappa}{3\varepsilon}}} \leqslant \frac{1}{\varepsilon^2}\left(M_1 + C_V\varepsilon^{-2}e^{-\frac{\kappa}{3\varepsilon}}\right).
\end{aligned} \tag{5.3}$$

Now we are going to make sure that the eigenfunctions $u^\varepsilon_{m\#}$ fast decays far from the point $\mathcal{P}^{++}$; the method of verifying this property echoes the proofs of Theorems 2.3 and 4.1.

Theorem 5.1. *Let $\lambda^\varepsilon_{m\#}$ be an eigenvalue of problem (1.1)–(1.3)$_\#$ in the subdomain $\Omega^\varepsilon_\#$ with some artificial boundary conditions and the inequality*

$$\varepsilon^2\lambda^\varepsilon_{m\#} \leqslant \Lambda_1 - \delta_\# \quad \textit{for} \quad \delta_\# > 0$$

holds. Then the associated normalized in the space $L^2(\Omega^\varepsilon_\#)$ eigenfunction $u^\varepsilon_{m\#}$ the weight estimate

$$\|e^{\frac{\kappa_{m\#}r}{\varepsilon}}\nabla_x u^\varepsilon_{m\#}; L^2(\Omega^\varepsilon_\#)\|^2 + \varepsilon^{-2}\|e^{\frac{\kappa_{m\#}r}{\varepsilon}}u^\varepsilon_{m\#}; L^2(\Omega^\varepsilon_\#)\|^2 \leqslant C_{m\#}\varepsilon^{-2}, \tag{5.4}$$

holds, where $\varepsilon \in (0, \varepsilon_{m\#}]$, and $\kappa_{m\#}$, $\varepsilon_{m\#}$ and $C_{m\#}$ are some positive numbers.

Proof. In the integral identity $(1.9)_\#$ corresponding to the problem (1.1)–$(1.3)_\#$ in the domain $\Omega^\varepsilon_\#$ we substitute the product $\psi^\varepsilon = e^{\frac{2\kappa r}{\varepsilon}} u^\varepsilon_{m\#}$ with some exponent $\kappa > 0$ and after simple transformations for the function $\mathbf{u}^\varepsilon_m = e^{\frac{\kappa r}{\varepsilon}} u^\varepsilon_{m\#}$ we obtain the identity

$$\|\nabla_x \mathbf{u}^\varepsilon_m; L^2(\Omega^\varepsilon_\#)\|^2 - \|\mathbf{u}^\varepsilon_m e^{-\frac{\kappa r}{\varepsilon}} \nabla_x e^{\frac{\kappa r}{\varepsilon}}; L^2(\Omega^\varepsilon_\#)\|^2 = \lambda^\varepsilon_{m\#} \|\mathbf{u}^\varepsilon_m; L^2(\Omega^\varepsilon_\#)\|^2. \tag{5.5}$$

We note that

$$e^{-\frac{\kappa r}{\varepsilon}} |\nabla_x e^{\frac{\kappa r}{\varepsilon}}| = \kappa \varepsilon^{-1}, \tag{5.6}$$

and partition the set $\Omega^\varepsilon_\#$ into four parts, namely, $\Xi^\varepsilon(R) = \{x : \xi \in \Xi(R)\}$ (cf. Definition (2.23)), $K^\varepsilon_R = \Omega^\varepsilon_\# \setminus (\Sigma^{1\varepsilon}_R \cup \Sigma^{2\varepsilon}_R)$ and

$$\Sigma^{1\varepsilon}_R = \{x \in \Omega^\varepsilon_\# : 1 + y_1 > \varepsilon(R-1), 1 + y_2 < \varepsilon R\},$$
$$\Sigma^{2\varepsilon}_R = \{x \in \Omega^\varepsilon_\# : 1 + y_1 < \varepsilon(R-1), 1 + y_2 > \varepsilon R\}.$$

As in Section 2.4, the size $R > 1$ is chosen so that in view of Proposition 2.1 to satisfy the estimates

$$\frac{1}{\varepsilon^2}\Big(\Lambda_1 - \frac{\delta_\#}{2}\Big) \|\mathbf{u}^\varepsilon_m; L^2(\Sigma^{j\varepsilon}_R)\|^2 \leqslant \|\nabla_x \mathbf{u}^\varepsilon_m; L^2(\Sigma^{j\varepsilon}_R)\|^2, \quad j = 1, 2.$$

Moreover, the one–dimensional Friedrichs inequality on the segment $(0, \varepsilon) \ni z$ shows that

$$\|\mathbf{u}^\varepsilon_m; L^2(K^\varepsilon_R)\|^2 \leqslant \frac{\varepsilon^2}{\pi^2} \|\partial_z \mathbf{u}^\varepsilon_m; L^2(K^\varepsilon_R)\|^2.$$

Now the formulas (5.5) and (5.6) imply the relation

$$\begin{aligned}
e^{2\sqrt{2}\kappa R}\big(\lambda^\varepsilon_{m\#} + \kappa^2\varepsilon^{-2}\big) \geqslant & e^{2\sqrt{2}\kappa R}\big(\lambda^\varepsilon_{m\#} + \kappa^2\varepsilon^{-2}\big)\|u^\varepsilon_{m\#}; L^2(\Xi^\varepsilon(R))\|^2 \\
\geqslant & \big(\lambda^\varepsilon_{m\#} + \kappa^2\varepsilon^{-2}\big)\|\mathbf{u}^\varepsilon_m; L^2(\Xi^\varepsilon(R))\|^2 = \|\nabla_x \mathbf{u}^\varepsilon_m; L^2(\Omega^\varepsilon_\#)\|^2 \\
& - \lambda^\varepsilon_{m\#}\|\mathbf{u}^\varepsilon_m; L^2(\Omega^\varepsilon_\# \setminus \Xi^\varepsilon(R))\|^2 - \|\mathbf{u}^\varepsilon_m e^{-\frac{\kappa r}{\varepsilon}} \nabla_x e^{\frac{\kappa r}{\varepsilon}}; L^2(\Omega^\varepsilon_\# \setminus \Xi^\varepsilon(R))\|^2 \\
\geqslant & \delta\|\nabla_x \mathbf{u}^\varepsilon_m; L^2(\Omega^\varepsilon_\#)\|^2 + \Big((1-\delta)\frac{\pi^2}{\varepsilon^2} - \lambda^\varepsilon_{m\#} - \frac{\kappa^2}{\varepsilon^2}\Big)\|\mathbf{u}^\varepsilon_m; L^2(K^\varepsilon_R)\|^2 \\
& + \Big(\frac{1-\delta}{\varepsilon^2}\Big(\Lambda_1 - \frac{\delta_\#}{2}\Big) - \lambda^\varepsilon_{m\#} - \frac{\kappa^2}{\varepsilon^2}\Big) \sum_{j=1,2} \|\mathbf{u}^\varepsilon_m; L^2(\Sigma^{j\varepsilon}_R)\|^2.
\end{aligned}$$

It remains to take positive $\delta = \delta_{m\#} > 0$ and $\kappa = \kappa_{m\#} > 0$ so small that the coefficients at the squares of Lebesgue norms of the functions $\mathbf{u}^\varepsilon_m$ in the right hand side exceed the quantity $c_{\delta,\kappa}\varepsilon^{-2}$ with some factor $c_{\delta,\kappa} > 0$. The proof is complete. □

Now we apply the minimax principle [30, Thm. 10.2.1] to the operator of problem (2.13) in the quarter of layer Ξ

$$M_1 = \min_{\Psi \in H^1(\Xi;\Upsilon)} \frac{\|\nabla_\xi \Psi; L^2(\Xi)\|^2}{\|\Psi; L^2(\Xi)\|^2}. \tag{5.7}$$

As the test function we take the function $\Xi \ni \xi \mapsto \Psi^\varepsilon(\xi) = \chi_\#(y) u^\varepsilon_{1\#}(x)$ (the relation between the coordinate systems ξ and x is given by the formula (2.12)). In view of Theorem 5.1 we have

$$\begin{aligned}
\|\Psi^\varepsilon; L^2(\Xi)\|^2 &\geqslant \varepsilon^{-3}\|u^\varepsilon_{1\#}; L^2(\Omega^\varepsilon_\#)\|^2 - \varepsilon^{-3}\|(1-\chi^2_\#)^{\frac12} u^\varepsilon_{1\#}; L^2(\Omega^\varepsilon_\#)\|^2 \\
&\geqslant \varepsilon^{-3} - c_0\varepsilon^{-3} e^{-\frac{2\kappa_{1\#}}{3\varepsilon}}, \\
\|\nabla_\xi \Psi^\varepsilon; L^2(\Xi)\|^2 &\leqslant \varepsilon^{-1}\|\nabla_x u^\varepsilon_{1\#}; L^2(\Omega^\varepsilon_\#)\|^2 + \varepsilon^{-1}\big(2\chi_\# \nabla_x u^\varepsilon_{1\#} + u^\varepsilon_{1\#}\nabla_x \chi_\#, u^\varepsilon_{1\#}\nabla_x \chi_\#\big)_{\Omega^\varepsilon_\#} \\
&\leqslant \varepsilon^{-1}\lambda^\varepsilon_{1\#} + c_1\varepsilon^{-3} e^{-\frac{2\kappa_{1\#}}{3\varepsilon}}.
\end{aligned}$$

These estimates and identity (5.7) imply the relation

$$\frac{M_1}{\varepsilon^2} \leqslant \frac{\lambda^\varepsilon_{1\#} + c_1\varepsilon^{-2}e^{-\frac{2\kappa_{1\#}}{3\varepsilon}}}{1 - c_0 e^{-\frac{2\kappa_\#}{3\varepsilon}}} \leqslant \lambda^\varepsilon_{1\#} + C\varepsilon^{-2}e^{-2\kappa_\#/3\varepsilon}. \tag{5.8}$$

The formulas (5.3) and (5.8), and the inequality (5.9), which will be verified later, that in the estimate we can take $n(\varepsilon) = 1$. Using even and odd continuation of eigenfunctions $u^\varepsilon_{1\#}$ through the sections (3.38), on which the artificial Dirichlet and Neumann boundary conditions, we obtain the asymptotics of first four eigenvalues of problem (1.1)–(1.3) in the entire domain Ω^ε; owing to the parity properties the associated eigenfunctions are linearly independent.

Theorem 5.2. *The first four terms of sequence* (1.8) *of eigenvalues of problem* (1.1)–(1.3) *in the domain* (1.23) *satisfy the asymptotic formulas* (1.25), *and the remainders* $\widetilde{\lambda}^\varepsilon_k$ *obey the estimates*

$$\big|\widetilde{\lambda}^\varepsilon_k\big| = \big|\lambda^\varepsilon_k - \varepsilon^{-2}M_1\big| \leqslant c_\sharp\varepsilon^{-2}e^{-\frac{2\kappa_\sharp}{3\varepsilon}} \quad \textit{for} \quad \varepsilon \in (0, \varepsilon_\sharp] \quad \textit{and} \quad k = 1,2,3,4,$$

where M_1 *is the first eigenvalue of problem* (2.13), *and* $c_\sharp$, $\kappa_\sharp$ *and* $\varepsilon_\sharp$ *are some positive quantities.*

Proof. It remains to make sure that for the second eigenvalue $\lambda^\varepsilon_{2\#}$ of problem (1.1)–(1.3)$_\#$ the inequality

$$\lambda^\varepsilon_{2\#} \geqslant \varepsilon^{-2}M_\perp \tag{5.9}$$

holds with some $M_\perp \in (M_1, \Lambda_1)$. Suppose that the relation (5.9) fails, that is, there exists an infinitesimal positive sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$, for which

$$\varepsilon_j^2\lambda^{\varepsilon_j}_{2\#} \to M_1 \quad \text{as} \quad j \to +\infty \quad (\text{or} \quad \varepsilon_j \to +0). \tag{5.10}$$

Omitting several first terms from the sequence, we suppose that $\varepsilon_j^2\lambda^{\varepsilon_j}_{2\#} \leqslant (M_1 + \Lambda_1)/2$ for all $j \in \mathbb{N}$ and in what follows we do not write this subscript. By the eigenfunctions $u^\varepsilon_{1\#}$ and $u^\varepsilon_{2\#}$ obeying the relations

$$\big(u^\varepsilon_{j\#}, u^\varepsilon_{k\#}\big)_{\Omega^\varepsilon_\#} = \delta_{j,k}, \qquad j,k = 1,2,$$

we define the functions on the quarter of layer Ξ

$$w^\varepsilon_{j\#}(\xi) = \varepsilon^{\frac{3}{2}}\chi_\#(y)u^\varepsilon_{j\#}(x), \qquad j = 1,2.$$

The inequalities

$$\begin{aligned} &\big|\big(w^\varepsilon_{j\#}, w^\varepsilon_{k\#}\big)_\Xi - \delta_{j,k}\big| \leqslant c\varepsilon^{-2}e^{-\frac{\kappa_\#}{3\varepsilon}}, \\ &\big|\big(\nabla_\xi w^\varepsilon_{j\#}, \nabla_\xi w^\varepsilon_{k\#}\big)_\Xi - \lambda^\varepsilon_{j\#}\delta_{j,k}\big| \leqslant ce^{-\frac{\kappa_\#}{3\varepsilon}}, \quad j,k = 1,2. \end{aligned} \tag{5.11}$$

This first is implied immediately from the estimate (5.4), while to get the second inequality we should additionally take into consideration the integral identity

$$\big(\nabla_x u^\varepsilon_{j\#}, \nabla_x\psi^\varepsilon\big)_{\Omega^\varepsilon_\#} = \lambda^\varepsilon_{j\#}\big(u^\varepsilon_{j\#}, \psi^\varepsilon\big)_{\Omega^\varepsilon_\#} \tag{5.12}$$

with the test function $\psi^\varepsilon = \chi^2_\# u^\varepsilon_{j\#} \in H^1_0(\Omega^\varepsilon_\#; \Gamma^\varepsilon_D)$, which, as usually, is transformed into the identity

$$\begin{aligned} &\big(\nabla_x(\chi_\# u^\varepsilon_{j\#}), \nabla_x(\chi_\# u^\varepsilon_{k\#})\big)_{\Omega^\varepsilon_\#} - \lambda^\varepsilon_{j\#}\big(\chi_\# u^\varepsilon_{j\#}, \chi_\# u^\varepsilon_{j\#}\big)_{\Omega^\varepsilon_\#} \\ &\qquad = \big(u^\varepsilon_{j\#}\nabla_x\chi_\#, \nabla_x(\chi_\# u^\varepsilon_{k\#})\big)_{\Omega^\varepsilon_\#} - \big(\chi_\#\nabla_x u^\varepsilon_{j\#}, u^\varepsilon_{k\#}\nabla_x\chi_\#\big)_{\Omega^\varepsilon_\#}. \end{aligned}$$

We pass to the limit as $\varepsilon \to +0$ in the integral identity (5.12) with the test function $\psi^\varepsilon(x) = \varepsilon^{\frac{1}{2}}\Psi(\xi)$, where $\Psi \in C^\infty_c(\Xi \cup \Theta)$. As a result,

$$w^0_{j\#} = \lim_{\varepsilon\to+0} w^\varepsilon_{j\#} \quad \text{weakly in} \quad H^1_0(\Xi; \Upsilon)$$

in accordance with the formulas (5.3), (5.8), (5.10) and (5.11) we get the relations

$$(\nabla_\xi w^0_{p\#}, \nabla_\xi \Psi)_\Xi = M_1(w^0_{p\#}, \Psi)_\Xi \quad \forall \Psi \in C^\infty_c(\Xi \cup \Theta), \quad p = j, k,$$
$$(w^0_{j\#}, w^0_{k\#})_\Xi = \delta_{j,k}, \quad j,k = 1,2,$$

which are impossible due to the simplicity of first eigenvalue M_1. The found contradiction means the validity of inequality (5.9). The proof is complete. □

We are in position to complete the arguing from the end of Section 5.1. Namely, on the base of Theorem 5.2, the relations (5.9) and relation (3.15) of spectral parameter we conclude that for some, generally speaking small $h > 0$ the segment

$$[\varepsilon^2(M_1^{-1} - h), \varepsilon^2(M_1^{-1} + h)]$$

contains the unique eigenvalue $\tau^\varepsilon_{1\#}$ of the operator $\mathcal{A}^\varepsilon_\#$. Thus, the relation (3.17) in the second part of Lemma 3.1, in which we let $\delta^\varepsilon = c_1\varepsilon^{-3}e^{-\frac{\kappa_{1\#}}{3\varepsilon}}$ and $\delta^\varepsilon_* = h\varepsilon^2$, provides the estimate for Sobolev norm of the difference between the eigenfunction $u^\varepsilon_{1\#}$ and its approximation $U^\varepsilon_\#$ from the formula (5.1). Finally, recalling even and odd continuations of eigenfunctions in the quarter $\Omega^\varepsilon_\#$ on the entire domain Ω^ε (there are four of them), we formulate the obtained result.

Theorem 5.3. *For the first four eigenfunctions of problem* (1.1)–(1.3) *in the polyhedron* (1.23) *the asymptotic formulas*

$$\varepsilon\Big\|\nabla_x u^\varepsilon_k - \frac{1}{2}\varepsilon^{-\frac{3}{2}}\sum_{\alpha,\vartheta=\pm} C^k_{\alpha\vartheta}\chi_{\alpha\vartheta}\nabla_x V_1; L^2(\Omega^\varepsilon)\Big\|$$
$$+\Big\|u^\varepsilon_k - \frac{1}{2}\varepsilon^{-\frac{3}{2}}\sum_{\alpha,\vartheta=\pm} C^k_{\alpha\vartheta}\chi_{\alpha\vartheta} V_1; L^2(\Omega^\varepsilon)\Big\| \leqslant C_0\varepsilon^{-4}e^{-\frac{\kappa}{3\varepsilon}} \quad \text{for} \quad \varepsilon \in (0, \varepsilon_0].$$

hold. Here $\chi_{\alpha\vartheta}(y) = \chi(|y - P^{\alpha\vartheta}|)$ *are cut–off functions,* $P^{\alpha\vartheta}$ *are the vertices* (1.24) *of the square* $\mathbb{Q}_1$*,* $V_1 \in H^1_0(\Xi;\Upsilon)$ *is the first eigenfunction of the problem* (2.13) *in the quarter of layer* (1.20) *depending on the system of rescaled Cartesian coordinates* $\xi^{\alpha\vartheta} = \varepsilon^{-1}(y - P^{\alpha\vartheta}, z)$ *appropriately rotated (see the formula* (2.12) *in the case* $\alpha = \vartheta = -1$*). Moreover,* $C^1_{\alpha\vartheta} = 1$*, while other columns of the coefficients* $C^k = \big(C^k_{++}, C^k_{-+}, C^k_{+-}, C^k_{--}\big)$ *are taken from the list*

$$(1,-1,-1,1), \quad (-1,1,-1,1), \quad (-1,-1,1,1).$$

5.3. Other asymptotic series of eigenvalues. Theorems 5.2 and 5.3 provide no complete information on asymptotics of the spectrum of problem (1.1)–(1.3) in the polyhedron (1.23). At first glance, it seems that the asymptotics procedure allow us to find the series of eigenvalues with other stable asymptotics. Indeed, by means of calculations and arguing from Section 3.3 we can verify the following statement.

Theorem 5.4. *For all* $p, q \in \mathbb{N}$ *there exist positive numbers* $c_{(p,q)}$ *and* $\varepsilon_{(p,q)}$*, as well as the index* $n_{(p,q)}(\varepsilon) \in \mathbb{N}$*, such that for the eigenvalue of problem* (1.1)–(1.3) *in the polyhedron* (1.23) *the relation holds*

$$\big|\lambda^\varepsilon_{n_{(p,q)}(\varepsilon)} - \varepsilon^{-2}\pi^2 - \mu_{(p,q)}\big| \leqslant c_{(p,q)}\varepsilon \quad \text{for} \quad \varepsilon \in (0, \varepsilon_{(p,q)}]. \tag{5.13}$$

Here $\mu_{(p,q)} = \pi^2\big(p^2 + q^2\big)/4$ *are the eigenvalues of the Dirichlet problem for the Laplace operator in the square* $\square_1$.

We stress that the Dirichlet conditions on the boundary $\partial\square_1$ are due to the slope of all four lateral sides and the absence of threshold resonance in the problem (2.1)–(2.3)$_N$, see Section 4.5. At the same time, as in Theorem 4.3, the eigenvalues in the formula (5.13), having large indices $n_{(p,q)}(\varepsilon)$, belong to the middle–frequency range of spectrum.

One can try to obtain the formal asymptotic representations of the eigenvalues λ_5^ε, λ_6^ε, λ_7^ε, ... by means of the asymptotic procedure from Section 4.1. The ordinary differential equations on four segments $(-1,1)$ are obtained by the same scheme, but in this work we fail to justify the imposing of boundary conditions or transmission conditions at the points $\mathcal{P}^{\pm\vartheta}$, $\vartheta = \pm$ since the author does not know whether there is the threshold resonance in the problem (2.13) on the quarter of layer (1.20). If it is absent, then the mentioned equations are equipped with the Dirichlet condition at $y_k = \pm 1$, however, the emergence of resonance can give rise, for instance, to the classical Kirchhoff transmission conditions at the vertices (1.24) (see, for instance, [41]), which join the differential equation on the sides (3.33) of square $\square_1$ into a single spectral problem.

We especially stress that for the problem (2.13) with $M = \Lambda_1$ in the quarter of layer (1.20) with skewed lateral sides the notion of the threshold resonance is to be specified since the asymptotic behavior at infinity of its solution is unknown: the Fourier method does not work by clear reasons, while the known results on the behavior of solutions in layer–type domains (see, for instance, [48] and others) concern mostly the Neumann condition and do not serve the specific mixed boundary value problem appeared in the present work.

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